B. Michalak

Dynamic modeling thin skeletal shallow shells as 2D structures with nonuniform microstructures

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Abstract The subject of this consideration is a thin skeletal elastic shallow shell with an orthogonal beam-grid microstructures. The important feature of the considered shells is that a dimension of the microstructure is of an order of the shell thickness. The formulation of 2D-macroscopic mathematical model of these shells, based on a tolerance averaging approximation (Woźniak et al., 2008), is the aim of the paper. During the modeling procedure, the shell under consideration is treated as a structure with a nonuniform microstructure. The general results of the contribution will be illustrated by the analysis of natural vibrations of a cylindrical thin skeletal shallow shell.

Keywords Composites shell · Modeling · Dynamics

1 Introduction

The object of this paper is a thin composite shallow shell. The considered skeletal shell is made of two families of thin beams. The regions situated between the beams are filled with a homogeneous matrix material (Fig. 1). From a formal point of view, the shell under consideration can be described in the framework of the well-known theories for thin elastic shallow shells (e.g. Gradowczyk [7]). However, due to the inhomogeneous microstructure of the shell, this direct description of the structure leads to shell equations with non-continuous and highly oscillating coefficients. These are too complicated to be used in the analysis of engineering problems and numerical calculations.

Modeling of composites shells reinforced by various constituents such as fibers or particles has been a subject of interest of many researches. The earlier models can be found in for example Hill [9], Hashin and Rosen [8], Budiansky [3]. The dynamic analysis of the stiffened shells can also be found in many papers, for example, see Mead and Bardell [16], Mecitoglu and Dokmeci [17], Wang et al. [25], Nayak and Bandyopadhyay [21], Solaroli et al. [23]. The above-mentioned studies are concerned with analysis shells with low number of the stiffeners where distances between stiffeners are large compared to the width of stiffeners. In contrast to these papers, the shells under consideration have microstructure with a period much smaller than their characteristic dimensions. Shells of this kind are described by partial differential equations with rapidly oscillating coefficients, which are not a good tool to apply to engineering problems. Therefore, simplified models of heterogeneous shells have been proposed in which material properties are represented by effective stiffnesses. The modeling problem is how to describe microheterogeneous shell by certain averaged equations.

The best known among averaging approaches are those based on the homogenization of differential equations with highly oscillating coefficients. The mathematical framework of asymptotic homogenization can be found in, see for example Bensoussan et al. [2], Jikov et al. [10]. The asymptotic homogenization technique

B. Michalak (✉)

Department of Structural Mechanics, Technical University of Lodz, al. Politechniki 6, 93-590 Lodz, Poland
E-mail: bmichala@p.lodz.pl

Fig. 1 Fragment of skeletal shell

has been used by Kalamkarov et al. [12], Challagulla et al. [4] to analyze the shells reinforced by fibers grid and by Lewiński and Telega [13] to analyze rib-reinforced shells and fiber-reinforced composite shells. Saha et al. [22] analyzed the hexagonal-core composite shells. Because the formulation of averaged models by using the asymptotic homogenization methods is rather complicated from the computational point of view, these asymptotic techniques are restricted to the first approximation. Then we obtain averaged models neglecting the length-scale effect on the shell behavior. In order to detect this effect on the dynamic behavior of the shell under consideration, we applied tolerance averaging technique presented in Woźniak et al. [28].

The aim of the analysis is to formulate averaged model describing dynamic behavior of a microheterogeneous shallow shell. The generalized period λ of inhomogeneity is assumed to be sufficiently small compared to the minimum characteristic length dimension $L_{\min}$ of the projection of the shell on the plane Ox_1x_2 . At the same time, the microstructure length parameter λ is of an order of the shell thickness h . The formulation of the macroscopic mathematical model for the analysis of dynamic behavior of the shells under consideration will be based on the tolerance averaging technique. The general modeling procedures of this technique are given by Woźniak et al. in book [28] and for periodic structures by Woźniak and Wierzbicki [29]. Some applications of this technique for the modeling of various dynamic problems of elastic periodic composites are given in series of paper by Baron [1], Cielecka and Jędrusiak [5], Dell'Isolla [6], Mazur-Śniady and Śniady [15], Michalak et al. [19], Tomczyk [24], Wierzbicki and Woźniak [27]. In the book edited by Woźniak et al. [28], the list of references on this subject can be found. The above modeling technique describes the effect of the microstructure size on the overall behavior of a composite structure. This effect plays a crucial role in a description of some initial and boundary layer phenomena (Łaciński [14], Wierzbicki et al. [26]), as well as in the analysis of dispersion phenomena in periodic structures (Jędrusiak [11], Michalak [18]). The general conception to the 2D modeling of the thin skeletal elastic shallow shells can be found in Michalak and Woźniak [20].

The aim of this contribution is two-fold. First, to formulate the 2-D macroscopic mathematical model of the shells under consideration. Second, to investigate a free-vibration problem in the framework of derived model and in the framework of an approximate model, where we deal with approximate uniform cell distribution in the plane Π . Throughout the paper, indices $i, j, k, \dots$ run over 1, 2, 3, indices $\alpha, \beta, \gamma, \dots$ run over 1, 2. We also introduce non-tensorial indices $A, B, C, \dots$ which run over the sequence 1, $\dots$, N . The summation convention holds all aforementioned sub- and superscripts.

2 Direct description of composite shell

The model equations for the dynamic behavior of the skeletal shell will be obtained in the framework of a linear theory for thin elastic shallow shell. The starting point of the contribution in the direct description of the shell under consideration is a well-known theory of thin shallow shells. The displacement field of the arbitrary point of the shallow shell is given in form

$$\begin{aligned} w_3(x, z) &= w_3(x), \\ w_\alpha(x, z) &= w_\alpha^0(x) + d_\alpha(x)z. \end{aligned} \quad (1)$$

Let $\mathbf{p}(x, t)$ stand for the external forces and ρ for the mass density of the shell. Denoting by $b_{\alpha\beta}$ the curvature tensor of the midsurface of the undeformed shallow shell and by $\partial_k = \partial/\partial x^k$, in the framework of the linear approximated theory for thin shallow shells, we obtain the following system of equations:

(i) strain-displacement relations

$$\begin{aligned}\varepsilon_{\alpha\beta}(x, z) &= \varepsilon_{\alpha\beta}^0(x) + \kappa_{\alpha\beta}(x)z, \\ \varepsilon_{\alpha\beta}^0 &= \partial_{(\beta} w_{\alpha)}^0 - b_{\alpha\beta} w_3, \quad \kappa_{\alpha\beta} = \partial_{\beta} d_{\alpha}.\end{aligned}\quad (2)$$

(ii) strain energy

$$E_z(x, z) = \frac{1}{2} C^{\alpha\beta\gamma\delta} \varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta}, \quad (3)$$

(iii) kinetic energy

$$K_z(x, z) = \frac{1}{2} \rho (\dot{w}_3 \dot{w}_3 + \dot{w}_{\alpha} \dot{w}_{\beta} \delta^{\alpha\beta}), \quad (4)$$

for $z \in (-h/2, h/2)$.

The strain energy averaged over the shell thickness is given by (for the rectangular plane Π of the shell projection on $x_3 = 0$, the covariant differentiation with respect to Π is replaced by partial derivative)

$$E(x) = \frac{1}{2} B^{\alpha\beta\gamma\delta} \partial_{\beta} d_{\alpha} \partial_{\gamma} d_{\delta} + \frac{1}{2} D^{\alpha\beta\gamma\delta} (\partial_{(\beta} w_{\alpha)}^0 - b_{\alpha\beta} w_3) (\partial_{(\delta} w_{\gamma)}^0 - b_{\gamma\delta} w_3) \quad (5)$$

where

$$\begin{aligned}B^{\alpha\beta\gamma\eta} &= \frac{Eh^3}{12(1-\nu^2)} 0.5 (\delta^{\alpha\eta} \delta^{\beta\gamma} + \delta^{\alpha\gamma} \delta^{\beta\eta} + \nu (e^{\alpha\gamma} e^{\beta\eta} + e^{\alpha\eta} e^{\beta\gamma})), \\ D^{\alpha\beta\gamma\eta} &= \frac{Eh}{1-\nu^2} 0.5 (\delta^{\alpha\eta} \delta^{\beta\gamma} + \delta^{\alpha\gamma} \delta^{\beta\eta} + \nu (e^{\alpha\gamma} e^{\beta\eta} + e^{\alpha\eta} e^{\beta\gamma})).\end{aligned}\quad (6)$$

It can be seen that the coefficients in formulae (4, 5) are discontinuous and highly oscillating. Hence, a direct application of the above equations to the analysis of engineering problems and numerical calculations is too complicated. That is why we shall formulate a averaged model of the skeletal shell under considerations. The above equations will be used as the starting point of the modeling procedure.

3 Modeling technique

Let the midsurface of the undeformed shell under consideration be given by $x^i = R^i(\theta^1, \theta^2)$, $(\theta^1, \theta^2) \in \Pi$, where Π is a regular plane region. In the sequel $\mathbf{x} = (x^1, x^2) = (\theta^1, \theta^2)$ stands for an arbitrary point belonging to Π , where $\mathbf{x}$ belong to the orthogonal Cartesian coordinate system $O(x_1, x_2, x_3)$.

For shallow shells will be considered the non-uniform cell distribution in Π (Fig. 2a), where a two-dimensional plane $\Pi = (-L_1/2, L_1/2) \times (-L_2/2, L_2/2)$ is the shell projection on $x_3 = 0$ (Fig. 2b). Every $\Delta(\mathbf{x})$, where $\mathbf{x}$ belong to Π , will be referred to the cell in Π with center at $\mathbf{x}$

$$\begin{aligned}\Delta(\mathbf{x}) &= \left(x_1 - \frac{\lambda_1}{2}, x_1 + \frac{\lambda_1}{2}\right) \times \left(x_2 - \frac{\lambda_2}{2}, x_2 + \frac{\lambda_2}{2}\right), \\ \mathbf{x} &= (x_1, x_2) \in \Pi : \Delta \equiv \{\Delta(\mathbf{x}), \mathbf{x} \in \Pi\}.\end{aligned}\quad (7)$$

Denoting by $\lambda_{\alpha}(x_{\alpha})$ the diameter of $\Delta(x)$ and define $\lambda = \sup(\lambda_{\alpha})$ as a microstructure length parameter, we assume that $\lambda_{\alpha}(x_{\alpha})$ are small compared to the minimum characteristic length dimension of region Π ($\lambda_{\alpha} \ll L_{\alpha}$). At the same time, the thickness h of the shell under consideration is supposed to be constant and of an order of the microstructure length parameter λ ($\lambda \approx h \ll \min\{L_1, L_2\}$).

In order to derive averaged model equations for skeletal shell under consideration, we applied tolerance averaging approach Woźniak at el. [28]. We mention some basic concepts of this technique, as an averaging operator, a tolerance parameter, a tolerance periodic function, a slowly varying function, a highly oscillating function.

The fundamental concept of the modeling technique is *the averaging operation*

$$\langle f \rangle(x) = \frac{1}{|\Delta|} \int_{\Delta(x)} f(y, x) dy, \quad x \in \bar{\Pi}. \quad (8)$$

Fig. 2 The shell projection on plane $x_3 = 0$

We shall refer (8) to as averaging of arbitrary integrable function $f(\cdot)$ for an arbitrary $\mathbf{x} \in \bar{\Pi}$.

Let δ stand for an arbitrary positive number and X be a linear normed space. Tolerance relation for a certain δ is defined by

$$(\forall (x_1, x_2) \in X^2) [x_1 \approx x_2 \Leftrightarrow \|x_1 - x_2\|_X \leq \delta], \quad (9)$$

where δ is said to be *the tolerance parameter*, and this relation will be denoted by $\approx$.

Let $\partial^k f$ be the k -th gradient of function $f = f(\mathbf{x})$, $\mathbf{x} \in \Pi$, $k = 0, 1, \dots, \alpha$, ($\alpha \geq 0$), $\partial^0 f \equiv f$.

Function $f \in H^\alpha(\Pi)$ is called *the tolerance periodic function* (with respect to cell Δ and tolerance parameter δ), $f \in TP_\delta^\alpha(\Pi, \Delta)$, if for $k = 0, 1, \dots, \alpha$, the following conditions hold

$$(\forall \mathbf{x} \in \Pi) \left(\exists \tilde{f}^{(k)}(\mathbf{x}, \cdot) \in H^0(\Delta) \right) \left[\left\| \partial^k f|_{\Pi_{\mathbf{x}}}(\cdot) - \tilde{f}^{(k)}(\mathbf{x}, \cdot) \right\|_{H^0(\Pi_{\mathbf{x}})} \leq \delta \right], \quad (10)$$

$$\int_{\Delta(\cdot)} \tilde{f}^{(k)}(\cdot, y) dy \in C^0(\bar{\Pi}).$$

Function $\tilde{f}^{(k)}(\mathbf{x}, \cdot)$ is referred to as *the periodic approximation of $\partial^k f$ in $\Delta(\mathbf{x})$* , $\mathbf{x} \in \Pi$, $k = 0, 1, \dots, \alpha$.

Function $F \in H^\alpha(\Pi)$ is called *the slowly varying function* (with respect to the cell Δ and tolerance parameter δ), $F \in SV_\delta^\alpha(\Pi, \Delta)$, if

$$F \in TP_\delta^\alpha(\Pi, \Delta),$$

$$(\forall \mathbf{x} \in \Pi) [\tilde{F}^{(k)}(\mathbf{x}, \cdot)|_{\Delta(\mathbf{x})} = \partial^k F(\mathbf{x}), \quad k = 0, \dots, \alpha]. \quad (11)$$

It can be observed that periodic approximation $\tilde{F}^{(k)}$ of $\partial^k F(\cdot)$ in $\Delta(\mathbf{x})$ is a constant function for every $\mathbf{x} \in \Pi$. If $F \in SV_\delta^\alpha(\Pi, \Delta)$, then $(\forall \mathbf{x} \in \Pi) \left(\left\| \partial^k F(\cdot) - \partial^k F(\mathbf{x}) \right\|_{H^0(\Delta(\mathbf{x}))} \leq \delta, k = 0, 1, \dots, \alpha \right)$.

Function $\phi \in H^\alpha(\Pi)$ is called *the highly oscillating function* (with respect to the cell Δ and tolerance parameter δ), $\phi \in HO_\delta^\alpha(\Pi, \Delta)$, if

$$\phi \in TP_\delta^\alpha(\Pi, \Delta),$$

$$\forall F \in SV_\delta^\alpha(\Pi, \Delta) (f \equiv \phi F \in TP_\delta^\alpha(\Pi, \Delta)), \quad (12)$$

and for $k = 1, \dots, \alpha$ these functions satisfy conditions

$$\begin{aligned} (\forall \mathbf{x} \in \Pi) \quad & \left[\tilde{\phi}^{(k)}(\mathbf{x}, \cdot) \Big|_{\Delta(\mathbf{x})} = \partial^k \tilde{\phi}(\mathbf{x}) \right], \\ & \tilde{f}^{(k)}(\mathbf{x}, \cdot) \Big|_{\Delta(\mathbf{x})} = F(\mathbf{x}) \partial^k \tilde{\phi}(\mathbf{x}) \Big|_{\Delta(\mathbf{x})}. \end{aligned} \quad (13)$$

If $\alpha = 0$, then we denote $\tilde{f} \equiv \tilde{f}^{(0)}$.

Let by $\varphi(\cdot)$ we denote a highly oscillating function, $\varphi \in H O_{\delta}^2(\Pi, \Delta)$, defined on $\bar{\Pi}$, continuous together with gradient $\partial^1 \varphi$. Its gradient $\partial^2 \varphi$ is a piecewise continuous and bounded. Function $\varphi(\cdot)$ is called *the fluctuation shape function* of the 2-nd kind, if it depends on λ as a parameter and satisfies conditions:

- 1° $\partial^k \varphi \in O(\lambda^{\alpha-k})$ for $k = 1, \dots, \alpha, \alpha = 2$,
- 2° $\langle \varphi \rangle(\mathbf{x}) \approx 0$ for every $\mathbf{x} \in \Pi_{\Delta}$.

Set of all fluctuation shape functions of the 2-nd kind is denoted by $FS_{\delta}^2(\Pi, \Delta)$. Condition (2°) can be replaced by $\langle \rho \varphi \rangle(\mathbf{x}) \approx 0$ for every $\mathbf{x} \in \Pi_{\Delta}$, where $\rho > 0$ is a certain tolerance periodic function.

The starting point of the modeling procedure is a decomposition of displacement fields. The modeling technique will be based on the tolerance averaging approximation and on the restriction of the displacement field under consideration given by

$$\begin{aligned} w_3(x, t) &= V_3(x, t), \\ w_{\alpha}^0(x, t) &= V_{\alpha}(x, t) + \varphi^A(x) V_{\alpha}^A(x, t), \\ d_{\alpha}(x, t) &= -\partial_{\alpha} V_3(x, t) + \varphi^A(x) u_{\alpha}^A(x, t), \end{aligned} \quad (14)$$

for $x = (x_1, x_2), z \in (-h/2, h/2), A = 1, \dots, N$, (summation over A holds) and every time t .

The first modeling assumption, related to the above decomposition, states that $V_3(\cdot, t), u_{\alpha}^A(\cdot, t), V_{\alpha}(\cdot, t), V_{\alpha}^A(\cdot, t)$ are slowly varying functions together with all partial derivatives. Functions $V_3(\cdot, t) \in SV_{\delta}^2(\Pi, \Delta)$, $u_{\alpha}^A(\cdot, t) \in SV_{\delta}^1(\Pi, \Delta)$, $V_{\alpha}(\cdot, t) \in SV_{\delta}^1(\Pi, \Delta)$, $V_{\alpha}^A(\cdot, t) \in SV_{\delta}^1(\Pi, \Delta)$, are the basic unknowns of the modeling problem. Moreover, let $\varphi^A(\cdot), A = 1, \dots, N$ be a system of certain given a priori linear independent functions. The postulated functions $\varphi^A(\cdot)$, which are known dependent on the microstructure length parameter λ , are referred to as fluctuation shape functions.

The second modeling assumption of tolerance averaging approximation states that terms $O(\delta)$ in the course of averaging can be neglected see Woźniak et al. [28]. Under this approximation for an arbitrary integrable function f , slowly varying function F and fluctuation shape function φ , we have

$$\begin{aligned} \langle fF \rangle &= \langle f \rangle F + O(\delta) \\ \langle f \partial_{\alpha}(\varphi F) \rangle &= \langle f \partial_{\alpha} \varphi \rangle F + O(\delta). \end{aligned} \quad (15)$$

The second from conditions (15) is implied by the fact that for every differentiable slowly varying function, F terms $f \varphi \partial_{\alpha} F$ are of an order $O(\delta)$.

4 Averaging description

The tolerance averaging approach can be applied to Eqs. (2–5) by using decomposition of displacement fields $w_3(\mathbf{x}, z, t), w_{\alpha}(\mathbf{x}, z, t)$. Substituting the right-hand sides of Eqs. (14) into (5), on the basis of tolerance averaging approximation, we finally arrive the strain energy averaged over the cell $\Delta(x)$

$$\begin{aligned} \langle E \rangle &= \frac{1}{2} \tilde{B}^{\alpha\beta\gamma\delta} \partial_{\alpha\beta} V_3 \partial_{\gamma\delta} V_3 - \tilde{B}^{\alpha A \gamma \delta} u_{\alpha}^A \partial_{\gamma\delta} V_3 + \frac{1}{2} \tilde{B}^{\alpha A \gamma B} u_{\alpha}^A u_{\gamma}^B + \frac{1}{2} \tilde{D}^{\alpha\beta\gamma\delta} \partial_{\beta} V_{\alpha} \partial_{\delta} V_{\gamma} \\ &\quad + \tilde{D}^{\alpha A \gamma \delta} V_{\alpha}^A \partial_{\delta} V_{\gamma} + \frac{1}{2} \tilde{D}^{\alpha A \gamma B} V_{\alpha}^A V_{\gamma}^B - \tilde{D}^{\alpha\beta} \partial_{\beta} V_{\alpha} V_3 - \tilde{D}^{\alpha A} V_{\alpha}^A V_3 + \frac{1}{2} \tilde{D}^{33} V_3 V_3, \end{aligned} \quad (16)$$

where we have denoted

$$\begin{aligned} \tilde{B}^{\alpha\beta\gamma\delta} &= \langle B^{\alpha\beta\gamma\delta} \rangle, & \tilde{B}^{\alpha A \gamma \delta} &= \langle B^{\alpha\beta\gamma\delta} \partial_{\beta} \varphi^A \rangle, & \tilde{B}^{\alpha A \gamma B} &= \langle B^{\alpha\beta\gamma\delta} \partial_{\beta} \varphi^A \partial_{\delta} \varphi^B \rangle, \\ \tilde{D}^{\alpha\beta\gamma\delta} &= \langle D^{\alpha\beta\gamma\delta} \rangle, & \tilde{D}^{\alpha A \gamma \delta} &= \langle D^{\alpha\beta\gamma\delta} \partial_{\beta} \varphi^A \rangle, & \tilde{D}^{\alpha A \gamma B} &= \langle D^{\alpha\beta\gamma\delta} \partial_{\beta} \varphi^A \partial_{\delta} \varphi^B \rangle, \\ \tilde{D}^{\alpha\beta} &= \langle D^{\alpha\beta\gamma\delta} b_{\gamma\delta} \rangle, & \tilde{D}^{\alpha A} &= \langle D^{\alpha\beta\gamma\delta} b_{\gamma\delta} \partial_{\beta} \varphi^A \rangle, & \tilde{D}^{33} &= \langle D^{\alpha\beta\gamma\delta} b_{\alpha\beta} b_{\gamma\delta} \rangle \end{aligned} \quad (17)$$

Substituting displacement field given by decomposition (14) into formula for kinetic energy (4), we obtain

$$K_z(x, z) = \frac{1}{2} \rho \left[\dot{V}_3 \dot{V}_3 + \left(\dot{V}_\alpha(x) + \varphi^A(x) \dot{V}_\alpha^A(x) \right) \right] + \underbrace{\left(-\partial_\alpha \dot{V}_3(x) + \varphi^A(x) \dot{u}_\alpha^A(x) z \right)}_{\text{underlined}} + \left(\dot{V}_\beta(x) + \varphi^B(x) \dot{V}_\beta^B(x) + \left(-\partial_\beta \dot{V}_3(x) + \varphi^B(x) \dot{u}_\beta^B(x) \right) z \delta^{\alpha\beta} \right) \quad (18)$$

After averaging over the shell thickness and neglecting for thin shells terms given rotation inertia (underlined terms in Eq. 18), we have formula for kinetic energy averaged over cell $\Delta(x)$

$$\langle K \rangle(x) = \frac{1}{2} \langle \rho h \rangle \dot{V}_3 \dot{V}_3 + \frac{1}{2} \langle \rho h \rangle \dot{V}_\alpha \dot{V}_\beta \delta^{\alpha\beta} + \frac{1}{2} \langle \rho h \varphi^A \varphi^B \rangle \dot{V}_\alpha^A \dot{V}_\beta^B \delta^{\alpha\beta}. \quad (19)$$

Averaging work of external forces over the cell, we have formula

$$\langle F \rangle(x) = \langle f^3 \rangle V_3 + \langle f^\alpha \rangle V_\alpha + \langle f^\alpha \varphi^A \rangle V_\alpha^A. \quad (20)$$

In order to derive the governing equations, we shall introduce tolerance averaged Lagrangian $\langle L \rangle$

$$\langle L \rangle = \langle K \rangle - \langle E \rangle + \langle F \rangle. \quad (21)$$

4.1 Tolerance model

Let $\tilde{\phi}^A(\mathbf{x}, \cdot)$ and $\partial_\alpha \tilde{\phi}^A(\mathbf{x}, \cdot)$ stand for periodic approximations of $\phi^A(\cdot)$ and $\partial_\alpha \phi^A(\cdot)$ in cell $\Delta(\mathbf{x})$, $\mathbf{x} \in \Pi$. It can be observed that the periodic approximations of $w_{3h}(\cdot, \mathbf{x}, t)$, $w_{\alpha h}^0(\cdot, \mathbf{x}, t)$, $d_{\alpha h}(\cdot, \mathbf{x}, t)$ and $\partial_\beta w_{3h}(\cdot, \mathbf{x}, t)$, $\partial_\beta w_{\alpha h}^0(\cdot, \mathbf{x}, t)$, $\partial_\beta d_{\alpha h}(\cdot, \mathbf{x}, t)$ have the form

$$\begin{aligned} w_{3h}(x, y, t) &= V_3(s, t), \\ w_{\alpha h}^0(x, y, t) &= V_\alpha(x, t) + \tilde{\varphi}^A(x, y) V_\alpha^A(x, t), \\ d_{\alpha h}(x, y, t) &= -\partial_\alpha V_3(x, t) + \tilde{\varphi}^A(x, y) u_\alpha^A(x, t), \\ \partial_\beta w_{3h}(x, y, t) &= \partial_\beta V_3(x, t), \\ \partial_\beta w_{\alpha h}^0(x, y, t) &= \partial_\beta V_\alpha(x, t) + \partial_\beta \tilde{\varphi}^A(x, y) V_\alpha^A(x, t), \\ \partial_\beta d_{\alpha h}(x, y, t) &= -\partial_{\alpha\beta} V_3(x, t) + \partial_\beta \tilde{\varphi}^A(x, y) u_\alpha^A(x, t), \end{aligned} \quad (22)$$

for $y \in \Delta(\mathbf{x})$, $\mathbf{x} \in \Pi$.

Setting $w_3 = w_{3h}$, $w_\alpha^0 = w_{\alpha h}^0$, $d_\alpha = d_{\alpha h}$ in Eq. (21), we define $L_h = L(x, \dot{w}_{3h}, \dot{w}_{\alpha h}, w_{3h}, \partial_\beta w_{\alpha h}^0, \partial_\beta d_{\alpha h})$. Periodic approximation of function L_h in every $\Delta(\mathbf{x})$ is denoted by

$$\begin{aligned} \tilde{L}_h(x, y, V_3(x, t), \partial_\beta V_3(x, t), V_\alpha(x, t) + \tilde{\varphi}^A(x, y) V_\alpha^A(x, t) + (-\partial_\alpha V_3(x, t) \\ + \tilde{\varphi}^A(x, y) u_\alpha^A(x, t)) z, \partial_\beta V_\alpha(x, t) + \partial_\beta \tilde{\varphi}^A(x, y) V_\alpha^A(x, t) \\ + (-\partial_{\alpha\beta} V_3(x, t) + \partial_\beta \tilde{\varphi}^A(x, y) u_\alpha^A(x, t)) z) \end{aligned} \quad (23)$$

The tolerance averaging of L_h under decomposition (14) is given by

$$\begin{aligned} \langle L_h \rangle &\left(x, V_3, \partial_{\alpha\beta} V_3, \partial_\alpha V_\beta, V_\alpha^A, u_\alpha^A, \dot{V}_3, \dot{V}_\alpha, \dot{V}_\alpha^A \right) \\ &= \frac{1}{|\Delta|} \int_{\Delta(\mathbf{x})} \tilde{L}_h(x, y, V_3(x, t), \partial_\beta V_\alpha(x, t) + \partial_\beta \tilde{\varphi}^A(x, y) V_\alpha^A(x, t), -\partial_{\alpha\beta} V_3(x, t) \\ &\quad + \partial_\beta \tilde{\varphi}^A(x, y) u_\alpha^A(x, t), \dot{V}_3(x, t), \dot{V}_\alpha(x, t) + \tilde{\varphi}^A(x, y) \dot{V}_\alpha^A(x, t)) dy. \end{aligned} \quad (24)$$

Equations of motion we arrive from the principle of stationary action for functional

$$\begin{aligned} \mathcal{A}_h &\left(V_3(\cdot), V_\alpha(\cdot), V_\alpha^A(\cdot), u_\alpha^A(\cdot) \right) \\ &= \int_{t_0}^{t_1} \int_{\Pi} \langle L_h \rangle \left(x, V_3, \partial_{\alpha\beta} V_3, \partial_\alpha V_\beta, V_\alpha^A, u_\alpha^A, \dot{V}_3, \dot{V}_\alpha, \dot{V}_\alpha^A \right) dx dt, \end{aligned} \quad (25)$$

where Lagrangian $\langle L \rangle$ is given by Eq.(21).

Applying the principle of stationary action to functional $\mathcal{A}_h$, we obtain the following system equations for the shell under consideration

$$\begin{aligned} \partial_{\alpha\beta} \left(\tilde{B}^{\alpha\beta\gamma\delta} \partial_{\gamma\delta} V_3 - \tilde{B}^{\gamma A\alpha\beta} u_\gamma^A \right) - \tilde{D}^{\alpha\beta} \partial_\beta V_\alpha - \tilde{D}^{\alpha A} V_\alpha^A + \tilde{D}^{33} V_3 + \langle \rho h \rangle \ddot{V}_3 &= \langle f^3 \rangle, \\ \tilde{B}^{\alpha A \gamma \delta} \partial_{\gamma\delta} V_3 - \tilde{B}^{\alpha A \gamma B} u_\gamma^B &= 0, \\ \partial_\delta \left(\tilde{D}^{\alpha\beta\gamma\delta} \partial_\beta V_\alpha + \tilde{D}^{\alpha A \gamma \delta} V_\alpha^A - \tilde{D}^{\gamma\delta} V_3 \right) - \langle \rho h \rangle \ddot{V}_\gamma &= -\langle f^\gamma \rangle, \\ \tilde{D}^{\alpha A \gamma \delta} \partial_\delta V_\gamma + \tilde{D}^{\alpha A \gamma B} V_\gamma^B - \tilde{D}^{\alpha A} V_3 + \langle \rho h \varphi^A \varphi^B \rangle \ddot{V}_\alpha^B &= \langle f^\alpha \varphi^A \rangle. \end{aligned} \quad (26)$$

It is to observe that u_γ^B can be eliminated from above equations

$$u_\mu^B = K^{\mu B \alpha A} \tilde{B}^{\alpha A \gamma \delta} \partial_{\gamma\delta} V_3, \quad (27)$$

where $K^{\mu B \alpha A}$ determine the linear transformation, inverse to that given by $\tilde{B}^{\mu B \alpha A} \left(K^{\mu B \alpha A} \tilde{B}^{\alpha A \gamma C} = \delta_{\mu\gamma}^{BC} \right)$.

After simple manipulations, we obtain finally the following system of equations for the averaged displacements $V_3(\mathbf{x}, t)$, $V_\alpha(\mathbf{x}, t)$, and displacements fluctuation amplitudes $V_\alpha^A(\mathbf{x}, t)$

$$\begin{aligned} \partial_{\alpha\beta} \left(\left(\tilde{B}^{\alpha\beta\gamma\delta} - \tilde{B}^{\mu B \alpha \beta} K^{\mu B \alpha A} \tilde{B}^{\alpha A \gamma \delta} \right) \partial_{\gamma\delta} V_3 \right) - \tilde{D}^{\alpha\beta} \partial_\beta V_\alpha - \tilde{D}^{\alpha A} V_\alpha^A + \tilde{D}^{33} V_3 + \langle \rho h \rangle \ddot{V}_3 &= \langle f^3 \rangle, \\ \partial_\delta \left(\tilde{D}^{\alpha\beta\gamma\delta} \partial_\beta V_\alpha + \tilde{D}^{\alpha A \gamma \delta} V_\alpha^A - \tilde{D}^{\gamma\delta} V_3 \right) - \langle \rho h \rangle \ddot{V}_\gamma &= -\langle f^\gamma \rangle, \\ \tilde{D}^{\alpha A \gamma \delta} \partial_\delta V_\gamma + \tilde{D}^{\alpha A \gamma B} V_\gamma^B - \tilde{D}^{\alpha A} V_3 + \langle \rho h \varphi^A \varphi^B \rangle \ddot{V}_\alpha^B &= \langle f^\alpha \varphi^A \rangle. \end{aligned} \quad (28)$$

Coefficients in the above equations are smooth and functional in contrast to equations in direct description with the discontinuous and highly oscillating coefficients.

The above equations together with micro-macro decomposition of displacement fields (14) and physical reliability that solutions are slowly varying functions constitute the tolerance model of the skeletal shells. The underlined terms are depended explicit on the microstructure length parameter λ . It means that derived model describes on a macroscopic level the effect of microstructure length on the solution of initial/boundary value problem for macroscopic fields $V_3(\mathbf{x}, t), V_\alpha(\mathbf{x}, t), V_\alpha^A(\mathbf{x}, t)$. Neglecting these phenomena, we can derive simplified asymptotic model equations.

4.2 Asymptotic model

The equations of the asymptotic model will be derived similarly to tolerance model equations applying principle stationary action.

Using asymptotic approach, we introduce parameter $\varepsilon = 1/n, n = 1, 2, \dots$

Let $\Delta_\varepsilon \equiv (-\varepsilon\lambda_1/2, \varepsilon\lambda_1/2) \times (-\varepsilon\lambda_2/2, \varepsilon\lambda_2/2)$ be a scaled cell and $\Delta_\varepsilon(\mathbf{x}) \equiv \mathbf{x} + \Delta_\varepsilon$ be a set of the cell with a center at $x \in \bar{\Pi}$.

The fundamental assumption of the proposed modeling procedure is that we introduce family of displacement fields

$$\begin{aligned} w_{3\varepsilon}(x, y, t) &= V_3(x, t), \quad y \in \Delta_\varepsilon(x) \\ w_{\alpha\varepsilon}^0(x, y, t) &= V_\alpha(x, t) + \varepsilon \tilde{\Phi}^A \left(x, \frac{y}{\varepsilon} \right) V_\alpha^A(w, t), \\ d_{\alpha\varepsilon}(x, y, t) &= -\partial_\alpha V_3(x, t) + \varepsilon \tilde{\Phi}^A \left(x, \frac{y}{\varepsilon} \right) u_\alpha^A(w, t), \end{aligned} \quad (29)$$

where $\tilde{\varphi}^A(x, \cdot)$ is a periodic approximation of highly oscillating function $\varphi^A(\cdot)$ in $\Delta_\varepsilon(\mathbf{x})$. It follows that

$$\begin{aligned}\partial_\beta w_{3\varepsilon}(x, y, t) &= \partial_\beta V_3(x, t), \\ \partial_\beta w_{\alpha\varepsilon}^0(x, y, t) &= \partial_\beta V_\alpha(x, t) + \partial_\beta \tilde{\varphi}^A\left(x, \frac{y}{\varepsilon}\right) V_\alpha^A(x, t) + \varepsilon \tilde{\varphi}^A\left(x, \frac{y}{\varepsilon}\right) \partial_\beta V_\alpha^A(x, t), \\ \partial_\beta d_{\alpha\varepsilon}(x, y, t) &= \partial_{\alpha\beta} V_3(x, t) + \partial_\beta \tilde{\varphi}^A\left(x, \frac{y}{\varepsilon}\right) u_\alpha^A(x, t) + \varepsilon \tilde{\varphi}^A\left(x, \frac{y}{\varepsilon}\right) \partial_\beta u_\alpha^A(x, t), \\ \dot{w}_{3\varepsilon}(x, y, t) &= \dot{V}_3(x, t), \\ \dot{w}_{\alpha\varepsilon}^0(x, y, t) &= \dot{V}_\alpha(x, t) + \varepsilon \tilde{\varphi}^A\left(x, \frac{y}{\varepsilon}\right) \partial_\beta V_\alpha^A(x, t).\end{aligned}\quad (30)$$

Under limit passage $\varepsilon \rightarrow 0$ for $y \in \Delta_\varepsilon(\mathbf{x})$, we obtain

$$\begin{aligned}\partial_\beta w_{3\varepsilon}(x, y, t) &= \partial_\beta V_3(x, t), \\ \partial_\beta w_{\alpha\varepsilon}^0(x, y, t) &= \partial_\beta V_\alpha(x, t) + \partial_\beta \tilde{\varphi}^A\left(x, \frac{y}{\varepsilon}\right) V_\alpha^A(x, t) + O(\varepsilon), \\ \partial_\beta d_{\alpha\varepsilon}(x, y, t) &= \partial_{\alpha\beta} V_3(x, t) + \partial_\beta \tilde{\varphi}^A\left(x, \frac{y}{\varepsilon}\right) u_\alpha^A(x, t) + O(\varepsilon), \\ \dot{w}_{3\varepsilon}(x, y, t) &= \dot{V}_3(x, t), \\ \dot{w}_{\alpha\varepsilon}^0(x, y, t) &= \dot{V}_\alpha(x, t) + O(\varepsilon).\end{aligned}\quad (31)$$

Let $\tilde{L}(x, y, w_3, \partial_\beta w_\alpha^0, \partial_\beta d_\alpha, \dot{w}_3, \dot{w}_\alpha^0)$, $y \in \Delta(\mathbf{x})$ stands for periodic approximation of Lagrangian L . Hence,

$$\begin{aligned}\tilde{L}_\varepsilon\left(\frac{y}{\varepsilon}, x, V_3\left(\frac{y}{\varepsilon}, x, t\right), \partial_\beta V_\alpha\left(\frac{y}{\varepsilon}, x, t\right) + \partial_\beta \tilde{\varphi}^A\left(\frac{y}{\varepsilon}, x\right) V_\alpha^A\left(\frac{y}{\varepsilon}, x, t\right) + O(\varepsilon), \right. \\ \left. -\partial_{\alpha\beta} V_3\left(\frac{y}{\varepsilon}, x, t\right) + \partial_\beta \tilde{\varphi}^A\left(\frac{y}{\varepsilon}, x\right) u_\alpha^A\left(\frac{y}{\varepsilon}, x, t\right) + O(\varepsilon), \dot{V}_3\left(\frac{y}{\varepsilon}, x, t\right), \dot{V}_\alpha\left(\frac{y}{\varepsilon}, x, t\right) + O(\varepsilon)\right).\end{aligned}\quad (32)$$

Under limit passage $\varepsilon \rightarrow 0$ $\tilde{L}_\varepsilon$ as a function of $\left(\frac{y}{\varepsilon}\right)$, $y \in \Delta_\varepsilon(\mathbf{x})$, by means of a property of the mean value, Jikov et al. [10], tends weakly in $L_{loc}^\gamma(R^m)$, $\gamma \geq 1$ to

$$\begin{aligned}L_0\left(x, \dot{V}_3, \dot{V}_\alpha, V_3, \partial_{\alpha\beta} V_3, \partial_\beta V_\alpha, v_\alpha^A, u_\alpha^A\right) = \\ \frac{1}{|\Delta|} \int_{\Delta(\mathbf{x})} \tilde{L}_\varepsilon\left(y, x, \dot{V}_3\left(\frac{y}{\varepsilon}, x, t\right), \dot{V}_\alpha\left(\frac{y}{\varepsilon}, x, t\right), V_3\left(\frac{y}{\varepsilon}, x, t\right), \partial_\beta V_\alpha\left(\frac{y}{\varepsilon}, x, t\right) + \partial_\beta \tilde{\varphi}^A\left(\frac{y}{\varepsilon}, x\right) V_\alpha^A\left(\frac{y}{\varepsilon}, x, t\right), \right. \\ \left. -\partial_{\alpha\beta} V_3\left(\frac{y}{\varepsilon}, x, t\right) + \partial_\beta \tilde{\varphi}^A\left(\frac{y}{\varepsilon}, x\right) u_\alpha^A\left(\frac{y}{\varepsilon}, x, t\right)\right) dy.\end{aligned}\quad (33)$$

Asymptotic action functional will be defined by

$$\mathcal{A}_h^0\left(V_3(\cdot), V_\alpha(\cdot), V_\alpha^A(\cdot), u_\alpha^A(\cdot)\right) = \int_{t_0}^{t_1} \int_{\Pi} \langle L_0 \rangle(x, V_3, \partial_{\alpha\beta} V_3, \partial_\alpha V_\beta, V_\alpha^A, u_\alpha^A, \dot{V}_3, \dot{V}_\alpha) dx dt. \quad (34)$$

From the principle of stationary action for functional $\mathcal{A}_h^0$, we obtain equations of asymptotic model

$$\begin{aligned}\partial_{\alpha\beta} \left(\left(\tilde{B}^{\alpha\beta\gamma\delta} - \tilde{B}^{\mu B\alpha\beta} K^{\mu B\alpha A} \tilde{B}^{\alpha A\gamma\delta} \right) \partial_{\gamma\delta} V_3 \right) \\ - \tilde{D}^{\alpha\beta} \partial_\beta V_\alpha - \tilde{D}^{\alpha A} V_\alpha^A + \tilde{D}^{33} V_3 + \langle \rho h \rangle \ddot{V}_3 = \langle f^3 \rangle \\ \partial_\delta \left(\tilde{D}^{\alpha\beta\gamma\delta} \partial_\beta V_\alpha + \tilde{D}^{\alpha A\gamma\delta} V_\alpha^A - \tilde{D}^{\gamma\delta} V_3 \right) - \langle \rho h \rangle \ddot{V}_\gamma = -\langle f^\gamma \rangle, \\ \tilde{D}^{\alpha A\gamma\delta} \partial_\delta V_\gamma + \tilde{D}^{\alpha A\gamma B} V_\gamma^B - \tilde{D}^{\alpha A} V_3 = 0.\end{aligned}\quad (35)$$

Now we can eliminate V_γ^B from Eq. (35)

$$V_\gamma^B = M^{\gamma B\alpha A} \tilde{D}^{\alpha A} V_3 - M^{\gamma B\alpha A} \tilde{D}^{\alpha A\mu\delta} \partial_\delta V_\mu, \quad (36)$$

where $M^{\mu B\alpha A}$ determine the linear transformation, inverse to that given by $\tilde{D}^{\mu B\alpha A} \left(M^{\mu B\alpha A} \tilde{D}^{\alpha A\gamma C} = \delta_{\mu\gamma}^{BC} \right)$.

Denoting

$$\begin{aligned} F^{\alpha\beta\gamma\delta} &= \tilde{B}^{\alpha\beta\gamma\delta} - \tilde{B}^{\mu B\alpha\beta} K^{\mu B\alpha A} \tilde{B}^{\alpha A\gamma\delta} & E^{\alpha\beta\gamma\delta} &= \tilde{D}^{\alpha\beta\gamma\delta} - \tilde{D}^{\tau A\gamma\delta} M^{\tau A\mu B} \tilde{D}^{\mu B\alpha\beta} \\ E^{\alpha\beta} &= \tilde{D}^{\alpha\beta} - \tilde{D}^{\gamma B} M^{\gamma B\mu A} \tilde{D}^{\mu A\alpha\beta} & E^{33} &= \tilde{D}^{33} - \tilde{D}^{\alpha A} M^{\alpha A\gamma B} \tilde{D}^{\gamma B}, \end{aligned} \quad (37)$$

we arrive from (35) the following system equations of motion for the averaged displacements of the shell midsurface $V_3(x, t)$ and $V_\alpha(x, t)$

$$\begin{aligned} \partial_{\alpha\beta} (F^{\alpha\beta\gamma\delta} \partial_{\gamma\delta} V_3) - E^{\alpha\beta} \partial_\beta V_\alpha + E^{33} V_3 + \langle \rho h \rangle \ddot{V}_3 &= \langle f^3 \rangle, \\ \partial_\delta (E^{\alpha\beta\gamma\delta} \partial_\beta V_\alpha - E^{\gamma\delta} V_3) - \langle \rho h \rangle \ddot{V}_\gamma &= \langle f^\gamma \rangle. \end{aligned} \quad (38)$$

Equations (36–38) represent what can be called the asymptotic model of the skeletal shells under consideration. It has to be emphasized that for the asymptotic model only averaged displacement fields $V_3(\mathbf{x}, t), V_\alpha(\mathbf{x}, t)$ can be taken as the basic unknown.

5 Applications

We shall investigate free vibrations of open cylindrical skeletal shell band simply supported on its boundary (Fig. 3).

The crucial point of the tolerance modeling technique is a determination of fluctuation shape functions (FSF). In a dynamic processes system of FSF, $\varphi^A(\cdot)$, $A = 1, \dots, N$, can be taken as a representing eigenvibrations form of free vibrations of the cell $\Delta(\mathbf{x})$, $\mathbf{x} \in \Pi$. Our analysis will be restricted to the simplest case $A = I, II$ and for the cell under considered as the fluctuation shape functions, we assume functions (Fig. 4)

Fig. 3 Fragment of skeletal cylindrical shell

Fig. 4 Fluctuation shape functions

$$\begin{aligned}\varphi^1(x_1, \mathbf{y}) &= s_1(x_1, y_1) \left[1 - \left(\frac{2y_2}{b_2} \right)^2 \right], \\ \varphi^2(x_1, \mathbf{y}) &= s_2(y_2) \left[1 - \left(\frac{2y_1}{b_1(x_1)} \right)^2 \right],\end{aligned}\quad (39)$$

where

$$\begin{aligned}s_1(x_1, y_1) &= \begin{cases} -(y_1 + \lambda_1(x_1)/2); & y_1 \in \langle -\lambda_1(x_1)/2; -b_1(x_1)/2 \rangle \\ d_1 y_1 / (\lambda_1(x_1) - d_1); & y_1 \in \langle -b_1(x_1)/2; b_1(x_1)/2 \rangle \\ -(y_1 - \lambda_1(x_1)/2); & y_1 \in \langle b_1(x_1)/2; \lambda_1(x_1)/2 \rangle \end{cases} \\ s_2(y_2) &= \begin{cases} -(y_2 + \lambda_2/2); & y_2 \in \langle -\lambda_2/2; -b_2/2 \rangle \\ d_2 y_2 / (\lambda_2 - d_2); & y_2 \in \langle -b_2/2; b_2/2 \rangle \\ -(y_2 - \lambda_2/2); & y_2 \in \langle b_2/2; \lambda_2/2 \rangle \end{cases}.\end{aligned}$$

Tolerance model. After simple manipulation, we obtain from Eq. (26) the following system of four differential equations describing dynamic behavior of the cylindrical shell band

$$\begin{aligned}\partial_{11} (F^{1111} \partial_{11} V_3) - \tilde{D}^{11} \partial_1 V_1 - \tilde{D}^{1I} V_1^I + \tilde{D}^{33} V_3 + \langle \rho h \rangle \ddot{V}_3 &= 0, \\ \partial_1 (\tilde{D}^{1111} \partial_1 V_1 + \tilde{D}^{1I11} V_1^I + \tilde{D}^{2I11} V_2^{II} - \tilde{D}^{11} V_3) - \langle \rho h \rangle \ddot{V}_1 &= 0, \\ \tilde{D}^{1I11} \partial_1 V_1 + \tilde{D}^{1I1I} V_1^I + \tilde{D}^{1I2II} V_2^{II} - \tilde{D}^{1I} V_3 + \langle \rho h \varphi^I \varphi^I \rangle \ddot{V}_1^I &= 0, \\ \tilde{D}^{2I11} \partial_1 V_1 + \tilde{D}^{2I1I} V_1^I + \tilde{D}^{2I2II} V_2^{II} - \tilde{D}^{2II} V_3 + \langle \rho h \varphi^{II} \varphi^{II} \rangle \ddot{V}_2^{II} &= 0,\end{aligned}\quad (40)$$

where

$$F^{1111} = \tilde{B}^{1111} - \frac{\tilde{B}^{1I11} (\tilde{B}^{1I11} \tilde{B}^{2I2II} - \tilde{B}^{2I11} \tilde{B}^{1I2II}) + \tilde{B}^{2I11} (\tilde{B}^{1I11} \tilde{B}^{2I11} - \tilde{B}^{2I1I} \tilde{B}^{1I11})}{\tilde{B}^{1I11} \tilde{B}^{2I2II} - \tilde{B}^{2I11} \tilde{B}^{1I2II}}.$$

Explicit form of averaged modules in Eq. (40) will be given, for example, for $\tilde{B}^{1111}$ and $\tilde{B}^{1I11}$. Denoting by d -the beams width, B_m -matrix stiffness, B_b -beams stiffness, $(\alpha_1 = B_m/B_b)$, $v_m = v_b$ and by λ_1, λ_2 - diameter of the cell, we have

$$\begin{aligned}\tilde{B}^{1111} &= B_b(\lambda_1 d + d(\lambda_1 - d) + \alpha_1(\lambda_1 - d)(\lambda_2 - d))/\lambda_1 \lambda_2, \\ \tilde{B}^{1I11} &= 2B_b(\lambda_2 - d)(\alpha_1 - 1)d/3\lambda_1 \lambda_2.\end{aligned}$$

Equation (40) represent a system of four partial differential equations for the averaged displacements $V_1(\cdot, t)$, $V_3(\cdot, t)$ and fluctuation amplitudes $V_1^I(\cdot, t)$, $V_2^{II}(\cdot, t)$. We look for solution to Eq. (40) in the form of a function with separable variables

$$\begin{aligned}V_1(x_1, t) &= \tilde{V}_1(x_1) e^{i\omega t} & V_3(x_1, t) &= \tilde{V}_3(x_1) e^{i\omega t}, \\ V_1^I(x_1, t) &= \tilde{V}_1^I(x_1) e^{i\omega t} & V_2^{II}(x_1, t) &= \tilde{V}_2^{II}(x_1) e^{i\omega t}.\end{aligned}\quad (41)$$

Substituting (41) into (40), we obtain equations for $\tilde{V}_1(x_1)$, $\tilde{V}_3(x_1)$ and $\tilde{V}_1^I(x_1)$, $\tilde{V}_2^{II}(x_1)$

$$\begin{aligned}\partial_{11} (F^{1111} \partial_{11} \tilde{V}_3) - \tilde{D}^{11} \partial_1 \tilde{V}_1 - \tilde{D}^{1I} \tilde{V}_1^I + \tilde{D}^{33} \tilde{V}_3 - \langle \rho h \rangle \omega^2 \tilde{V}_3 &= 0, \\ \partial_1 (\tilde{D}^{1111} \partial_1 \tilde{V}_1 + \tilde{D}^{1I11} \tilde{V}_1^I + \tilde{D}^{2I11} \tilde{V}_2^{II} - \tilde{D}^{11} \tilde{V}_3) + \langle \rho h \rangle \omega^2 \tilde{V}_1 &= 0, \\ \tilde{D}^{1I11} \partial_1 \tilde{V}_1 + \tilde{D}^{1I1I} \tilde{V}_1^I + \tilde{D}^{1I2II} \tilde{V}_2^{II} - \tilde{D}^{1I} \tilde{V}_3 - \langle \rho h \varphi^I \varphi^I \rangle \omega^2 \tilde{V}_1^I &= 0, \\ \tilde{D}^{2I11} \partial_1 \tilde{V}_1 + \tilde{D}^{2I1I} \tilde{V}_1^I + \tilde{D}^{2I2II} \tilde{V}_2^{II} - \tilde{D}^{2II} \tilde{V}_3 - \langle \rho h \varphi^{II} \varphi^{II} \rangle \omega^2 \tilde{V}_2^{II} &= 0.\end{aligned}\quad (42)$$

Since $\varphi^A(\cdot) \in O(\lambda)$, the inertial module $\langle \mu \varphi^A \varphi^B \rangle$ depends on the microstructure length parameter λ ; hence, aforementioned equations describe the microstructure length-scale effect on the natural frequencies of the shell under consideration.

Asymptotic model. Using Equation (38), we derive equations for the asymptotic model

$$\begin{aligned}\partial_{11} (F^{1111} \partial_{11} V_3) - E^{11} \partial_1 V_1 + E^{33} V_3 + \langle \rho h \rangle \ddot{V}_3 &= 0, \\ \partial_1 (E^{1111} \partial_1 V_1 - E^{11} V_3) - \langle \rho h \rangle \ddot{V}_1 &= 0.\end{aligned}\quad (43)$$

The basic unknowns in the above equations are averaged displacements $V_1(x_1, t)$, $V_3(x_1, t)$.

In the case of shallow shell, we have the non-uniform cell distribution in plane Π , where Π is the shell projection on $x_3 = 0$ and the Eqs. (40, 43) have smooth and slow functional coefficients. Hence, in most cases, solutions to specific problems for skeletal shallow shells have to be obtained using approximate methods. When we consider approximate uniform cell distribution in plane Π , then the coefficients in model equations are constant and Eqs. (40, 43) take more simple form.

Numerical calculations. The aim of numerical calculations is to compare free vibrations frequencies derived from model equations for the model with non-uniform cell distribution and model with approximate uniform cell distribution. For the sake of simplicity, we consider asymptotic model equations. We look for a solution to Eq. (43) in the form of a function with separable variables

$$V_1(x_1, t) = \tilde{V}_1(x_1)e^{i\omega t} \quad V_3(x_1, t) = \tilde{V}_3(x_1)e^{i\omega t}. \quad (44)$$

Substituting (44) into (43), we obtain

$$L_1(\tilde{V}_1, \tilde{V}_3) = 0 \quad L_2(\tilde{V}_1, \tilde{V}_3) = 0 \quad (45)$$

with differential operators

$$\begin{aligned} L_1(\tilde{V}_1, \tilde{V}_3, x_1) &= \partial_{11}(F^{1111}\partial_{11}\tilde{V}_3) - E^{11}\partial_1\tilde{V}_1 + E^{33}\tilde{V}_3 - \langle\rho h\rangle\omega^2\tilde{V}_3, \\ L_2(\tilde{V}_1, \tilde{V}_3, x_1) &= \partial_1(E^{1111}\partial_1\tilde{V}_1 - E^{11}\tilde{V}_3) + \langle\rho h\rangle\omega^2\tilde{V}_1. \end{aligned} \quad (46)$$

For uniform distribution of the cell, the differential operators (46) take the form

$$\begin{aligned} L_1(\tilde{V}_1, \tilde{V}_3, x_1) &= F^{1111}\partial_{1111}\tilde{V}_3 - E^{11}\partial_1\tilde{V}_1 + E^{33}\tilde{V}_3 - \langle\rho h\rangle\omega^2\tilde{V}_3, \\ L_2(\tilde{V}_1, \tilde{V}_3, x_1) &= E^{1111}\partial_{11}\tilde{V}_1 - E^{11}\partial_1\tilde{V}_3 + \langle\rho h\rangle\omega^2\tilde{V}_1. \end{aligned} \quad (47)$$

In order to obtain the approximate solution of Eq. (45) for the shell under consideration will be used the Galerkin method

$$\int_0^L L_1(\cdot)\delta\tilde{V}_3(x_1)dx_1 = 0 \quad \int_0^L L_2(\cdot)\delta\tilde{V}_1(x_1)dx_1 = 0, \quad (48)$$

where L is a span of the cylindrical shell band.

For simply supported shell band as trial functions $\delta\tilde{V}_1(x_1)$, $\delta\tilde{V}_3(x_1)$, we assume

$$\delta\tilde{V}_1(x_1) = \sin(2\pi x_1/L) \quad \delta\tilde{V}_3(x_1) = \sin(\pi x_1/L). \quad (49)$$

These functions satisfy the boundary conditions for simply supported shell band.

Substituting the operators (46) or (47) with functions (49) into Eq. (48), we derive the free vibration frequencies for the shell under consideration.

The numerical results are derived for skeletal shells made from steel-aluminum composite. Material properties assigned to steel beams and aluminum matrix are as follows: steel— $E = 210$ GPa, $\rho = 7,800$ kg/m³, $\nu = 0.3$; aluminum— $E = 69$ GPa, $\rho = 2,720$ kg/m³, $\nu = 0.3$.

In Figs. 5, 6 are presented results for cylindrical shallow shell band with the thickness $h = 0.1$ m, shell span $L = 5$ m and ratio $f/L = 1/10$ where f is the maximum rise of the shell. For the shallow shell with ratio $f/L = 1/10$, we have assumed the cell distribution given by function $\lambda_1(x_1) = (\lambda_1^0 - \lambda_1^k)\sin(\pi x_1/L) + \lambda_1^k = 0.05\lambda_1^0\sin(\pi x_1/L) + 0.95\lambda_1^0$, where λ_1^k , λ_1^0 are the cell dimensions on boundary and in the middle of the shell, respectively.

In Fig. 5, there are shown diagrams of the first transverse vibration frequency for non-uniform (continuous line) and approximate uniform (dot line) the cell distribution versus dimensionless microstructure parameter $m = d/\lambda$, where $\lambda = \lambda_1^0$ is the cell dimension in the middle of the shell and d is the beams width.

The differences between both above frequencies are smaller than 0.5%.

In Fig. 6, there are shown diagrams of the frequency of vibration in direction tangential to the midsurface for non-uniform (continuous line) and approximate uniform (dot line) the cell distribution versus dimensionless microstructure parameter $m = d/\lambda$.

Similar to transverse vibrations, the differences between above frequencies are smaller than 1.5%.

Fig. 5 Transverse vibration frequencies

Fig. 6 Frequencies vibration in the middle surface

6 Conclusions

The results obtained in this contribution justify to formulate the following conclusions.

1. The skeletal shallow shells (cf. Fig. 1) are described by the model equations involving only smooth coefficients in contrast to equations in direct description with the discontinuous and highly oscillating coefficients.
2. From the general tolerance model Eq. (24), by neglecting terms involving coefficients depending on the micro-structure length parameter λ , the simplified asymptotic model Eqs. (26, 28) were derived.
3. Solutions to the boundary/initial value problems formulated in the framework of the proposed tolerance model have the physical sense only if they are slowly varying in the distinguish directions. This requirement can be verified only a posteriori and determines the range of the physical applicability of the proposed model.
4. Since the proposed model equations have smooth and slow functional coefficients, then in most cases, solutions to specific problems for shells under consideration have to be obtained using well-known numerical methods.
5. The diagrams in Figs. 5, 6 show the natural vibration frequencies of the cylindrical skeletal shallow shell for symmetric mode. It was shown that differences between first frequency obtained for the model with non-uniform cell distribution and for the approximate uniform cell distribution are very small (Figs. 5, 6).
6. For the approximate uniform cell distribution, the model equations have constant coefficients and solutions to specific problems for shells under consideration can be obtained more simply than solutions for exact non-uniform cell distribution.

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