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Reformulation of strain invariants at incompressibility

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Abstract The description of (quasi-)incompressible materials such as elastomers is well established in modern continuum mechanics for many years now as well as from a theoretical background as in the numerical implementation in commercial software packages in the context of finite elements. Nevertheless, some questions arise in the practical application of that matter describing technical components, e.g., in the discussion *What are valid equivalent measures in order to compare different deformations?* Here, one could request for expressions that arise in a deformation intensity and an indication of the deformation mode locally at each material point. We propose an extension of the well-known description of incompressible kinematics. We reformulate the strain invariants at incompressibility in terms of (I_1, I_2) leading to an equivalent pair (λ, m) in order to determine a *distance* of an arbitrary deformation state, e.g., to its equivalent shearing state. Therefore, we postulate and define an associated deformation state and give the mathematical derivation of quite nice relationships, which we demonstrate on a shear example using the finite elements method to visualize the “new” measure quantity.

Keywords Rubber elasticity · Deformation invariants · Equivalent deformation · Life cycle

1 Introduction

This article deals with the description of incompressible materials by deformation invariants and principle stretches as given in modern textbooks on continuum mechanics and tensor analysis as [4] or [5]. Exemplarily, we think on rubber (based) materials, but this treatise is not restricted to such type of material; it is just a very nice representation of (nearly) incompressible behavior and has quite a big relevance in industrial applications.

In this context, most material tests are still conducted in uniaxial tension mode (because of arguments in quickness and cheapness), although the technical applications try to overcome that deformation mode just because of life cycle reasons.

Nevertheless, as the applications should enforce shear or nearly shear deformation states, the question often arises, whether different deformation states can be compared, e.g., with respect to each pre-deformation or with respect to an equivalent shear mode, see [3]. Thinking on rubber materials, the first question is concerned with the well-known MULLINS or softening effect due to the arrangement of the polymer network and the filler components, see [9], while the second question deals with the methodology of determining the dynamical,

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Fig. 1 Mapping of each material point denoted by $\mathbf{X}$ of reference configuration $\mathcal{B}_0$ at time t_0 onto $\mathbf{x}$ of current configuration $\mathcal{B}$ at time t by the deformation gradient $\mathbf{F}(\mathbf{X})$

i.e., viscoelastic, time-dependent behavior of elastomers at different frequencies, e.g., by a DMA (*dynamic mechanical analysis*), see [6] or [7].

The clear and defined loading situation in a test rig will not act in a technical application in its pure “prototype” form. So, often a lack of comparability is obvious, especially in life cycle predictions. Here, we try to refine the deformation kinematics of that material class and give some hints with the help of the continuum mechanics for further examinations, e.g., development of material models and general prediction methods.

Doing so, we analyze an arbitrary deformation state with respect to its position in the *plane of invariants*. The plane of invariants represents every deformation state of incompressible materials as a precise point in its domain of physically relevant positions. This domain is limited on the one hand by the uniaxial tension states and on the other hand by the conditions of (equi-)biaxial deformations. Shear deformations are represented by the (first) bisectrix in that plane. So, just graphical discussions raise the question of a *distance measure* comparing any deformation with its *equivalent (shear) deformation*, which now can be followed in a straight forward manner. This approach ends up in a nice representation from a continuum mechanics point of view, where we point out some mathematical details of deformation kinematics within the plane of invariants and give an example of a shear specimen analyzed by the finite element method.

Preliminary we denote the following vector and matrix operations, partly in index notation, which will be used basically following the modern textbook [4] on nonlinear continuum mechanics. The transpose of a vector or tensorial quantity in matrix representation is indicated by $(\bullet)^T$, and the inverse of a matrix or a tensor by $(\bullet)^{-1}$. The dot-product contracting two indices as $\mathbf{a} \cdot \mathbf{b}$ or $\mathbf{A} \cdot \mathbf{B}$ is given by $a_k b_k$ or $A_{ik} B_{kj}$, respectively, with the EINSTEIN summation convention. Furthermore, the trace operator is symbolized by $\text{tr}(\mathbf{A}) = A_{ii} = A_{11} + A_{22} + A_{33}$. As long as it is possible, first- and second-order tensor quantities are represented by small or capital roman or Greek symbols in boldface.

2 Description of deformation

2.1 Deformation gradient

Given a general deformation of a body represented by material points $\mathbf{X}$, we denote the new position in (EUKLIDIAN) space $\mathbb{E}^3$ by $\mathbf{x} = \mathbf{X} + \mathbf{u}$, see [4,5]. Here, the vector $\mathbf{u}$ gives the displacement of each point, see Fig. 1. So, the mapping of $\mathbf{X}$ onto $\mathbf{x}$ is given by $\mathbf{x} = \mathbf{F} \cdot \mathbf{X}$ with the deformation gradient

$$\mathbf{F} := \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{I} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}}, \quad (1)$$

where—in general— $\mathbf{F}$ is a non-symmetric 3×3 field in matrix representation.

2.2 Deformation invariants

Most material models for rubber(-like) materials are given in terms of the major deformation invariants, see [11,8], which is mainly due to material objectivity reasons, see [1,4], or [5]. We consider these invariants of the

symmetric *right* CAUCHY–GREEN deformation tensor $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$ with respect to the reference configuration $\mathcal{B}_0$, see Fig. 1, as

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \quad (2)$$

$$I_2 = \lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2 \quad (3)$$

$$I_3 = \lambda_1^2 \lambda_2^2 \lambda_3^2, \quad (4)$$

where the stretches along the axes are denoted by λ_1 , λ_2 , and λ_3 .

The invariants (2–4) are identical to the three invariants of the symmetric *left* CAUCHY–GREEN deformation tensor $\mathbf{b} = \mathbf{F} \cdot \mathbf{F}^T$ with respect to the current configuration $\mathcal{B}$ and are also given as

$$I_1 = \text{tr } \mathbf{b} \quad (5)$$

$$I_2 = \frac{1}{2} [(\text{tr } \mathbf{b})^2 - \text{tr } \mathbf{b}^2] = \text{tr } \mathbf{b}^{-1} \det \mathbf{b} \quad (6)$$

$$I_3 = \det \mathbf{b}. \quad (7)$$

2.3 Incompressibility

Dealing with incompressible materials, the constant volume leads to $J := \det \mathbf{F} \equiv 1$ and therefore $I_3 \equiv 1$, which concludes directly $\lambda_3 = \frac{1}{\lambda_1 \lambda_2}$ and with (2) and (3)

$$I_1 = \lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2 \lambda_2^2} \quad (8)$$

$$I_2 = \lambda_1^2 \lambda_2^2 + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}. \quad (9)$$

2.4 Graphical representation and separation of tension- and compression-dominated deformations

Already in [11], a graphical representation of the I_1 – I_2 -dependencies for typical deformation modes—called here *prototype deformations*—is presented. Following some mathematical costly, but basic operations by applying the CARDANO formulas in order to solve the resulting cubic equation, one obtains the $I_2(I_1)$ -dependency by (8) in (9) directly. Exemplarily, here we give

$$I_2 = 4 \sqrt{\frac{I_1}{3}} \cos \left[\frac{1}{3} \arccos \left(-3 \sqrt{\frac{3}{I_1^3}} \right) \right] + \frac{3}{4 I_1 \cos^2 \left[\frac{1}{3} \arccos \left(-3 \sqrt{\frac{3}{I_1^3}} \right) \right]} \quad (10)$$

for the uniaxial case with $\lambda_2 = \lambda_3 = \frac{1}{\sqrt{\lambda_1}}$ as already given in [10]. The function $I_2(I_1)$ for the equi-biaxial case with $\lambda_1 = \lambda_2$ and $\lambda_3 = \frac{1}{\lambda_1^2}$ is obtained in the same manner and represents a curve mirrored on the first bisectrix of the diagram. In [2], a detailed proceeding for experiments on rubber specimen is documented in order to realize physically unique results and parameter calibrations for such materials. In Fig. 2, we show the same visualization because of its impressive character and the consequences as basics for our following argumentation. As depicted in Fig. 2, the “plane of invariants” is divided into two regions by the shear states represented by the (first) bisectrix. All possible and physically admissible deformation states are represented by points in that region in between the curve “uniaxial tension” and “equi-biaxial tension.” So, principally, the two regions can be addressed by $I_1 > I_2$ for the “tension dominated” states and by $I_2 > I_1$ for the “compression dominated” deformation states, respectively.

As consequence, here we focus firstly on tension modes, i.e., $I_1 > I_2$ and discuss these in this article.

Fig. 2 Plane of invariants for incompressibility ($I_3 \equiv 1$): The *prototype modes* uni- and biaxial tension form the boundary of all arbitrary deformation states within the semi-infinite plane. Dots ($\cdot$) denote arbitrary deformation states

3 Reformulation and consequence for an equivalent measure

3.1 Derivation and solution of λ and m

Recalling (8) and (9) and again $\lambda_3 = \frac{1}{\lambda_1 \lambda_2}$, we yield a representation in the form

$$I_1 = \lambda^2 + \lambda^{2m} + \lambda^{-2(m+1)} = \mu + \mu^m + \mu^{-m-1} \quad (11)$$

$$I_2 = \lambda^{-2} \lambda^{-2m} + \lambda^{2(m+1)} = \mu^{-1} + \mu^{-m} + \mu^{m+1} \quad (12)$$

with m as *strain mode exponent* and λ as *characteristic stretch level* or its square $\mu = \lambda^2$. This representation can be recognized as generalization of the derivations in [11]. Therefore, that gives a new view onto the deformation kinematics of rubber-like materials in order to propose an associated deformation state in Sect. 3.3 later on. Nevertheless, this *new formulation* gives a representation (λ, m) with one parameter fixed for the prototype deformations, while the classical invariants (I_1, I_2) both change for that cases.

The set of Equations (11)–(12) can be solved for the two unknown λ and m for given I_1 and I_2 : The characteristic polynomial of the right CAUCHY–GREEN tensor $\mathbf{C}$ or the left CAUCHY–GREEN tensor $\mathbf{b}$

$$\det[\mathbf{C} - \mu \mathbf{I}] = \det[\mathbf{b} - \mu \mathbf{I}] \stackrel{!}{=} 0 \quad (13)$$

leads to the three eigenvalues μ_1, μ_2 , and μ_3 , see [5], directly from its representation

$$\mu^3 - I_1 \mu^2 + I_2 \mu - I_3 = 0 \quad (14)$$

with $I_3 = 1$ for assumed incompressibility and $\mu = \lambda^2$. The only valid solution—a non-complex, real value—is obtained as function of the invariants I_1 and I_2 , see (11)–(12), as

$$\mu(I_1, I_2) = \frac{1}{3} I_1 - \frac{a}{\sqrt[3]{c}} + \sqrt[3]{c} \quad (15)$$

with

$$a = \frac{1}{3} I_2 - \frac{1}{9} I_1^2 \quad (16)$$

$$b = \frac{1}{27} I_1^3 - \frac{1}{6} I_1 I_2 + \frac{1}{2} \quad (17)$$

$$c = b + \sqrt{b^2 + a^2}. \quad (18)$$

Fig. 3 Deformation exponent m as function of the invariants I_1, I_2 in the interval $[-\frac{1}{2} \dots 0 \dots 1]$ indicating uniaxial, shear, and biaxial deformation modes

The valid solution (15) is characterized by the fact that only for physical deformations within Fig. 2, its imaginary part of the solution vanishes.

Following (15), the strain mode exponent m results as

$$m(I_1, I_2) = \frac{\ln \left[\frac{I_1}{3} + \frac{a}{2\sqrt[3]{c}} - \frac{\sqrt[3]{c}}{2} + \sqrt{\frac{1}{4} \left(\frac{2I_1}{3} + \frac{a}{\sqrt[3]{c}} - \sqrt[3]{c} \right)^2 - \left(\frac{I_1}{3} - \frac{a}{\sqrt[3]{c}} + \sqrt[3]{c} \right)^{-1}} \right]}{\ln \left[\frac{I_1}{3} - \frac{a}{\sqrt[3]{c}} + \sqrt[3]{c} \right]} \quad (19)$$

in a closed form as given in [12].

As an illustration of (19), we give Fig. 3 as representation of m with respect to I_1 and I_2 . For the prototype deformation "uniaxial tension," the strain mode exponent results in $m \equiv -\frac{1}{2}$, for biaxial tension in $m \equiv 1$ and for shear deformations with $I_1 = I_2$, it is given as $m \equiv 0$, respectively.

3.2 Iso-lines for $\lambda = \text{constant}$

As a further illustration, the general correlation between the principle stretches and the invariants is described. This correlation can be found by transforming the characteristic polynomial (13) into (14), see [5]. By that, the linear equation (14) represents iso-lines in the I_1 - I_2 -diagram for constant values of λ as

$$I_2 = I_1 \mu - \mu^2 + \frac{1}{\mu} = I_1 \lambda^2 - \lambda^4 + \frac{1}{\lambda^2}, \quad (20)$$

with $I_3 \equiv 1$.

The slope of the linear equation (20) results in

$$\frac{\partial I_2}{\partial I_1} = \mu = \lambda^2. \quad (21)$$

Consequently, Fig. 4 graphically shows this correlation in the plane of invariants exemplarily for three different constant values of λ . To generate this figure, one principle stretch (λ_1) is kept constant while the stretch in the second principle direction (λ_2) is increased. Due to the incompressibility, λ_3 results directly. Naturally, this approach represents all possible deformation conditions.

Fig. 4 Plane of invariants with iso-lines for $\lambda = 2, 2.5, 3$

3.3 Associated deformation states

Based on the above given correlations, here we propose and define the *associated deformation states*

$$\mathbf{F}_{assoc} := \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-(m+1)} & 0 \\ 0 & 0 & \lambda^m \end{bmatrix} = \mathbf{U}_{assoc} \quad (22)$$

as a general deformation gradient for physical deformations at incompressibility based on λ and m as solution (15) and (19), respectively.

Please note: For the proposed prototype deformations 1–3, see Fig. 2, the equivalence of $\mathbf{F}_{assoc}$ and $\mathbf{U}_{assoc}$ results from the polar decomposition $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$ with $\mathbf{R} \equiv \mathbf{I}$. Giving an alternative interpretation of the result (22) above, one could argue λ as local *intensity* and m as *indicator* for the given deformation mode. Obtained by life cycle experiments for different states of m , the corresponding intensity λ for any deformation does *not* end up in the same cycle number. So, it is a challenge to define different experimental setups in order to separate failure modes of rubber components and to correlate their life cycle numbers to λ and m , respectively. In that case, it is also worth to mention the simplicity and nice structure of accounting for

$$m = \frac{\ln \lambda_2}{\ln \lambda_1} \quad (23)$$

in the case of, e.g., $\lambda_1 > \lambda_2 > \lambda_3$. Please compare this result with the representation of $m = m(I_1, I_2)$ in (19).

From (22) follows, e.g., for $m \equiv 0$ in the *plane strain* mode

$$\mathbf{F}_{ps} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (24)$$

and for $m \equiv -\frac{1}{2}$ in the *uniaxial* mode with

$$\mathbf{F}_{1ax} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-1/2} \end{bmatrix}, \quad (25)$$

where the (local) coordinate system is oriented, such that the main stretch is along the 1-direction without loss of generality.

Fig. 5 FE result on shear specimen for λ_{eqv} as contour from outside view (1) and in the symmetry cutting plane (2). Deflection: 20 mm

Fig. 6 FE result on shear specimen for strain mode exponent m

4 Example of a shear specimen

To demonstrate the applicability of the proposed scheme, here we show a finite element computation of an elastomeric shear buffer with an height of about 20 mm, which is loaded by a deflection of 20 mm. Figure 5 represents the contours of $\lambda_{eqv} = \sqrt{\mu}$ from (13) or (20), while Fig. 6 shows m from (19). In each case, the outer view and a view onto the cutting plane are given. Especially in Fig. 6(2), the dominance of the shear state can be observed for the most of the volume, while Fig. 6(1) shows a region with an uniaxial deformation mode and a zone of biaxial deformation for $m = 1$.

5 Summary and conclusions

Based on the major invariants of the CAUCHY–GREEN deformation tensors, an equivalent deformation measure λ and its strain mode exponent m are proposed for the case of incompressibility. By that, the pair (λ, m) is reformulated out of the invariants I_1, I_2 with the advantage that deformation modes with $m = \text{constant}$ could be identified and interpreted for test procedures in order to calibrate material parameters for hyperelasticity:

In Sect. 3.3, we address the aspect of separating deformation modes in different experimental setups, which can now be investigated in the view of the proposed deformation intensity λ and deformation-type indication $m = [-\frac{1}{2} \dots 0 \dots 1]$.

This enables us to discuss the question which deformation is comparable in the sense that a (pre)conditioning of rubber material introduces a softening effect due to arrangement of the polymer network and the filler components.

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References

1. Altenbach, J., Altenbach, H.: Einführung in die Kontinuumsmechanik. ISBN 3-519-03096-9, Teubner (1994)
2. Baaser, H.: FEM-Simulation von Elastomerbauteilen. ATZ-Automobiltechnische Zeitschrift 5 (2010)
3. Baaser, H., Hohmann, G.: Representation of dynamic elastomer behaviour with focus on amplitude dependency. In: ECCMR, ISBN 978-0-415-56327-7 (2009)
4. Holzapfel, G.: Nonlinear Solid Mechanics. ISBN 0-471-82304, Wiley (2000)
5. Itskov, M.: Tensor Algebra and Tensor Analysis for Engineers: With Applications to Continuum Mechanics, 2nd edn. ISBN 3540939067, Springer (2009)
6. Kaliske, M., Rothert, H.: Constitutive approach to rate-independent properties of filled elastomers. Int. J. Solids Struct. **35**(17), 2057–2071 (1998)
7. Lion, A., Kardelky, C.: The payne effect in finite viscoelasticity: Constitutive modelling based on fractional derivatives and intrinsic time scales. Int. J. Plast. **20**, 1313–1345 (2004)
8. Marckmann, G., Verron, E.: Comparison of hyperelastic models for rubberlike materials. Rubber Chem. Technol. **79**(5), 835–858 (2006)
9. Mullins, L.: Effect of stretching on the properties of rubber. J. Rubber Res. **16**(12), 275–289 (1949)
10. Nasdala, L.: Private Communications (2010)
11. Treloar, L.: The Physics of Rubber Elasticity, 3rd edn. ISBN 0198570279, Oxford University Press (1975)
12. van Haag, J.: Entwicklung und Implementierung eines hyperelastischen Materialmodells zur genauen Abbildung des Beanspruchungszustandes von Elastomeren. Diploma Thesis, Institut für Kunststoffverarbeitung, RWTH Aachen University (2011)