Edge-Distinguishing Index of a Graph

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Abstract We introduce a concept of edge-distinguishing colourings of graphs. A closed neighbourhood of an edge $e \in E(G)$ is a subgraph $N[e]$ induced by e and all edges adjacent to it. We say that a colouring $c : E(G) \rightarrow C$ does not distinguish two edges e_1 and e_2 if there exists an isomorphism φ of $N[e_1]$ onto $N[e_2]$ such that $\varphi(e_1) = e_2$ and φ preserves colours of c . An edge-distinguishing index of a graph G is the minimum number of colours in a proper colouring which distinguishes every two distinct edges of G . We determine the edge-distinguishing index for cycles, paths and complete graphs.

Keywords Proper edge colouring · Chromatic index · Euler tours in multigraphs

1 Introduction

We use standard terminology and notation of graph theory. Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. As usual, $\chi'(G)$ denotes the chromatic index of G .

In a graph G , the *closed neighbourhood of an edge* $e \in E(G)$ is the subgraph $N[e]$ induced by e and all edges adjacent to e . Observe that $N[e]$ is always a subgraph of a graph depicted in Fig. 1. We say that a *colouring* $c : E(G) \rightarrow C$ *does not distinguish two edges* e_1 and e_2 if there exists an isomorphism φ of $N[e_1]$ onto $N[e_2]$ such that $\varphi(e_1) = e_2$, and φ preserves colours of c .

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Fig. 1 The closed neighbourhood $N[e]$ of an edge e in any graph G is a subgraph of this graph

An *edge-distinguishing index* of a graph G is the smallest number $\text{edi}(G)$ of colours in a proper colouring $c : E(G) \rightarrow C$ that distinguishes every two distinct edges of G . Such a colouring is called *edge-distinguishing*. Clearly, $\text{edi}(G) \geq \chi'(G)$. In this paper, we determine the edge-distinguishing index for cycles, paths and complete graphs.

Our concept of the edge-distinguishing index is analogous to the concept of the vertex-distinguishing index introduced by Černý et al. [6] as *observability*, and independently by Burriss and Schelp [5] as *strong chromatic index* χ'_s . We adopt the notation vdi . To exhibit the analogy, we formulate the definition of vdi as follows. By the *closed neighbourhood of a vertex* $v \in V(G)$ we mean the star $S[v]$ induced by all edges incident to v . A colouring $c : E(G) \rightarrow C$ *does not distinguish two vertices* v_1 and v_2 if there exists an isomorphism φ of $S[v_1]$ onto $S[v_2]$ that preserves colours of edges. The *vertex-distinguishing index* of a graph G is the smallest number $\text{vdi}(G)$ of colours in a proper colouring of edges of G that distinguishes every two distinct vertices. Vertex-distinguishing colourings were investigated in a number of papers (e.g., [1–6]), and the vertex distinguishing index has been determined for some classes of graphs, e.g., complete graphs, cycles, paths, disjoint unions of cycles, disjoint unions of paths.

2 Cycles

In a cycle C_3 , the closed neighbourhood of any edge is a triangle, and trivially, $\text{edi}(C_3) = 3$. In every cycle C_n of length $n \geq 4$, the closed neighbourhood of any edge is a path P_4 of length three.

Let G_1 and G_2 be two graphs isomorphic to P_4 and let $c_i : E(G_i) \rightarrow \{1, \dots, k\}$ be a proper edge colouring of G_i , $i = 1, 2$. The colouring c_1 is *isomorphic* to the colouring c_2 if there is an isomorphism $\varphi : V(G_1) \rightarrow V(G_2)$ with $c_1(xy) = c_2(\varphi(x)\varphi(y))$ for any $xy \in E(G_1)$.

If $c : E(C_n) \rightarrow \{1, \dots, k\}$ is an edge-distinguishing colouring, it induces n colourings $c_i : E(G_i) \rightarrow \{1, \dots, k\}$, $i = 1, \dots, n$, where each G_i is isomorphic to P_4 (and determined by one of n edges of C_n). From the definition it follows that the colourings $c_1, \dots, c_n$ are pairwise nonisomorphic.

Clearly, the maximum number of pairwise nonisomorphic proper edge colourings of P_4 with k colours is equal to $k \binom{k-1}{2} + k(k-1) = \frac{1}{2}k^2(k-1)$; the first (the second) summand corresponds to colourings with distinct (identical, respectively) colours of hanging edges of P_4 . Hence, the number $\gamma(C_n) := \min\{k|k^2(k-1) \geq 2n\}$ satisfies the inequality $\text{edi}(C_n) \geq \gamma(C_n)$.

Fig. 2 An edge-distinguishing colouring of C_9 with three colours

For small cycles, one can easily verify that $\text{edi}(C_i) = 3$ for $i = 3, 5, 7, 9$, while $\text{edi}(C_i) = 4$ for $i = 4, 6, 8$ (this can be also checked using Proposition 1 below). See Fig. 2 for $i = 9$.

Recall that the vertex-distinguishing index $\text{vdi}(C_n)$ of a cycle of order n could be easily determined due to the following crucial observation [5, 6]. Namely, $\text{vdi}(C_n) \leq k$ if and only if the complete graph K_k (representing k colours) contains an Eulerian subgraph of size n . This technique adopted to edge-distinguishing colourings is more complicated.

Given an edge-distinguishing colouring of a cycle C_n with k colours, each unordered pair of distinct colours may appear at most k times as colours of two adjacent edges. Indeed, suppose e_1, e_2, e_3 are three subsequent edges on C_n and e_1, e_2 are coloured with c_1, c_2 , respectively. Then there are $k - 1$ colours admissible for e_3 . Moreover, if the colour of e_3 is c_1 , then we have another, k th, pair of colours c_2, c_1 .

Therefore, we define the *multigraph M_k of k colours*, as a multigraph of order $k \geq 3$ such that each pair of vertices is joined by k edges. Thus, the size of M_k equals $n_k = \frac{1}{2}k^2(k - 1)$.

An Eulerian multigraph is called *good* if it contains an Euler tour $W = (v_1 \dots v_n v_1)$, in which every trail of length two (i.e., either a path P_3 or a 2-cycle) appears at most once as a subtrail of W , in the following sense: for every i and j with $1 \leq i + 2 \leq j \leq n$, the sequence $(v_i v_{i+1} v_{i+2})$ does not coincide neither with $(v_j v_{j+1} v_{j+2})$ nor with $(v_{j+2} v_{j+1} v_j)$, where the indices are to be read modulo n . Such an Euler tour W is called *good*, too.

Observe that to every edge-distinguishing colouring of a cycle C_n with k colours, there corresponds a good Eulerian subgraph of size n of the multigraph M_k . The following observation follows directly from the definition of the edge-distinguishing index.

Proposition 1 $\text{edi}(C_n) \leq k$ if and only if the multigraph M_k contains a good Eulerian subgraph of size n . $\square$

Let M be a multigraph with $V(M) = \{u_1, \dots, u_p\}$, M' an Eulerian submultigraph of M , $F = (v_1 \dots v_r v_1)$ an Euler tour of M and M'' the complement of M' in M , i.e., the multigraph with $V(M'') = V(M)$ and $E(M'') = E(M) \setminus E(M')$. Let $\mu_{M''}(u_i u_j)$ denote the multiplicity of the edge $u_i u_j$ in M'' . Suppose that $\mu_{M''}(u_{i-1} u_i) \geq 1$, $\mu_{M''}(u_{i-1} u_s) \geq 1$ and $\mu_{M''}(u_i u_s) \geq 3$ for some $i \in \{1, \dots, r\}$ and $s \in \{1, \dots, p\}$ satisfying $u_s \notin \{v_{i-1}, v_i\}$ (where indices of v are taken modulo r). Then by $TU(i, s)$

we shall denote the operation of substituting the subtrail $(v_{i-1}v_i)$ of F with the trail $(v_{i-1}v_iu_s v_iu_s v_{i-1}v_i)$. Another operation, denoted by $U(i, s)$, represents the substitution of $(v_{i-1}v_i)$ with $(v_{i-1}v_iu_s v_i)$. The third operation $T(i, s)$ means substituting $(v_{i-1}v_i)$ with $(v_{i-1}v_iu_s v_{i-1}v_i)$. Note that the minimal multiplicity requirements for applying $U(i, s)$ and $T(i, s)$ are weaker than those above, namely $(0, 0, 2)$ (concerning subsequently the edges $v_{i-1}v_i$, $v_{i-1}u_s$ and v_iu_s) for $U(i, s)$ and $(1, 1, 1)$ for $T(i, s)$. The operations $U(i, s)$, $T(i, s)$ and $TU(i, s)$ yield Euler tours of a submultigraph of M that is of size greater than that of M , namely by two, three and five, respectively.

We shall also make use of the following lemma.

Lemma 1 *If k, l are integers with $k \geq 3$ and $l \in \{3, \dots, 3\binom{k}{2} + 2k - 2\} \cup \{3\binom{k}{2} + 2k\}$, there exist nonnegative integers a and b such that $a \leq \binom{k}{2}$, $b \leq k$ and $l = 3a + 2b$. Moreover, $a \geq b$ if $l \notin \{4, 7\}$ and $(k, l) \neq (3, 12)$.*

Proof If $l = 3\binom{k}{2} + 2k$, then $a = \binom{k}{2} \geq k = b$, hence for the rest of the proof we may suppose $l \leq 3\binom{k}{2} + 2k - 2$.

By the well-known Frobenius theorem (see, e.g., Theorem 2.1.1 of [7]), if p and q are two relatively prime natural numbers, then every integer l such that $l \geq (p-1)(q-1)$ can be represented in the form $l = p\alpha + q\beta$ for some nonnegative integers α and β . Thus, each $l \geq 2$ is of the form $l = 3\alpha + 2\beta$. If $\alpha \leq \binom{k}{2}$ and $\beta \leq k$, we set $a = \alpha$ and $b = \beta$.

Assume $\alpha > \binom{k}{2}$. Hence $\alpha = \binom{k}{2} + d$ with $d = 2r - \epsilon$, where r is a positive integer and $\epsilon \in \{0, 1\}$ depending on the parity of d . Thus

$$l = 3\left(\binom{k}{2} + 2r - \epsilon\right) + 2\beta = 3\left(\binom{k}{2} - \epsilon\right) + 2(3r + \beta).$$

Suppose $3r + \beta \geq k + 1$. Then

$$l \geq 3\binom{k}{2} - 3\epsilon + 2(k+1) \geq 3\binom{k}{2} + 2k + 2 - 3\epsilon \geq 3\binom{k}{2} + 2k - 1,$$

contrary to the assumption that $l \leq 3\binom{k}{2} + 2k - 2$. The lemma holds with $a = \binom{k}{2} - \epsilon$ and $b = 3r + \beta$.

Now assume that $\beta > k$. Hence $\beta = k + d$ with $d = 3r - \epsilon$, where r is a positive integer, $\epsilon \in \{0, 1, 2\}$ and $-\epsilon \equiv d \pmod{3}$. Thus $l = 3(\alpha + 2r) + 2(k - \epsilon)$. If $\alpha + 2r \geq \binom{k}{2} + 1$, then

$$l \geq 3\binom{k}{2} + 3 + 2k - 2\epsilon \geq 3\binom{k}{2} + 2k - 1.$$

Again we obtain a contradiction, so we can put $a = \alpha + 2r$ and $b = k - \epsilon$.

To end the proof, suppose $a < b$. If $b \geq 3$, then we can increase a by two, decreasing b by three, since $l = 3(a+2) + 2(b-3)$. We can continue this procedure until $a \geq b$, or $b \leq 2$. In the latter case $a \leq 1$ and $b = 2$. It follows that $l = 4$ if $a = 0$, or $l = 7$ if $a = 1$. In the first case, we need $a \leq \binom{k}{2}$. Since $a < b \leq k$ before the last round of the

procedure, the final value of a is bounded from above by $k + 1$. It is easy to see that $k + 1 \leq \binom{k}{2}$ if $k \leq 4$; thus l is expressed as necessary in that case. The same is true if $k = 3$ and $a \leq 3 = \binom{3}{2}$. If $k = 3$ and $a = k + 1 = 4$, then $l = 3a + 2b \geq 12$ is an even number satisfying $l \leq 3\binom{3}{2} + 2 \cdot 3 - 2 = 13$; however, for the pair $(k, l) = (3, 12)$ the inequality is not required. $\square$

Put $L_3 = \{2, 3, 5, 7\}$ and $L_k = \{2, \dots, \frac{1}{2}k^2(k-1)\} \setminus \{\frac{1}{2}k^2(k-1) - 1\}$ for $k \geq 4$.

Theorem 1 *Let $k \geq 3, n \geq 2$ be integers. Then the multigraph M_k contains a good Eulerian subgraph of size n if and only if $n \in L_k$.*

Proof The multigraph M_k is Eulerian, since the degree of any vertex equals $k(k-1)$, an even number. The deletion of one edge gives two vertices of odd degree, therefore M_k does not contain an Eulerian subgraph of size $\frac{1}{2}k^2(k-1) - 1$. In particular, M_3 does not contain an Eulerian subgraph of size eight. Clearly, M_3 does not contain a good Eulerian subgraph of size four, because any closed trail of size four in M_3 is of the form $(uvuwu)$, where (vuw) is passed twice. Also, each closed trail of size six in M_3 is either of the form $(uvwuvwu)$ or $(uvwvwuwu)$, and in both cases (uvw) appears twice, therefore there is no good Eulerian subgraph of size six in M_3 .

Let $\{u_1, \dots, u_k\}$ be the set of vertices of M_k . We prove the existence of good Eulerian subgraphs by induction on k .

For $k = 3$, the closed trail $F_3 = (u_1u_2u_1u_2u_3u_2u_3u_1u_3u_1)$ is a good Euler tour of M_3 of size nine (see Fig. 3, where a colour i corresponds to a vertex u_i). Finding good Eulerian subgraphs of M_3 of sizes two, three, five and seven is an easy exercise left to the reader.

Assume that the claim is true for M_k with $k \geq 3$. Observe that to obtain M_{k+1} from M_k , we add a vertex u_{k+1} , next we add one new edge between u_i and u_j , for every $i = 1, \dots, k-1, j = i+1, \dots, k$, and finally $k+1$ edges connecting u_{k+1} and u_i , for each $i = 1, \dots, k$. Thus, the number of new edges equals

$$n_{k+1} - n_k = \binom{k}{2} + k(k+1) = 3\binom{k}{2} + 2k.$$

Let $F_k = (v_1v_2 \dots v_{n_k}v_1)$ be a good Euler tour of M_k . First, we construct a closed trail $F_{k,l}$ of length $n_k + l$ for $l \in \{1, 2, 4, 7\}$. For $l = 1$ we may take

Fig. 3 A multigraph M_3 of three colours

$F_{k,1} = (v_1 u_{k+1} v_2 \dots v_{n_k} v_1)$. It is easy to see that there are $i, j \in \{1, \dots, n_k\}$ such that $v_i \neq v_j$ and $\{v_{i-1}, v_i\} \neq \{v_{j-1}, v_j\}$ (indices of v are taken modulo n_k). Let $F_{k,2}$ be the result of applying $U(i, k+1)$ to F_k . Then apply $U(j, k+1)$ and $TU(j, k+1)$ to $F_{k,2}$ to obtain $F_{k,4}$ and $F_{k,7}$, respectively. One can easily check that $F_{k,l}$ induces a good Eulerian subgraph of M_{k+1} of size $n_k + l$ for every $l \in \{1, 2, 4, 7\}$.

Now let $S_3 = \{3, \dots, 15\} \setminus \{4, 7, 12, 14\}$ and $S_k = \{3, \dots, 3\binom{k}{2} + 2k\} \setminus \{4, 7, 3\binom{k}{2} + 2k - 1\}$ for $k \geq 4$. We are going to construct a good Eulerian subgraph of M_{k+1} of size $n_k + s$ for every $s \in S_k$. By Lemma 1, each such s can be expressed as $s = 3a + 2b$ for some integers a, b with $0 \leq a \leq \binom{k}{2}$, $0 \leq b \leq k$ and $a \leq b$.

For every pair (i, j) of distinct elements of the set $\{1, \dots, k\}$ we choose $t(i, j) \in \{1, \dots, n_k\}$ satisfying $t(j, i) = t(i, j)$ and $\{v_{t(i,j)-1}, v_{t(i,j)}\} = \{u_i, u_j\}$. The choice can be made in such a way that $v_{t(i-1,i)-1} = u_{i-1}$ and $v_{t(i-1,i)} = u_i$, $i = 1, \dots, k$ (indices of u are taken modulo k). To see that it suffices to find a choice for $k = 3$ and then to realise how F_{k+1} , which corresponds to parameters $a = \binom{k}{2}$ and $b = k$, is inductively constructed from F_k .

First let $F_k^0 := F_k$. Further, to obtain F_k^i , apply to F_k^{i-1} the operation $TU(i, k+1)$ for $i = 1, \dots, b$, and the operation $T(i, k+1)$ for $i = b+1, \dots, a$. The construction is performed in such a way that it adds to F_k for each 2-element subset $\{i, j\}$ of the set $\{1, \dots, k\}$ at most one of the subtrails $(u_i u_j)$, $(u_j u_i)$ and for each $i \in \{1, \dots, k\}$ at most $k+1$ of subtrails $(u_i u_{k+1})$, $(u_{k+1} u_i)$. Therefore, F_k^a induces a good Eulerian subgraph of M_{k+1} of size $n_k + 5b + 3(a-b) = n_k + 3a + 2b = l$. To finish the proof it is sufficient to consider the following good Euler tour of a submultigraph of M_4 (with vertex set $\{1, 2, 3, 4\}$) of size $n_3 + 12 = 21$: (1212424313141423234341). $\square$

Recall that $\gamma(C_n)$ is a lower bound for the number of colours in each edge-distinguishing colouring of the cycle C_n . An immediate consequence of the above theorem is the following Vizing-type result.

Theorem 2

$$\text{edi}(C_n) = \begin{cases} \gamma(C_n) + 1, & \text{if } n = \frac{1}{2}k^2(k-1) - 1 \text{ for some } k, \text{ or } n \in \{4, 6\} \\ \gamma(C_n), & \text{otherwise.} \end{cases}$$

$\square$

Problem 1 Determine the edge-distinguishing index for the disjoint union of cycles.

This problem seems to be difficult since the solution of the analogous problem for the vertex-distinguishing index occurred to be really non-trivial, as Balister's proof [1] is more than thirty pages long. Observe also that $\text{edi}(C_v \cup C_3) = \text{edi}(C_v)$ for any $v \geq 4$, because a triangle being the closed neighbourhood of any edge of C_3 , is not isomorphic to the path P_4 being the closed neighbourhood of each edge of C_v . Therefore our problem is most interesting for graphs being disjoint unions of cycles of lengths greater than three. However, even with this assumption, we have $\text{edi}(C_4 \cup C_5) \geq \text{edi}(C_4) = 4$, while $\text{edi}(C_9) = 3$. We do not know if this exception is unique.

3 Paths

In a path P_n of order $n \geq 3$, the closed neighbourhood for $n - 3$ edges is a path of length three, and for two hanging edges it is a path of length two. Two colours are necessary to distinguish the two hanging edges in the path. Hence,

$$\text{edi}(P_n) \geq \gamma(P_n) := \min\{k \mid \frac{1}{2}k^2(k-1) \geq n-3\} = \gamma(C_{n-3}).$$

Obviously, $\text{edi}(P_2) = 1$, $\text{edi}(P_3) = 2 = \gamma(P_3)$, and $\text{edi}(P_4) = 3$ while $\gamma(P_4) = 2$.

For cycles, there were infinitely many exceptions for the equality $\text{edi}(C_n) = \gamma(C_n)$. Surprisingly, P_4 occurs to be the only such exception for paths.

Theorem 3 $\text{edi}(P_n) = \gamma(P_n)$ if $n \geq 5$.

Proof Any proper colouring of P_5 that uses two colours is edge-distinguishing, hence $\text{edi}(P_5) = 2 = \gamma(P_5)$.

Assume $n \geq 6$. Observe first that $\text{edi}(P_n) \leq \text{edi}(C_{n-1})$. To see this, consider an edge-distinguishing colouring of C_{n-1} with $\text{edi}(C_{n-1})$ colours and disconnect the cycle C_{n-1} in an arbitrary vertex v (so that the two edges of C_{n-1} incident to v become hanging) to obtain an edge-distinguishing colouring of P_n . Indeed, the hanging edges of P_n have distinct colours since c is proper, and colourings of the closed neighbourhoods of all other edges remain unchanged. Thus, we have $\gamma(C_{n-3}) = \gamma(P_n) \leq \text{edi}(P_n) \leq \text{edi}(C_{n-1})$. Obviously, $0 \leq \gamma(C_{n-1}) - \gamma(C_{n-3}) \leq 1$.

Consequently, if $\text{edi}(C_{n-1}) = \gamma(C_{n-1})$ and $\gamma(P_n) = \gamma(C_{n-1})$, then $\text{edi}(P_n) = \gamma(P_n)$. Thus, it suffices to consider two cases, when $\text{edi}(C_{n-1}) = \gamma(C_{n-1}) + 1$, or $\gamma(P_n) = \gamma(C_{n-1}) - 1$.

First suppose that $\text{edi}(C_{n-1}) = \gamma(C_{n-1}) + 1$. By Theorem 2, we have $n - 1 = \frac{1}{2}k^2(k-1) - 1$ for some $k \geq 3$. Hence $\text{edi}(C_{n-2}) = k$, and taking an edge-distinguishing colouring of C_{n-2} with k colours, we can choose an edge e of C_{n-2} with distinct colours of adjacent edges, and replace e by two hanging edges (that receive the colour of the edge e) in an obvious way to obtain an edge-distinguishing colouring of a path P_n .

Now, let $\gamma(P_n) = \gamma(C_{n-1}) - 1$. It follows that there exists a positive integer k such that $n = \frac{1}{2}k^2(k-1) + s$ with $s \in \{2, 3\}$. Take an edge-distinguishing colouring c with k colours of a cycle $(v_1 \dots v_l v_1)$ of length $l = \frac{1}{2}k^2(k-1)$. Since $\text{edi}(C_l) = k \geq 3$, there exist three consecutive edges, without loss of generality $v_l v_1$, $v_1 v_2$, and $v_2 v_3$, that are coloured with three distinct colours. Pick s distinct vertices $v'_1 \dots v'_s$ out of $V(C_l)$ and consider the path $(v_1 \dots v_l v'_1 \dots v'_s)$ of order $n = l + s$, coloured so that $c'(v_i v_{i+1}) = c(v_i v_{i+1})$ for $i = 1, \dots, l-1$, $c'(v_l v'_1) = c(v_l v_1)$ and $c'(v'_i v'_{i+1}) = c(v_i v_{i+1})$ for $i = 1, \dots, s$. It is easy to see that c' is an edge-distinguishing colouring of P_n . $\square$

We can also formulate the problem for disjoint union of paths.

Problem 2 Determine the edge-distinguishing index for the disjoint union of paths.

A general solution does not seem to be simple, as the following examples show. By Theorem 3, $\text{edi}(P_{13}) = 4$, while for $G = P_5 \cup P_8$ we have $\text{edi}(G) = 3$, since the consecutive edges of P_5 can be coloured with 1212, and those of P_8 with 3131231. On the other hand, if $p = \binom{r}{2}$ for an integer $r \geq 6$, then $\text{edi}(pP_3) = r$, while clearly $\text{edi}(P_{3p}) < r$.

4 Complete Graphs

The closed neighbourhood of an edge e of a complete graph K_n , with $n \geq 3$, is a union of $n - 2$ triangles with e as their common edge. As we already noticed, $\text{edi}(K_3) = 3$, since $K_3 = C_3$. It is easy to verify that $\text{edi}(K_4) = 5$, while $\chi'(K_4) = 3$. It occurs that this is a unique exception for the equality $\text{edi}(K_n) = \chi'(K_n)$.

Theorem 4 *The edge-distinguishing index of a complete graph K_n of order $n \neq 4$ equals its chromatic index, i.e.,*

$$\text{edi}(K_n) = \begin{cases} n - 1, & \text{if } n \text{ is even,} \\ n, & \text{if } n \text{ is odd,} \\ 5, & \text{if } n = 4. \end{cases}$$

Proof Let $n \geq 6$ be any even number. To make the proof shorter and more readable, we shall first prove the claim for the complete graph K_{n-1} of odd order.

Let c be the standard colouring of edges of K_{n-1} with $n - 1$ colours. Namely, let $V(K_{n-1})$ consist of consecutive vertices $v_1, \dots, v_{n-1}$ of a regular $(n - 1)$ -gon R_{n-1} in a plane, colour each side with a distinct colour from the set $C = \{1, \dots, n - 1\}$, and colour each other edge of K_{n-1} , which is a chord of the $(n - 1)$ -gon, with a colour of the (unique) parallel side. Thus, any monochromatic class is a set of parallel edges. More precisely, define $c(v_i v_j) = l$ if and only if $i + j \equiv 2l \pmod{n - 1}$.

In such a colouring, there do not exist two triangles with the same set of colours of edges. To see this, observe that two triangles would have the same set of colours if and only if their edges were pairwise parallel. It is easily seen that a regular $(n - 1)$ -gon together with all chords does not contain two such triangles, when $n - 1$ is odd. Consequently, this colouring is edge-distinguishing.

For the complete graph K_n of even order, we also use the standard colouring c of its edges, as follows. At first, we colour the edges of a complete subgraph K_{n-1} in the way described above. In this colouring, for each vertex $v_i \in V(K_{n-1})$, the colour i of the side opposite to v_i on the $(n - 1)$ -gon is the unique colour missing within the edges incident to it. We put $c(v_i v_n) = i$ for $i = 1, \dots, n - 1$.

We shall prove that if $j, k \in \{1, \dots, n - 1\}$, $j \neq k$, then the edges $e = v_j v_{2k-j}$ (with indices taken modulo $n - 1$) and $e' = v_k v_n$, both coloured with k , are distinguished by c . Suppose they are not. Then for the triangle T with $V(T) = \{v_j, v_k, v_{2k-j}\}$ contained in $N[e]$ there is a triangle T' with $V(T') \supset \{v_k, v_n\}$ (contained in $N[e']$) having the same set of colours as T . Since c is proper, T' must share an edge with T .

If $v_j v_k \in E(T')$, then $j = c(v_j v_n) = c(v_k v_{2k-j})$, $k + (2k - j) \equiv 2j \pmod{n-1}$ and $3(k - j) \equiv 0 \pmod{n-1}$. If $v_{2k-j} v_k \in E(T')$, then $l = c(v_{2k-j} v_n) = c(v_j v_k)$, and so from $l \equiv 2k - j \pmod{n-1}$ and $j + k \equiv 2l \pmod{n-1}$ we obtain again $3(k - j) \equiv 0 \pmod{n-1}$.

Now, if $i \in \{1, \dots, n-1\} \setminus \{j, k\}$ and $i \not\equiv 2k - j \pmod{n-1}$, then the triangle T_i with $V(T_i) = \{v_i, v_j, v_{2k-j}\}$ is also contained in $N[e]$. Since c does not distinguish e from e' , there is $p \in \{1, \dots, n-1\} \setminus \{k\}$ such that the triangle T'_i with $V(T'_i) = \{v_p, v_k, v_n\}$ (contained in $N[e']$) has the same set of colours as T_i .

If $p = c(v_p v_n) = c(v_i v_j)$ and $l = c(v_p v_k) = c(v_i v_{2k-j})$, then $i + j \equiv 2p \pmod{n-1}$ and $p + k \equiv i + (2k - j) \equiv 2l \pmod{n-1}$, which, using the congruence $3(k - j) \equiv 0 \pmod{n-1}$, leads to $i \equiv k \pmod{n-1}$ and $i = k$, a contradiction.

If $p = c(v_p v_n) = c(v_i v_{2k-j})$ and $l = c(v_p v_k) = c(v_i v_j)$, then $i + (2k - j) \equiv 2p \pmod{n-1}$ and $p + k \equiv i + j \equiv 2l \pmod{n-1}$, which yields again $i \equiv k \pmod{n-1}$ and $i = k$ contradicting our assumptions. $\square$

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