Z. Mróz · D. Bojczuk

Shape and topology sensitivity analysis and its application to structural design

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Abstract The incremental sensitivity analysis associated with variation of structure material parameters, shape or topology variation is generally discussed by analyzing the evolution of potential and complementary energies, or arbitrary functionals of state fields. The concept of configuration and sensitivity generalized forces is used in presenting the sensitivity derivatives. The general reciprocity relations are derived for the case of potential or complementary energy variations. The topology variations in bar structures related to introduction of elements and introduction of inclusions and voids in plates are discussed, and the sensitivity forces are derived.

Keywords Sensitivity analysis · Topological derivative · Structural design · Bar structures · Plates

1 Introduction

The usual structural analysis problems are associated with specification of stress, strain and displacement fields for given structure configuration, topology, material parameters, loading and support conditions. In many cases the variations of state fields and of structure response functionals are required in redesign or optimal design problems. The sensitivity analysis provides a general methodology of derivation of sensitivity derivatives or variations with respect to structural parameters. In optimal design problems the sensitivity derivatives provide the gradients of the objective function and of design constraints that can be used in the incremental procedure tending to an optimal solution, cf. Haug et al. [1], Haftka et al. [2]. In the present paper first the incremental structure evolution will be discussed with introduced configuration and sensitivity forces associated with the variation of structural parameters. The reciprocity relations for the increments of configuration forces and structural parameters are derived in Sect. 2. In Sect. 3 the topological variation in truss or frame structures is considered with the sensitivity derivatives presented analytically. In Sect. 4 the sensitivity of an arbitrary displacement functional is provided for the case of an inclusion or void introduced in a homogeneous plate. In Sect. 5 some applications of sensitivity analysis to structural design are discussed.

2 Configurational and sensitivity forces, reciprocity relations

Consider an elastic structure for which the stress, strain and displacement fields are σ , ϵ , $\mathbf{u}$ and the structure parameter (or design) vector is denoted by $\mathbf{s}$. The structure is assumed to be loaded by the traction $\mathbf{T}^0 = \lambda^s \bar{\mathbf{T}}^0(\mathbf{x})$

Z. Mróz
Institute of Fundamental Technological Research, Warsaw, Poland

D. Bojczuk (✉)
Faculty of Management and Computer Modelling, Kielce University of Technology, Kielce, Poland
E-mail: mecdb@eden.tu.kielce.pl

on the boundary portion S_T and by the displacements $\mathbf{u}^0 = \lambda^k \bar{\mathbf{u}}^0(\mathbf{x})$ on the boundary portion S_u , where $\mathbf{x}$ denotes a certain point. The loading process is now induced by varying λ^s and λ^k with the structure parameter vector fixed. The structure transformation process occurs for varying vector $\mathbf{s}$ and fixed loading factors λ^s and λ^k . For the elastic structure the stationary conditions of the potential energy $\Pi = \Pi(\mathbf{u}, \lambda^s, \mathbf{s})$ or of the complementary energy $\bar{\Pi} = \bar{\Pi}(\boldsymbol{\sigma}, \lambda^k, \mathbf{s})$ specify solution of the boundary value problem, thus

$$\delta \Pi = \Pi_{,u}(\mathbf{u}, \lambda^s, \mathbf{s}) \cdot \delta \mathbf{u} = 0 \quad (1)$$

provides the equilibrium equations in a class of kinematically admissible fields $\mathbf{u}^k$ and variations $\delta \mathbf{u}$. Here $\Pi_{,u}$ is the Frechet derivative of the potential energy functional with respect to the displacement vector at the actual state $\mathbf{u}^k = \mathbf{u}$ and $\delta \Pi$ denotes its variation. The dot between symbols denotes the scalar product. For a class of statically admissible stress fields $\boldsymbol{\sigma}$ and variations $\delta \boldsymbol{\sigma}$, there is

$$\delta \bar{\Pi} = \bar{\Pi}_{,\sigma}(\boldsymbol{\sigma}, \lambda^k, \mathbf{s}) \cdot \delta \boldsymbol{\sigma} = 0, \quad (2)$$

where $\bar{\Pi}_{,\sigma}$ is the Frechet derivative with respect to $\boldsymbol{\sigma}$. Now (2) provides the compatibility conditions satisfied at the actual stress state $\boldsymbol{\sigma}$. When the loading or displacement factors vary, the structure follows the *loading path* C^l through the constitutive equilibrium states. The incremental equations along the loading path follow from (1) and (2), namely

$$\left(\Pi_{,uu} \dot{\mathbf{u}}^l + \Pi_{,u\lambda} \dot{\lambda}^s \right) \cdot \delta \mathbf{u} = 0 \quad (3)$$

and

$$\left(\bar{\Pi}_{,\sigma\sigma} \dot{\boldsymbol{\sigma}}^l + \bar{\Pi}_{,\sigma\lambda} \dot{\lambda}^k \right) \cdot \delta \boldsymbol{\sigma} = 0, \quad (4)$$

where $\dot{\mathbf{u}}^l$, $\dot{\boldsymbol{\sigma}}^l$ and $\dot{\lambda}^s$, $\dot{\lambda}^k$ are the increments or rates with respect to a selected evolution parameter. The product with respect to all indices of lower rank tensor applies for symbols put in juxtaposition. On the other hand, when the loading parameters are fixed but the structural vector $\mathbf{s}$ varies, the structure states evolve along the *transformation path* C_s . In view of (1) and (2), we have

$$\left(\Pi_{,uu} \dot{\mathbf{u}}^s + \Pi_{,us} \dot{\mathbf{s}} \right) \cdot \delta \mathbf{u} = 0 \quad (5)$$

and

$$\left(\bar{\Pi}_{,\sigma\sigma} \dot{\boldsymbol{\sigma}}^s + \bar{\Pi}_{,\sigma s} \dot{\mathbf{s}} \right) \cdot \delta \boldsymbol{\sigma} = 0, \quad (6)$$

where $\dot{\mathbf{u}}^s$, $\dot{\boldsymbol{\sigma}}^s$ are the displacement and stress increments induced by the structural vector increment $\dot{\mathbf{s}}$. The variation of potential and complementary energies along the transformation path are

$$\begin{aligned} \dot{\Pi}_{,s} &= \Pi_{,u} \cdot \dot{\mathbf{u}}^s + \dot{\Pi}_{,s} \cdot \dot{\mathbf{s}} = \Pi_{,s} \cdot \dot{\mathbf{s}} = -\mathbf{S} \cdot \dot{\mathbf{s}}, \\ \dot{\bar{\Pi}}_{,s} &= \bar{\Pi}_{,\sigma} \cdot \dot{\boldsymbol{\sigma}}^s + \dot{\bar{\Pi}}_{,s} \cdot \dot{\mathbf{s}} = \bar{\Pi}_{,s} \cdot \dot{\mathbf{s}} = \mathbf{S} \cdot \dot{\mathbf{s}}. \end{aligned} \quad (7)$$

As $\dot{\mathbf{u}}^s$ and $\dot{\boldsymbol{\sigma}}^s$ are the kinematically and statically admissible fields, the stationary conditions (1) and (2) are applied in (7). Further, since $\Pi = -\bar{\Pi}$, there is

$$-\Pi_{,s} = \bar{\Pi}_{,s} = \mathbf{S}, \quad \mathbf{S} = \mathbf{S}(\mathbf{u}, \mathbf{s}), \quad \text{or} \quad \mathbf{S} = \mathbf{S}(\boldsymbol{\sigma}, \mathbf{s}) \quad (8)$$

and $\mathbf{S}$ is the *generalized configurational force* conjugate to $\mathbf{s}$. We now have in view of (5) and (6)

$$\begin{aligned} \mathbf{D} &= \dot{\mathbf{S}} \cdot \dot{\mathbf{s}} = - \left(\Pi_{,us} \dot{\mathbf{u}}^s + \Pi_{,ss} \dot{\mathbf{s}} \right) \cdot \dot{\mathbf{s}} = - \left(\Pi_{,us} \dot{\mathbf{u}}^s \cdot \dot{\mathbf{s}} + \Pi_{,ss} \dot{\mathbf{s}} \cdot \dot{\mathbf{s}} \right) \\ &= \Pi_{,uu} \dot{\mathbf{u}}^s \cdot \dot{\mathbf{u}}^s - \Pi_{,ss} \dot{\mathbf{s}} \cdot \dot{\mathbf{s}} = -\bar{\Pi}_{,\sigma\sigma} \dot{\boldsymbol{\sigma}}^s \cdot \dot{\boldsymbol{\sigma}}^s + \bar{\Pi}_{,ss} \dot{\mathbf{s}} \cdot \dot{\mathbf{s}}. \end{aligned} \quad (9)$$

In deriving (9), the incremental equilibrium or compatibility Eqs. (5) and (6) have been applied with the fields $\dot{\mathbf{u}}^s$ and $\dot{\boldsymbol{\sigma}}^s$ put instead of $\delta \mathbf{u}$ and $\delta \boldsymbol{\sigma}$. The second variations of $\Pi = -\bar{\Pi}$ along the transformation path now are

$$\begin{aligned} \ddot{\Pi}_{,s} &= \Pi_{,su} \dot{\mathbf{u}}^s \cdot \dot{\mathbf{s}} + \Pi_{,ss} \dot{\mathbf{s}} \cdot \dot{\mathbf{s}} = -\Pi_{,uu} \dot{\mathbf{u}}^s \cdot \dot{\mathbf{u}}^s + \Pi_{,ss} \dot{\mathbf{s}} \cdot \dot{\mathbf{s}} = -\dot{\mathbf{S}} \cdot \dot{\mathbf{s}}, \\ \ddot{\bar{\Pi}}_{,s} &= \bar{\Pi}_{,\sigma\sigma} \dot{\boldsymbol{\sigma}}^s \cdot \dot{\mathbf{s}} + \bar{\Pi}_{,ss} \dot{\mathbf{s}} \cdot \dot{\mathbf{s}} = -\bar{\Pi}_{,\sigma\sigma} \dot{\boldsymbol{\sigma}}^s \cdot \dot{\boldsymbol{\sigma}}^s + \bar{\Pi}_{,ss} \dot{\mathbf{s}} \cdot \dot{\mathbf{s}} = \dot{\mathbf{S}} \cdot \dot{\mathbf{s}}. \end{aligned} \quad (10)$$

In view of (5) the following equalities can be written for incremental paths of two structural parameters s_1 and s_2

$$\begin{aligned} (\Pi_{,uu}\dot{\mathbf{u}}^{s_1} + \Pi_{,us}\dot{\mathbf{s}}_1) \cdot \dot{\mathbf{u}}^{s_2} &= 0, \\ (\Pi_{,uu}\dot{\mathbf{u}}^{s_2} + \Pi_{,us}\dot{\mathbf{s}}_2) \cdot \dot{\mathbf{u}}^{s_1} &= 0, \end{aligned} \quad (11)$$

or

$$\begin{aligned} -\dot{\mathbf{S}}_1 \cdot \dot{\mathbf{s}}_2 &= (\Pi_{,su}\dot{\mathbf{u}}^{s_1} + \Pi_{,ss}\dot{\mathbf{s}}_1) \cdot \dot{\mathbf{s}}_2 = -\Pi_{,uu}\dot{\mathbf{u}}^{s_1} \cdot \dot{\mathbf{u}}^{s_2} + \Pi_{,ss}\dot{\mathbf{s}}_1 \cdot \dot{\mathbf{s}}_2, \\ -\dot{\mathbf{S}}_2 \cdot \dot{\mathbf{s}}_1 &= (\Pi_{,su}\dot{\mathbf{u}}^{s_2} + \Pi_{,ss}\dot{\mathbf{s}}_2) \cdot \dot{\mathbf{s}}_1 = -\Pi_{,uu}\dot{\mathbf{u}}^{s_2} \cdot \dot{\mathbf{u}}^{s_1} + \Pi_{,ss}\dot{\mathbf{s}}_2 \cdot \dot{\mathbf{s}}_1. \end{aligned} \quad (12)$$

We have therefore

$$\dot{\mathbf{S}}_1 \cdot \dot{\mathbf{s}}_2 = \dot{\mathbf{S}}_2 \cdot \dot{\mathbf{s}}_1 \quad (13)$$

and the reciprocity relations occur for the increments or rates of configurational forces and increments or rates of conjugated structural parameters.

These reciprocity relations are valid for elastic linear or nonlinear materials. Such relations were already extensively discussed by Herrmann and Kienzler [3,4] and Kienzler and Herrmann [5], but the proof presented here is general and provides the expressions for the second-order sensitivities of energy functionals. Here the structure evolution vector $\mathbf{s}$ can present material parameter, shape and topology variations. Let us note that $\mathbf{s}$ can now be a scalar or vector parameter of structure modification. The reciprocity relations for rates of loading and generalized conjugate displacements are derived from the equilibrium Eq. (3) along the loading path. For two loading paths associated with load factors $\dot{\lambda}_1^s$ and $\dot{\lambda}_2^s$ we have

$$\begin{aligned} (\Pi_{,uu}\dot{\mathbf{u}}^{\lambda_1} + \Pi_{,u\lambda}\dot{\lambda}_1^s) \cdot \dot{\mathbf{u}}^{\lambda_2} &= 0, \\ (\Pi_{,uu}\dot{\mathbf{u}}^{\lambda_2} + \Pi_{,u\lambda}\dot{\lambda}_2^s) \cdot \dot{\mathbf{u}}^{\lambda_1} &= 0. \end{aligned} \quad (14)$$

Along the loading path the variation of potential energy is expressed as follows:

$$\dot{\Pi}_{,\lambda} = \Pi_{,u} \cdot \dot{\mathbf{u}}^\lambda + \Pi_{,\lambda}\dot{\lambda}^s = \Pi_{,\lambda}\dot{\lambda}^s = -q\dot{\lambda}^s, \quad (15)$$

where $q = -\Pi_{,\lambda}$ is the generalized displacement conjugate to the load factor λ^s . The second variation of Π in view of (3) equals

$$\ddot{\Pi}_{,\lambda} = \Pi_{,uu}\dot{\mathbf{u}}^\lambda \cdot \dot{\mathbf{u}}^\lambda + 2\Pi_{,u\lambda}\dot{\mathbf{u}}^\lambda \cdot \dot{\lambda}^s + \Pi_{,\lambda\lambda}(\dot{\lambda}^s)^2 = -\Pi_{,uu}\dot{\mathbf{u}}^\lambda \cdot \dot{\mathbf{u}}^\lambda + \Pi_{,\lambda\lambda}(\dot{\lambda}^s)^2 = -\dot{q}\dot{\lambda}^s. \quad (16)$$

The reciprocity relations are now expressed in the form

$$\begin{aligned} -\dot{q}_1 \cdot \dot{\lambda}_2^s &= (\Pi_{,\lambda u} \cdot \dot{\mathbf{u}}^{\lambda_1} + \Pi_{,\lambda\lambda}\dot{\lambda}_1^s) \dot{\lambda}_2^s = -\Pi_{,uu}\dot{\mathbf{u}}^{\lambda_1} \cdot \dot{\mathbf{u}}^{\lambda_2} + \Pi_{,\lambda\lambda}\dot{\lambda}_1^s \dot{\lambda}_2^s, \\ -\dot{q}_1 \cdot \dot{\lambda}_2^s &= (\Pi_{,\lambda u} \cdot \dot{\mathbf{u}}^{\lambda_2} + \Pi_{,\lambda\lambda}\dot{\lambda}_2^s) \dot{\lambda}_1^s = -\Pi_{,uu}\dot{\mathbf{u}}^{\lambda_2} \cdot \dot{\mathbf{u}}^{\lambda_1} + \Pi_{,\lambda\lambda}\dot{\lambda}_1^s \dot{\lambda}_2^s \end{aligned} \quad (17)$$

and

$$\dot{q}_1 \cdot \dot{\lambda}_2^s = \dot{q}_2 \cdot \dot{\lambda}_1^s. \quad (18)$$

The relations (13) and (18) apply to both linear and nonlinear elastic structures. In particular, when the load factor λ^s occurs linearly in the potential energy, the second terms in (16) and (17) vanish.

Consider now an arbitrary functional

$$G = G(\mathbf{u}, \lambda^s, \mathbf{s}) \quad (19)$$

and its variation induced by the structure evolution along the transformation path

$$\dot{G}_{,s} = G_{,u} \cdot \dot{\mathbf{u}}^s + G_{,s} \cdot \dot{\mathbf{s}}, \quad (20)$$

where $\dot{\mathbf{u}}^s$ satisfies the incremental equilibrium Eq. (3). The variation $\dot{\mathbf{u}}^s$ can be specified by solving (3) for each specific case of structure variation in a direct sensitivity approach. However, a more effective adjoint structure approach can be proposed by considering an augmented functional

$$\overline{G} = G(\mathbf{u}, \lambda^s, \mathbf{s}) - \Pi_{,u}(\mathbf{u}, \lambda^s, \mathbf{s}) \cdot \mathbf{u}^a, \quad (21)$$

where the multiplier $\mathbf{u}^a$ is a kinematically admissible displacement field. The variation of $\overline{G}$ along the transformation path now is expressed as follows:

$$\begin{aligned} \dot{\overline{G}}_{,s} &= G_{,u} \cdot \dot{\mathbf{u}}^s + G_{,s} \cdot \dot{\mathbf{s}} - (\Pi_{,uu} \dot{\mathbf{u}}^s + \Pi_{,us} \dot{\mathbf{s}}) \cdot \mathbf{u}^a \\ &= (G_{,u} - \Pi_{,uu} \mathbf{u}^a) \cdot \dot{\mathbf{u}}^s + G_{,s} \cdot \dot{\mathbf{s}} - \Pi_{,us} \dot{\mathbf{s}} \cdot \mathbf{u}^a. \end{aligned} \quad (22)$$

Let us specify an adjoint problem

$$\Pi_{,uu} \mathbf{u}^a - G_{,u} = \mathbf{0} \quad (23)$$

providing the adjoint field $\mathbf{u}^a$. The first-order sensitivity of the functional (19) is now presented in a simple form

$$\dot{\overline{G}}_{,s} = G_{,s} \cdot \dot{\mathbf{s}} - \Pi_{,us} \dot{\mathbf{s}} \cdot \mathbf{u}^a = (G_{,s} - \Pi_{,su} \mathbf{u}^a) \cdot \dot{\mathbf{s}} = \hat{\mathbf{S}} \cdot \dot{\mathbf{s}}, \quad (24)$$

where $\hat{\mathbf{S}} = G_{,s} - \Pi_{,us} \mathbf{u}^a$ is the *sensitivity force* conjugate to $\mathbf{s}$ and depending on both primary and adjoint fields. The reciprocity relations (13) now cannot be proved for a general case. In the following, the sensitivity expressions will be derived by applying the adjoint structure approach. The details of derivation of the second-order sensitivity for arbitrary state functionals and of the potential energy have been discussed by Mróz [6].

3 Topological sensitivity analysis for bar structures

In this section we shall derive the sensitivity forces associated with variation of topological parameters for bar structures. In the structural context this parameter corresponds to introduction or removal of a load carrying element or support. When the sensitivity derivative does not vanish for the vanishing cross-section of the element, it provides an assessment of the potential force versus the cost gradient and induces structure modification. The infinitesimal and finite topology modifications can then be considered, cf. Bojczuk and Mróz [7–9].

3.1 Topological derivative for bar structures in regular states

Consider now a plane bar structure made of $n - 1$ rectilinear elements and loaded in-plane. We assume that this structure can be modified by introduction of additional n -th element of length l . The sensitivity of functional G with respect to introduction of this element can be expressed as follows:

$$\delta G = T_{,S_A}^G \delta S_A + T_{,S_I}^G \delta S_I, \quad (25)$$

where $S_A = E_n A_n$ and $S_I = E_n I_n$ are the extensional and flexural stiffnesses of the virtual element; E_n , A_n and I_n denote Young's modulus, cross-section area and moment of inertia of virtual bar and

$$T_{,S_A}^G = \lim_{S_A \rightarrow 0} \frac{G(S_A) - G(0)}{S_A} = G_{,S_A} \big|_{S_A=0}, \quad T_{,S_I}^G = \lim_{S_I \rightarrow 0} \frac{G(S_I) - G(0)}{S_I} = G_{,S_I} \big|_{S_I=0} \quad (26)$$

are the topological derivatives of functional G with respect to extensional and flexural stiffnesses of virtual element.

Now, consider the functional of generalized displacements of the form

$$G = \sum_{i=1}^n \int_0^{l_i} f(\mathbf{U}, S_A, S_I) dx_i, \quad (27)$$

where analyzed structure consists of n elements including one virtual element, and f denotes function of stiffnesses moduli S_A , S_I of virtual element and of vector $\mathbf{U}$ of generalized displacements, whose components u and w are the longitudinal and transverse displacements and θ is the rotation angle. Introduce an adjoint structure of the same configuration and boundary conditions as the considered structure and respectively loaded by distributed longitudinal and transverse forces and distributed moments, namely

$$q_N^a = f_{,u}, \quad q_T^a = f_{,w}, \quad m^a = f_{,\theta}. \quad (28)$$

In view of (28) the first variation of functional G with respect to introduction of virtual element can be presented as follows:

$$\delta G = \sum_{i=1}^n \int_0^{l_i} f_{,\mathbf{U}} \cdot \delta \mathbf{U} dx_i = \sum_{i=1}^n \left(\int_0^{l_i} q_N^a \delta u dx_i + \int_0^{l_i} q_T^a \delta w dx_i + \int_0^{l_i} m^a \delta \theta dx_i \right). \quad (29)$$

Taking into account the virtual work equation

$$\sum_{i=1}^n \left(\int_0^{l_i} q_N^a \delta u dx_i + \int_0^{l_i} q_T^a \delta w dx_i + \int_0^{l_i} m^a \delta \theta dx_i \right) = \sum_{i=1}^n \left(\int_0^{l_i} M^a \delta \kappa dx_i + \int_0^{l_i} N^a \delta \varepsilon dx_i \right), \quad (30)$$

next the complementary virtual work equation

$$\sum_{i=1}^n \left(\int_0^{l_i} \delta M \kappa^a dx_i + \int_0^{l_i} \delta N \varepsilon^a dx_i \right) = 0, \quad (31)$$

and the relations

$$\begin{aligned} \sum_{i=1}^n \int_0^{l_i} \delta N \varepsilon^a dx_i &= \sum_{i=1}^n \int_0^{l_i} \delta (\varepsilon E A) \varepsilon^a dx_i = \sum_{i=1}^n \int_0^{l_i} N^a \delta \varepsilon dx_i + \int_0^l \varepsilon \varepsilon^a \delta S_A dx_n, \\ \sum_{i=1}^n \int_0^{l_i} \delta M \kappa^a dx_i &= \sum_{i=1}^n \int_0^{l_i} \delta (\kappa E I) \kappa^a dx_i = \sum_{i=1}^n \int_0^{l_i} M^a \delta \kappa dx_i + \int_0^l \kappa \kappa^a \delta S_I dx_n, \end{aligned} \quad (32)$$

the sensitivity of the functional G with respect to introduction of virtual bar now takes the form

$$\delta G = - \left(\int_0^l \varepsilon \varepsilon^a \delta S_A dx_n + \int_0^l \kappa \kappa^a \delta S_I dx_n \right), \quad (33)$$

so it is expressed in terms of axial strain fields ε , ε^a and curvature fields κ , κ^a in the virtual element of primary and adjoint structures. Moreover, N , N^a , and M , M^a are the conjugate normal forces and bending moments, and superscript a denotes quantities in the adjoint structure.

Now, we express fields of strains and curvatures of the virtual element by kinematic quantities on its ends. Assume that displacements in longitudinal and transverse directions and rotation angle at the end 1 of the virtual element n in local coordinate system (x_n, y_n) are u_1, w_1, θ_1 , while at the end 2 respectively u_2, w_2, θ_2

Fig. 1 Introduction of virtual element into bar structure

(Fig. 1). Using transformation formulae, the relations between generalized displacements in local coordinate systems (x_i, y_i) , (x_j, y_j) related to bar elements i, j and in the coordinate system (x_n, y_n) are of the form

$$\begin{cases} u_1 = u_{1i} \cos \alpha_1 + w_{1i} \sin \alpha_1, \\ w_1 = -u_{1i} \sin \alpha_1 + w_{1i} \cos \alpha_1, \\ \theta_1 = \theta_{1i}, \end{cases} \quad \begin{cases} u_2 = u_{2j} \cos \alpha_2 + w_{2j} \sin \alpha_2, \\ w_2 = -u_{2j} \sin \alpha_2 + w_{2j} \cos \alpha_2, \\ \theta_2 = \theta_{2j}, \end{cases} \quad (34)$$

where $u_{1i}, w_{1i}, \theta_{1i}$ are generalized displacements of the point 1 in coordinate system (x_i, y_i) , $u_{2j}, w_{2j}, \theta_{2j}$ are generalized displacements of the point 2 in coordinate system (x_j, y_j) , while α_1, α_2 are the angles between the axes x_i, x_j and the axis x_n .

Taking into account that $\delta S_A = \text{const}$, $\delta S_I = \text{const}$ along virtual element and that virtual element is unloaded, the equilibrium equations for this element are of the form

$$u'' = 0, \quad w^{IV} = 0. \quad (35)$$

Integrating the first Eq. (35) in view of the boundary conditions

$$u(0) = u_1, \quad u(l) = u_2, \quad (36)$$

where l denotes length of the virtual bar, we obtain the constant field of virtual strain, namely

$$\varepsilon(x_n) = \varepsilon_n = \frac{u_2 - u_1}{l}. \quad (37)$$

Next, integrating the second Eq. (35), we obtain curvature, transverse displacement and rotation angle of the virtual element in the form

$$\begin{aligned} \kappa &= w'' = Ax_n + B, \\ \theta &= w' = \frac{1}{2}Ax_n^2 + Bx_n + C, \\ w &= \frac{1}{6}Ax_n^3 + \frac{1}{2}Bx_n^2 + Cx_n + D. \end{aligned} \quad (38)$$

In order to determine constants A, B, C and D the boundary conditions on the ends of virtual element can be used, namely

$$w(0) = w_1, \quad w(l) = w_2, \quad \theta(0) = \theta_1, \quad \theta(l) = \theta_2. \quad (39)$$

Then, we get

$$A = \frac{6}{l^3} [-2(w_2 - w_1) + l(\theta_1 + \theta_2)], \quad B = \frac{2}{l^2} [3(w_2 - w_1) - l(2\theta_1 + \theta_2)], \quad C = \theta_1, \quad D = w_1. \quad (40)$$

Now, the field of the virtual curvature takes the form

$$\kappa(x_n) = \frac{6}{l^3} [-2(w_2 - w_1) + l(\theta_1 + \theta_2)]x_n + \frac{2}{l^2} [3(w_2 - w_1) - l(2\theta_1 + \theta_2)]. \quad (41)$$

Analogously, in the virtual element of the adjoint structure, we have

$$\begin{aligned}\varepsilon^a(x_n) &= \varepsilon_n^a = \frac{u_2^a - u_1^a}{l}, \\ \kappa^a(x_n) &= \frac{6}{l^3} [-2(w_2^a - w_1^a) + l(\theta_1^a + \theta_2^a)]x_n + \frac{2}{l^2} [3(w_2^a - w_1^a) - l(2\theta_1^a + \theta_2^a)].\end{aligned}\quad (42)$$

Moreover, curvatures at the ends 1 and 2 of the virtual element of the primary and adjoint structures are

$$\begin{aligned}\kappa_1 &= \kappa(0) = \frac{2}{l^2} [3(w_2 - w_1) - l(2\theta_1 + \theta_2)], \\ \kappa_1^a &= \kappa^a(0) = \frac{2}{l^2} [3(w_2^a - w_1^a) - l(2\theta_1^a + \theta_2^a)], \\ \kappa_2 &= \kappa(l) = \frac{2}{l^2} [-3(w_2 - w_1) - l(\theta_1 + 2\theta_2)], \\ \kappa_2^a &= \kappa^a(l) = \frac{2}{l^2} [-3(w_2^a - w_1^a) - l(\theta_1^a + 2\theta_2^a)].\end{aligned}\quad (43)$$

Now, taking into account (37), (42)₁ and (43), the sensitivity (33) of the functional G with respect to introduction of virtual bar can be finally presented in the form

$$\delta G = -l\varepsilon_n \varepsilon_n^a \delta S_A - \frac{l}{6} (2\kappa_1 \kappa_1^a + 2\kappa_2 \kappa_2^a + \kappa_1 \kappa_2^a + \kappa_2 \kappa_1^a) \delta S_I. \quad (44)$$

When the functional G expresses the potential energy Π , namely

$$\begin{aligned}\Pi &= \frac{1}{2} \sum_{i=1}^n \left(\int_0^{l_i} S_I \kappa^2 dx_i + \int_0^{l_i} S_A \varepsilon^2 dx_i \right) - \sum_{i=1}^n \left(\int_0^{l_i} q_N u dx_i + \int_0^{l_i} q_T w dx_i + \int_0^{l_i} m \theta dx_i \right) \\ &= -\frac{1}{2} \sum_{i=1}^n \left(\int_0^{l_i} q_N u dx_i + \int_0^{l_i} q_T w dx_i + \int_0^{l_i} m \theta dx_i \right),\end{aligned}\quad (45)$$

the problem becomes self-adjoint and sensitivity (44) takes the form

$$\delta \Pi = \frac{1}{2} l \varepsilon_n^2 \delta S_A + \frac{1}{6} l (\kappa_1^2 + \kappa_2^2 + \kappa_1 \kappa_2) \delta S_I. \quad (46)$$

3.2 Topological derivative of buckling load for bar structures

Analogously as in Sect. 3.1, consider now plane bar structure made of $n - 1$ rectilinear elements and loaded in-plane. We assume that this structure can be modified by introduction of additional n -th element of length l , extensional stiffness $S_A = E_n A_n$, located transversely to one of the existing elements and playing role of the elastic support simply connected at the ends. The critical state condition for buckling now is

$$\sum_{i=1}^{n-1} \int_0^{l_i} \left(\frac{1}{2} E I \kappa^2 + \frac{1}{2} \lambda_{cr} N \theta^2 \right) dx_i + \int_0^l \frac{1}{2} S_A \varepsilon_n^2 dx_n = 0, \quad (47)$$

where $\kappa = w''$ and $\theta = w'$ is the curvature and rotation angle of the eigenmode w , N denotes normal force, E , A , I are the Young's modulus, cross-sectional area and moment of inertia. Moreover, ε_n is the virtual strain and taking into account that

$$u(0) = 0, \quad u(l) = w_n, \quad (48)$$

where w_n denotes the eigenmode deflection of the structure at the supported point, and in view of (37), we have

$$\varepsilon_n = \frac{w_n}{l}. \quad (49)$$

Now, calculating variation of (47) with respect to introduction of virtual support along the path of critical states in view of (49) and assuming that $S_A = \text{const}$, we obtain

$$\sum_{i=1}^{n-1} \int_0^{l_i} \left[EI \kappa \delta \kappa + \frac{1}{2} \delta \lambda_{cr} N \theta^2 + \frac{1}{2} \lambda_{cr} \delta N \theta^2 + \lambda_{cr} N \theta \delta \theta \right] dx_i + \frac{1}{2} \frac{w_n^2}{l} \delta S_A = 0. \quad (50)$$

Using the virtual work equation along the path of critical states

$$\sum_{i=1}^n \int_0^{l_i} (M \delta \kappa + \lambda_{cr} N \theta \delta \theta) dx_i = 0, \quad (51)$$

and taking into account that the third term under the integral in (50) related to redistribution of normal forces vanishes in this case (cf. Mróz and Bojczuk [10]), the topological derivative of critical value of load parameter with respect to introduction of virtual element finally can be presented as follows:

$$T_{,S_A}^{\lambda_{cr}} = \lambda_{cr,S_A} |_{S_A=0} = - \frac{w_n^2}{l \sum_{i=1}^{n-1} \int_0^{l_i} N \theta^2 dx_i}. \quad (52)$$

4 Topological sensitivity analysis for plates

In this section we shall derive the sensitivity forces associated with variation of topological parameters for plates. Here, the composite material design is considered and the topological modification is related to introduction of inclusions of new materials or introduction of voids.

4.1 Topological derivative with respect to introduction of inclusions

Consider now an elastic plate, whose middle surface occupies the domain $A \subset \mathbb{R}^2$, with the boundary $\Gamma = \Gamma_u \cup \Gamma_T$, Fig. 2. The plate is loaded by tractions $\mathbf{T} = \mathbf{T}^0$ ($T_i^0 = \sigma_{ij} n_j^T$, $i, j = 1, 2$) on the boundary portion Γ_T and by body forces $\mathbf{p}_1^0, \mathbf{p}_2^0$ applied respectively in the plate domains A_ξ and A_0 ($A = A_\xi \cup A_0$), where $\mathbf{n}^T = [n_1^T, n_2^T]$ is the unit vector normal to the boundary Γ_T . The stress and strain states occurring in the plate domain can be presented in the vector form as $\boldsymbol{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{12}]^T$, $\boldsymbol{\varepsilon} = [\varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12}]^T$. Moreover, displacements $\mathbf{u} = \mathbf{u}^0$ are specified on the boundary portion Γ_u .

The topological derivative of the functional G with respect to the hole or inclusion area is defined as follows (cf. Sokołowski and Żochowski [11, 12], Bojczuk and Mróz [13])

$$T_{,A_0}^G(\mathbf{x}) = \lim_{A_0 \rightarrow 0} \frac{G(A - A_0) - G(A)}{A_0} = \lim_{\xi \rightarrow 0} \frac{G(A_\xi) - G(A)}{A_0^{(fix)} \xi^2}, \quad (53)$$

where $\mathbf{x}(\mathbf{x} \in A)$ is an arbitrary position in the plate domain, in which the derivative is specified, $A_0, A_0^{(fix)}$ denote respectively the domain of inclusion (void) and the reference domain, ξ is the expansion parameter, which specifies the size of the modification and $A_\xi = A - A_0$.

Let us consider the response functional of the form

$$G = \int_{A_\xi} f_1(\mathbf{u}_1) dA + \int_{A_0} f_2(\mathbf{u}_2) dA + \int_{\Gamma_T} g(\mathbf{u}_1) d\Gamma_T, \quad (54)$$

Fig. 2 Introduction of inclusion or void into plate

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are the displacement fields respectively in A_ξ and A_0 domains. In order to specify the sensitivity derivative with respect to introduction of inclusion the variational approach is applied here. Now, the first variation of functional (54) in the expansion process of inclusion located at the material point $\mathbf{x}$ is expressed as follows

$$\delta G = \int_{A_\xi} f_{1,\mathbf{u}_1} \cdot \delta \mathbf{u}_1 dA + \int_{A_0} f_{2,\mathbf{u}_2} \cdot \delta \mathbf{u}_2 dA + \int_{\Gamma_\xi} [f] n_k \delta \varphi_k d\Gamma_T + \int_{\Gamma_T} g_{1,\mathbf{u}_1} \cdot \delta \mathbf{u}_1 d\Gamma_T, \quad (55)$$

where $\mathbf{n} = [n_1, n_2]^T$ is the unit vector normal to the interface Γ_ξ , $\delta \varphi = [\delta \varphi_1, \delta \varphi_2]^T$ is the interface transformation vector, $[f] = f_1 - f_2$ denotes “jump” of the quantity f on the interface Γ_ξ , and comma preceding an index denotes partial derivative. Following the previous derivation for plates cf. Bojczuk and Mróz [9] and the general methodology of sensitivity analysis cf. Dems and Mróz [14], the variations of state fields can be eliminated by introducing an adjoint plate structure of the same form, as the primary plate, but with induced body force fields, namely

$$\mathbf{p}_1^{a0} = f_{1,\mathbf{u}_1} \text{ in } A_\xi, \quad \mathbf{p}_2^{a0} = f_{2,\mathbf{u}_2} \text{ in } A_0, \quad (56)$$

and satisfying the following boundary conditions

$$\mathbf{T}^{a0} = g_{1,\mathbf{u}_1} \text{ on } \Gamma_T, \quad \mathbf{u}_1^{a0} = \mathbf{0} \text{ on } \Gamma_u, \quad (57)$$

where $\boldsymbol{\sigma}^a$, $\mathbf{u}^a$, $\boldsymbol{\varepsilon}^a$ are the state fields in the adjoint structure. Taking into account continuity conditions on the interface Γ_ξ

$$[\varepsilon_{tt}] = 0, \quad [\sigma_{nn}] = 0, \quad [\sigma_{nt}] = 0, \quad (58)$$

where n, t denote respectively normal and tangential direction to this interface and using virtual work equation

$$\begin{aligned} \int_{A_\xi} \boldsymbol{\sigma}_1^a \cdot \delta \boldsymbol{\varepsilon}_1 dA + \int_{A_0} \boldsymbol{\sigma}_2^a \cdot \delta \boldsymbol{\varepsilon}_2 dA &= \int_{\Gamma_T} \mathbf{T}^{a0} \cdot \delta \mathbf{u}_1 d\Gamma_T + \int_{A_\xi} \mathbf{p}_1^{a0} \cdot \delta \mathbf{u}_1 dA + \int_{A_0} \mathbf{p}_2^{a0} \cdot \delta \mathbf{u}_2 dA \\ &\quad - \int_{\Gamma_\xi} (\sigma_{nn}^a [\varepsilon_{nn}] + 2\sigma_{nt}^a [\varepsilon_{nt}]) \delta \varphi_n d\Gamma_\xi, \end{aligned} \quad (59)$$

next, the complementary virtual work equation

$$\begin{aligned} \int_{A_\xi} \boldsymbol{\sigma}_1^a \cdot \delta \boldsymbol{\varepsilon}_1 dA + \int_{A_0} \boldsymbol{\sigma}_2^a \cdot \delta \boldsymbol{\varepsilon}_2 dA &= \int_{A_\xi} \boldsymbol{\varepsilon}_1^a \cdot \delta \boldsymbol{\sigma}_1 dA + \int_{A_0} \boldsymbol{\varepsilon}_2^a \cdot \delta \boldsymbol{\sigma}_2 dA \\ &= \int_{\Gamma_\xi} [\mathbf{p}^0] \cdot \mathbf{u}^a \delta \varphi_n d\Gamma_\xi - \int_{\Gamma_\xi} [\sigma_{tt}] \varepsilon_{tt}^a \delta \varphi_n d\Gamma_\xi, \end{aligned} \quad (60)$$

we finally obtain

$$\delta G = \int_{\Gamma_\xi} (-[\sigma_{tt}] \varepsilon_{tt}^a + \sigma_{nn}^a [\varepsilon_{nn}] + 2\sigma_{nt}^a [\varepsilon_{nt}] + [\mathbf{p}^0] \cdot \mathbf{u}^a + [f]) \delta \varphi_n d\Gamma_\xi. \quad (61)$$

Let us introduce a general curvilinear coordinate system, where the first of coordinates ξ attains constant value on the interface Γ_ξ and the second θ varies in the interval $\langle 0; 2\pi \rangle$. In the particular case let us choose elliptical coordinates, which are related with the local coordinate system x_{10}, x_{20} coinciding with the ellipse semi-axes $\xi a, \xi b$ by relationships

$$x_{10} = \xi a \cos \theta, \quad x_{20} = \xi b \sin \theta. \quad (62)$$

Then, taking ξ as the design parameter, (61) takes the form

$$\delta G = G_{,\xi} \delta \xi = \xi ab \int_0^{2\pi} ([\sigma_{tt}] \varepsilon_{tt}^a - \sigma_{nn}^a [\varepsilon_{nn}] - 2\sigma_{nt}^a [\varepsilon_{nt}] - [\mathbf{p}^0] \cdot \mathbf{u}^a - [f]) d\theta \delta \xi, \quad (63)$$

To express the sensitivity derivative with respect to the inclusion area $A = \pi \xi^2 ab$, the following incremental formula is used $dA = 2\pi \xi ab d\xi$, thus

$$T_{,A_0}^G(\mathbf{x}) = (G_{,\xi} \xi, A)|_{\xi=0} = \frac{1}{2\pi} \int_0^{2\pi} ([\sigma_{tt}] \varepsilon_{tt}^a - \sigma_{nn}^a [\varepsilon_{nn}] - 2\sigma_{nt}^a [\varepsilon_{nt}] - [\mathbf{p}^0] \cdot \mathbf{u}^a - [f]) d\theta. \quad (64)$$

When the last term in (54) disappears, $[\mathbf{p}^0] = \mathbf{0}$, while the two first terms correspond to the total strain energy U , we get

$$T_{,A_0}^U(\mathbf{x}) = \frac{1}{4\pi} \int_0^{2\pi} ([\sigma_{tt}] \varepsilon_{tt} - \sigma_{nn} [\varepsilon_{nn}] - 2\sigma_{nt} [\varepsilon_{nt}]) d\theta. \quad (65)$$

Now, let us consider the particular case, namely topological sensitivity derivative with respect to introduction of void. Taking into account that on the free boundary of the void $\sigma_{nn} = 0$, $\sigma_{nt} = 0$ the general formula (64) now takes the form

$$T_{,A_0}^G(\mathbf{x}) = \frac{1}{2\pi} \int_0^{2\pi} (\sigma_{tt} \varepsilon_{tt}^a - \mathbf{p}^0 \cdot \mathbf{u}^a - f) d\theta. \quad (66)$$

When the functional G represents the strain energy U and $\mathbf{p}^0 = \mathbf{0}$, the formula (65) can be rewritten as follows

$$T_{,A_0}^U(\mathbf{x}) = \frac{1}{4\pi} \int_0^{2\pi} \sigma_{tt} \varepsilon_{tt} d\theta = \frac{1}{4\pi E} \int_0^{2\pi} \sigma_{tt}^2 d\theta. \quad (67)$$

In order to use derived formulae for topological derivative, the stress or strain distributions on the interface should be known. In many cases, they can be analytically determined using methods of the elasticity theory (cf. Mura [15]; Muskhelishvili [16]), or they can be specified numerically.

Fig. 3 Introduction of circular inclusion or void into plate in plane state of stress—notation

4.2 Example: Topological sensitivity derivative of the total strain energy with respect to introduction of circular inclusion into plate

Let us consider an infinite plate under plane state of stress and containing a circular inclusion made of different material, with its center introduced at a point $\mathbf{x}$. We assume that Young's modulus, Kirchhoff's shear modulus and Poisson's ratio are denoted respectively for matrix by E_1 , G_1 , ν_1 and for inclusion by E_2 , G_2 , ν_2 .

The displacement and stress components appearing in the matrix, in the coordinate system n, t , are (cf. Muskhelishvili [16])

$$\begin{aligned} u_n^{(1)} &= \frac{\sigma_1 + \sigma_2}{8G_1 r} [(\kappa_1 - 1)r^2 + 2\gamma_1 R^2] + \frac{\sigma_1 - \sigma_2}{8G_1 r} \left[\beta_1(\kappa_1 - 1)r^2 + 2r^2 + \frac{2\delta_1 R^4}{r^2} \right] \cos 2\theta, \\ u_t^{(1)} &= -\frac{\sigma_1 - \sigma_2}{8G_1 r} \left[\beta_1(\kappa_1 - 1)r^2 + 2r^2 - \frac{2\delta_1 R^4}{r^2} \right] \sin 2\theta, \end{aligned} \quad (68)$$

and

$$\begin{aligned} \sigma_{nn}^{(1)} &= \frac{\sigma_1 + \sigma_2}{2} \left(1 - \frac{\gamma_1 R^2}{r^2} \right) + \frac{\sigma_1 - \sigma_2}{2} \left(1 - \frac{2\beta_1 R^2}{r^2} - \frac{3\delta_1 R^4}{r^4} \right) \cos 2\theta, \\ \sigma_{tt}^{(1)} &= \frac{\sigma_1 + \sigma_2}{2} \left(1 + \frac{\gamma_1 R^2}{r^2} \right) - \frac{\sigma_1 - \sigma_2}{2} \left(1 - \frac{3\delta_1 R^4}{r^4} \right) \cos 2\theta, \\ \sigma_{nt}^{(1)} &= -\frac{\sigma_1 - \sigma_2}{2} \left(1 + \frac{\beta_1 R^2}{r^2} + \frac{3\delta_1 R^4}{r^4} \right) \sin 2\theta, \end{aligned} \quad (69)$$

where σ_1, σ_2 denote the principal stresses in the neighbourhood of the point $\mathbf{x}$ for plate without inclusion (hole), r, θ are the polar coordinates, R is the radius of inclusion (Fig. 3) and $\beta_1, \gamma_1, \delta_1$ are the unknown constants.

The displacement and stress components expressed by (68), (69), can be presented on the boundary between matrix and inclusion i.e. for $r = R$, in the form

$$\begin{aligned} u_n^{(1)} \Big|_{r=R} &= \frac{\sigma_1 + \sigma_2}{8G_1} [(\kappa_1 - 1)R + 2\gamma_1 R] + \frac{\sigma_1 - \sigma_2}{8G_1} [\beta_1(\kappa_1 - 1)R + 2R + 2\delta_1 R] \cos 2\theta, \\ u_t^{(1)} \Big|_{r=R} &= -\frac{\sigma_1 - \sigma_2}{8G_1} [\beta_1(\kappa_1 - 1)R + 2R - 2\delta_1 R] \sin 2\theta, \\ \sigma_{nn}^{(1)} \Big|_{r=R} &= \frac{\sigma_1 + \sigma_2}{2} (1 - \gamma_1) + \frac{\sigma_1 - \sigma_2}{2} (1 - 2\beta_1 - 3\delta_1) \cos 2\theta, \end{aligned}$$

$$\begin{aligned}\sigma_{tt}^{(1)}\Big|_{r=R} &= \frac{\sigma_1 + \sigma_2}{2}(1 + \gamma_1) - \frac{\sigma_1 - \sigma_2}{2}(1 - 3\delta_1) \cos 2\theta, \\ \sigma_{nt}^{(1)}\Big|_{r=R} &= -\frac{\sigma_1 - \sigma_2}{2}(1 + \beta_1 + 3\delta_1) \sin 2\theta.\end{aligned}\quad (70)$$

The displacement and stress components occurring in the inclusion are (cf. Muskhelishvili [16])

$$\begin{aligned}u_n^{(2)} &= \frac{\sigma_1 + \sigma_2}{8G_2}\beta_2(\kappa_2 - 1)r + 2\delta_2 r - \frac{\sigma_1 - \sigma_2}{4G_2}\delta_2 r \cos 2\theta, \\ u_t^{(2)} &= -\frac{\sigma_1 - \sigma_2}{4G_2}\delta_2 r \sin 2\theta, \\ \sigma_{nn}^{(2)} &= \frac{\sigma_1 + \sigma_2}{2}\beta_2 + \frac{\sigma_1 - \sigma_2}{2}\delta_2 \cos 2\theta, \\ \sigma_{tt}^{(2)} &= \frac{\sigma_1 + \sigma_2}{2}\beta_2 - \frac{\sigma_1 - \sigma_2}{2}\delta_2 \cos 2\theta, \\ \sigma_{nt}^{(2)} &= -\frac{\sigma_1 - \sigma_2}{2}\delta_2 \sin 2\theta,\end{aligned}\quad (71)$$

where β_2, δ_2 are the unknown constants. It is easy to notice that the stress state in the inclusion is homogeneous. Using the following conditions on the boundary between the matrix and the inclusion

$$\begin{aligned}\sigma_{nn}^{(1)}\Big|_{r=R} &= \sigma_{nn}^{(2)}\Big|_{r=R}, & \sigma_{nt}^{(1)}\Big|_{r=R} &= \sigma_{nt}^{(2)}\Big|_{r=R}, \\ u_n^{(1)}\Big|_{r=R} &= u_n^{(2)}\Big|_{r=R}, & u_t^{(1)}\Big|_{r=R} &= u_t^{(2)}\Big|_{r=R},\end{aligned}\quad (72)$$

the values of unknown constants can be expressed as follows

$$\begin{aligned}\beta_1 &= -\frac{2(G_2 - G_1)}{G_1 + G_2\kappa_1}, & \delta_1 &= \frac{G_2 - G_1}{G_1 + G_2\kappa_1}, & \gamma_1 &= \frac{G_1(\kappa_2 - 1) - G_2(\kappa_1 - 1)}{2G_2 + G_1(\kappa_2 - 1)}, \\ \beta_2 &= \frac{G_2(\kappa_1 + 1)}{2G_2 + G_1(\kappa_2 - 1)}, & \delta_2 &= \frac{G_2(\kappa_1 + 1)}{G_1 + G_2\kappa_1},\end{aligned}\quad (73)$$

where for the plane state of stress there is

$$\kappa_1 = \frac{3 - \nu_1}{1 + \nu_1}, \quad \kappa_2 = \frac{3 - \nu_2}{1 + \nu_2}. \quad (74)$$

Now, taking into account the Hooke's law for the matrix

$$\begin{aligned}\varepsilon_{nn}^{(1)} &= \frac{1}{E_1}(\sigma_{nn}^{(1)} - \nu_1 \sigma_{tt}^{(1)}), \\ \varepsilon_{tt}^{(1)} &= \frac{1}{E_1}(\sigma_{tt}^{(1)} - \nu_1 \sigma_{nn}^{(1)}), \\ \varepsilon_{nt}^{(1)} &= \frac{\sigma_{nt}^{(1)}}{2G_1},\end{aligned}\quad (75)$$

next the analogous relations for the inclusion, and substituting (70) and (71) into (65), the topological sensitivity derivative of the total strain energy with respect to introduction of circular inclusion, is finally obtained as follows

$$\begin{aligned}T_{,A_0}^U(\mathbf{x}) &= \frac{1}{2E_1} \left[(\sigma_1 + \sigma_2)^2 \frac{G_1(\kappa_2 - 1) - G_2(\kappa_1 - 1)}{2G_2 + G_1(\kappa_2 - 1)} \right. \\ &\quad \left. + (\sigma_1 - \sigma_2)^2 (G_1 - G_2) \frac{8G_1 + 4G_2(\kappa_1 - 1) + G_2(1 + \nu_1)(\kappa_1 + 1)^2}{4(G_1 + G_2\kappa_1)^2} \right].\end{aligned}\quad (76)$$

The diagram of topological derivative variation in function of ratio of Kirchhoff's moduli $\mu = G_2/G_1$ for different values of stress ratio $\zeta = \sigma_2/\sigma_1$ and in the case, when $\nu_1 = \nu_2 = 0.3$ is presented in Fig. 4. It is easy to notice that, as we expect, for $0 \leq \mu < 1$ the topological derivative is positive, for $\mu = 1$ equals zero, while for $\mu > 1$ is negative and when $\mu \rightarrow \infty$ attains a finite value. Similar analysis can also be done for the plane state of strain and in this case, we have $\kappa_1 = 3 - 4\nu_1$, $\kappa_2 = 3 - 4\nu_2$.

Fig. 4 Non-dimensional topological derivative $\frac{E_1}{\sigma_1^2} \cdot T_{,A_0}^U$ variation in function of ratio of Kirchhoff's moduli $\mu = G_2/G_1$ for different values of stress ratio $\zeta = \sigma_2/\sigma_1$

5 Application of shape and topology sensitivity analysis to structural design

Consider now general optimization problem defined as follows

$$\min_{s_i, i=1,2,\dots,n-1} C, \quad \text{subject to } G - G_0 \leq 0, \quad (77)$$

where $C(s_i)$ is the global cost, $G(s_i)$ denotes an arbitrary functional expressed by (27) or (54), and G_0 is the upper bound imposed on this functional. Introducing the Lagrangian functional

$$L(s_i, \lambda) = C + \mu(G - G_0), \quad \mu \geq 0 \quad (78)$$

the stationary conditions are

$$\begin{aligned} C_{,s_i} + \mu G_{,s_i} &= 0, \quad i = 1, 2, \dots, n-1, \\ \mu(G - G_0) &= 0. \end{aligned} \quad (79)$$

where μ ($\mu \geq 0$) is the Lagrange multiplier and $s_i, i = 1, 2, \dots, n-1$ are the design parameters. The sensitivity derivatives $G_{,s_i}$ are specified by the formulae derived in the paper. The cost derivatives $C_{,s_i}$ can easily be derived for the assumed cost function expressed in terms of design parameters. The optimal values of the parameters $s_i, i = 1, 2, \dots, n-1$ and the multiplier μ are determined in the incremental process of gradient optimization.

Next, using the sensitivity derivative with respect to topological parameter s_n the condition of topology modification can be formulated. When the condition is satisfied, standard optimization with respect to $s_i, i = 1, 2, \dots, n$ should be performed again.

5.1 Formulation of topology modification conditions for bar structures

Consider first topology modification by introduction of virtual element into a bar structure in the regular state. Assume, that the functional G is expressed by (27) and the cost of structure is

$$C = C^{(n-1)} + cS_A l, \quad (80)$$

where $C^{(n-1)}$ denotes cost of the structure without additional bar, while c, S_A, l are respectively unit cost, extensional stiffness and length of virtual element. Now, the condition of topology modification by introduction of a new bar, in view of (44), takes the form

$$T_{,S_A}^L = T_{,S_A}^C + \mu T_{,S_A}^G = l(c - \mu \varepsilon_n \varepsilon_n^a) \leq 0 \quad (81)$$

and it corresponds to non-positive value of topological derivative of the Lagrangian (78), where $\mu \geq 0$ is the Lagrange multiplier calculated for the unmodified structure.

Consider now the problem of the form (77), but with buckling constraint cost $\lambda - \lambda_{cr} \leq 0$, i.e. $G \equiv \lambda$, $G_0 \equiv \lambda_{cr}$, and λ_{cr} denotes critical value of load parameter λ . We assume that the bar structure is modified by introduction of additional n -th element of unit cost c , length l , extensional stiffness S_A , which plays the role of the elastic, transverse support simply connected on the ends. Now, the topological derivative of the Lagrangian $L = C + \mu(\lambda - \lambda_{cr})$, in view of (52), provides the condition of new support introduction in the form

$$T_{,S_A}^L = T_{,S_A}^C - \mu T_{,S_A}^{\lambda_{cr}} = cl + \mu \frac{w_n^2}{l \sum_{i=1}^{n-1} \int_0^{l_i} N \theta^2 dx_i} \leq 0, \quad (82)$$

where w_n is the eigenmode deflection of the beam at the supported point, $\theta = w'$ denotes the slope and $\mu \geq 0$, as previously, is the Lagrange multiplier calculated for the unmodified structure.

5.2 Formulation of topology modification conditions for plates

In the case of the problem (77) for plates, the condition of introduction of an infinitesimally small inclusion (hole) at the point $\mathbf{x}$, takes the general form (Bojczuk and Mróz [8])

$$T_{,A_0}^L(\mathbf{x}) = T_{,A_0}^C(\mathbf{x}) + \mu T_{,A_0}^G(\mathbf{x}) \leq 0, \quad (83)$$

where $T_{,A_0}^L(\mathbf{x})$, $T_{,A_0}^C(\mathbf{x})$, $T_{,A_0}^G(\mathbf{x})$ are the topological derivatives at the point $\mathbf{x}$ respectively of Lagrangian, cost functional C and functional G , and this last derivative is expressed by (64) - (67) and (76). So, an inclusion (hole) should be introduced at a point, where the condition (83) is satisfied and $T_{,A_0}^L(\mathbf{x})$ attains minimal value (cf. bubble method, Eschenauer et al. [17]). This condition enables localization of introduced new structural element.

However, in order to accelerate the optimization process, finite topology modifications can be applied. Now, the problem consists in introduction of finite holes of unknown size and shape together with introduction of finite changes of other boundaries. It is assumed that domains of relatively small values of the topological derivative of Lagrangian, which is expressed by (83), should be eliminated. It can be done using level-set method by removal of all domains, where this topological derivative is smaller than an adequately chosen negative iso-value (cf. Burger et al. [18], Wang et al. [19]). However, also alternative approach can be applied, where the specially constructed design quality function is used, which contains the scaling factor controlling amount of removed domain related to the respective iso-value (cf. Bojczuk and Mróz [8]).

6 Concluding remarks

The present paper provides the general framework of specification of sensitivity derivatives of potential or complementary energies and of arbitrary state functionals, especially displacement functionals. Here, also topological derivatives with respect to introduction of additional elements into bar structures and inclusions or voids into plates are taken into account. These sensitivity derivatives can be regarded as configurational or sensitivity forces generating effective redesign or optimal design procedures. The reciprocity relations naturally follow from the second-order sensitivity expressions of energies and increments of configurational forces.

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