A Note on Counting Homomorphisms of Paths

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Abstract We obtain two identities and an explicit formula for the number of homomorphisms of a finite path into a finite path. For the number of endomorphisms of a finite path these give over-count and under-count identities yielding the closed-form formulae of Myers. We also derive finite Laurent series as generating functions which count homomorphisms of a finite path into any path, finite or infinite.

Keywords Path · Homomorphism · Generating function · Catalan number · Endomorphism

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1 Introduction

Path homomorphisms appear in many contexts in combinatorics, semigroup theory and the theory of stochastic processes. In this note we count homomorphisms of a finite path into any path. Our formulae for the number of endomorphisms of a finite path over-count or under-count by a sum involving weighted central binomial coefficients; using an identity recently proved by Lin and Zeng [3], we note that each yields the elegant closed-form formulae of Myers [4]. For homomorphisms of a finite path into a finite path we use a result of Sato and Cong [5] to obtain an explicit enumeration formula.

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We also obtain relevant generating function expressions that yield, as a particular case, the recent result obtained by Arworn and Wojtylak [2] for counting finite path homomorphisms.

2 Notation and Terminology

A graph X has vertex set $V(X)$ and edge set $E(X)$. Given two graphs G and H , a mapping $f : V(G) \rightarrow V(H)$ is a *homomorphism* from G to H if $\{f(u), f(v)\} \in E(H)$ whenever $\{u, v\} \in E(G)$. We write $f : G \rightarrow H$ and denote the set of all homomorphisms from G to H by $\text{Hom}(G, H)$.

Let the two-way infinite path P_∞ be the graph with vertices all integers i and edges all pairs $\{i, i + 1\}$. The one-way infinite path P_∞^+ and the finite path P_n of length n are defined by restricting the vertices to nonnegative integers or to nonnegative integers less than $n + 1$, respectively. (We follow the convention that P_n has n edges and $n + 1$ vertices.) In this paper G will always be the finite path P_n , and H will be a finite or infinite path.

Let $a, b \in V(H)$, possibly with $a = b$. Then we write

$$\text{Hom}(P_n, H; a, b) = \{f \in \text{Hom}(P_n, H) : f(0) = a, f(n) = b\},$$

and

$$\text{Hom}(P_n, H; a) = \bigcup_{b \in V(H)} \text{Hom}(P_n, H; a, b).$$

The cardinalities of these families are denoted by

$$\pi(P_n, H; a, b) = |\text{Hom}(P_n, H; a, b)|, \quad \pi(P_n, H; a) = |\text{Hom}(P_n, H; a)|.$$

Of course,

$$\pi(P_n, H; a) = \sum_{b \in V(H)} \pi(P_n, H; a, b).$$

We also define the potentially infinite cardinality

$$\pi(P_n, H) = \sum_{a \in V(H)} \pi(P_n, H; a).$$

3 Homomorphisms of P_n into P_r

For $r \geq 1$, consider homomorphisms $f : P_n \rightarrow P_r$. Evidently

$$\pi(P_n, P_r; a, 0) = \pi(P_{n-1}, P_r; a, 1) \text{ and } \pi(P_n, P_r; a, r) = \pi(P_{n-1}, P_r; a, r-1) \quad (1)$$

and if $0 < b < r$ then

$$\pi(P_n, P_r; a, b) = \pi(P_{n-1}, P_r; a, b-1) + \pi(P_{n-1}, P_r; a, b+1). \quad (2)$$

From (1) and (2), summation of $\pi(P_n, P_r; a, b)$ over $0 \leq b \leq r$ gives

$$\pi(P_n, P_r; a) = 2\pi(P_{n-1}, P_r; a) - \pi(P_{n-1}, P_r; a, 0) - \pi(P_{n-1}, P_r; a, r), \quad (3)$$

and then

$$\pi(P_n, P_r) = 2\pi(P_{n-1}, P_r) - \sum_{a=0}^r \pi(P_{n-1}, P_r; a, 0) - \sum_{a=0}^r \pi(P_{n-1}, P_r; a, r). \quad (4)$$

By symmetry,

$$\pi(P_{n-1}, P_r; a, 0) = \pi(P_{n-1}, P_r; r-a, r) = \pi(P_{n-1}, P_r; 0, a).$$

Hence (4) yields

$$\pi(P_n, P_r) = 2\pi(P_{n-1}, P_r) - 2\pi(P_{n-1}, P_r; 0), \quad (5)$$

and iteration of (5) gives us

$$\pi(P_n, P_r) = 2^n \pi(P_0, P_r) - \sum_{k=1}^n 2^k \pi(P_{n-k}, P_r; 0), \quad \pi(P_0, P_r) = r+1.$$

Hence we have the *over-count identity*:

Theorem 3.1 *The number of homomorphisms from P_n to P_r satisfies the identity*

$$\pi(P_n, P_r) = 2^n(r+1) - \sum_{k=1}^n 2^k \pi(P_{n-k}, P_r; 0).$$

□

Since $\pi(P_0, P_r; 0) = 1$, iterating (3) with $a = 0$ yields

$$\pi(P_n, P_r; 0) = 2^n - \sum_{i=1}^n 2^{i-1} \pi(P_{n-i}, P_r; 0, 0) - \sum_{i=1}^n 2^{i-1} \pi(P_{n-i}, P_r; 0, r). \quad (6)$$

With (6), the over-count identity (Theorem 3.1) yields

$$2^n(r+1) - \pi(P_n, P_r)$$

$$\begin{aligned}
&= \sum_{i=1}^n 2^i \left(2^{n-i} - \sum_{j=1}^{n-i} 2^{j-1} \pi(P_{n-i-j}, P_r; 0, 0) - \sum_{j=1}^{n-i} 2^{j-1} \pi(P_{n-i-j}, P_r; 0, r) \right) \\
&= 2^n n - \sum_{i=1}^n \sum_{j=1}^{n-i} 2^{i+j-1} (\pi(P_{n-i-j}, P_r; 0, 0) + \pi(P_{n-i-j}, P_r; 0, r)).
\end{aligned}$$

Writing $i + j = k$, we deduce the *under-count identity*:

Theorem 3.2 *The number of homomorphisms from P_n to P_r satisfies the identity*

$$\pi(P_n, P_r) = 2^n(r-n+1) + \sum_{k=2}^n 2^{k-1}(k-1) \left(\pi(P_{n-k}, P_r; 0, 0) + \pi(P_{n-k}, P_r; 0, r) \right).$$

□

Let us consider the over-count and under-count identities in the special case $r = n$, the case of endomorphisms $f : P_n \rightarrow P_n$. Catalan numbers (sequence A000108 in OEIS [6]) are given by

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

and play a key role in counting endomorphisms of P_n . For example

$$\pi(P_{2n}, P_{2n}; 0, 0) = C_n. \quad (7)$$

We shall need the following elementary identity for weighted sums of Catalan numbers.

Lemma 3.3 *For any positive integer n , let $s = \lfloor \frac{n-1}{2} \rfloor$ and $t = \lfloor \frac{n}{2} \rfloor$. Then*

$$2^{n-1} \sum_{k=0}^s \frac{1}{4^k} C_k = 2^n - \binom{n}{t}.$$

Proof The identity is easily verified for $n = 1, 2$. Suppose it holds for every positive integer up to some particular value of n . We now show that it also holds for $n + 2$, whence the claim follows by induction. By hypothesis,

$$\begin{aligned}
2^{n+1} \sum_{k=0}^{s+1} \frac{1}{4^k(k+1)} \binom{2k}{k} &= 4 \left[2^n - \binom{n}{t} \right] + \frac{2^{n+1}}{4^{s+1}(s+2)} \binom{2s+2}{s+1} \\
&= 2^{n+2} - D(n),
\end{aligned}$$

where

$$D(n) = 4 \binom{n}{t} - \frac{2^{n-2s-1}}{s+2} \binom{2s+2}{s+1}.$$

There are two cases. If $n = 2s + 1$ then $t = s$ and we have

$$D(n) = D(2s + 1) = 4 \binom{2s+1}{s} - \frac{1}{s+2} \binom{2s+2}{s+1} = \binom{2s+3}{s+1} = \binom{n+2}{t+1}.$$

If $n = 2s + 2$ then $t = s + 1$ and we have

$$D(n) = D(2s + 2) = 4 \binom{2s+2}{s+1} - \frac{2}{s+2} \binom{2s+2}{s+1} = \binom{2s+4}{s+2} = \binom{n+2}{t+1}.$$

Together these cases verify the induction. $\square$

When $r = n$ the second sum in (6) vanishes, because $\pi(P_m, P_n; 0, n) = 0$ when $0 \leq m < n$. Because of (7), the first sum becomes a weighted sum of Catalan numbers, since $\pi(P_m, P_n; 0, 0) = \pi(P_m, P_m; 0, 0)$ if $0 \leq m \leq n$, and this is zero when m is odd. With $s = \lfloor \frac{n-1}{2} \rfloor$ equation (6) yields

$$\pi(P_n, P_n; 0) = 2^n - \sum_{i=0}^{n-1} 2^{n-i-1} \pi(P_i, P_n; 0, 0) = 2^n - \sum_{k=0}^s 2^{n-2k-1} \frac{1}{k+1} \binom{2k}{k}. \quad (8)$$

Simplifying with Lemma 3.3 this becomes

$$\pi(P_n, P_n; 0) = \binom{n}{\lfloor n/2 \rfloor}. \quad (9)$$

This result was obtained by Arworn in [1]. With (9) and Theorem 3.1 we have

Theorem 3.4 *The number of endomorphisms of P_n is given by*

$$\pi(P_n, P_n) = 2^n(n+1) - \sum_{k=0}^{n-1} 2^{n-k} \binom{k}{\lfloor k/2 \rfloor}.$$

$\square$

Again the case $r = n$ of Theorem 3.2 yields the companion result:

Theorem 3.5 *The number of endomorphisms of P_n is given by*

$$\pi(P_n, P_n) = 2^n + \sum_{k=0}^s 2^{n-2k-1} \frac{n-2k-1}{k+1} \binom{2k}{k},$$

where $s = \lfloor \frac{n-1}{2} \rfloor$.

$\square$

The equivalence of the expressions for $\pi(P_n, P_n)$ in Theorems 3.4 and 3.5 implies the following identity for weighted sums of central binomial coefficients and Catalan numbers (with a “correction” applying on the right when n is odd):

Corollary 3.6 *For any integer $n \geq 2$, let $t = \lfloor \frac{n}{2} \rfloor$ and $\varepsilon = n - 2t$. Then*

$$2^n \sum_{k=0}^{t-1} \frac{1}{4^k} \binom{2k}{k} + n2^{n-1} \sum_{k=0}^{t-1} \frac{1}{4^k} C_k = n2^n - (n+1)\varepsilon C_t.$$

□

Applying Lemma 3.3 to Corollary 3.6, we have

$$2^{2t-1} \sum_{k=0}^{t-1} \frac{1}{4^k} \binom{2k}{k} = t \binom{2t}{t}. \quad (10)$$

Since $2 \binom{2k+1}{k} = \binom{2k+2}{k+1}$ this immediately implies

$$2^{2t-1} \sum_{k=0}^{t-1} \frac{1}{4^k} \binom{2k+1}{k} = (t+1) \binom{2t+1}{t} - 2^{2t}. \quad (11)$$

Using (10) and (11) to evaluate the sum in Theorem 3.4, we obtain

Theorem 3.7 *For any integer $n \geq 0$, let $t = \lfloor \frac{n}{2} \rfloor$ and $\varepsilon = n - 2t$. Then the number of endomorphisms of P_n is*

$$\pi(P_n, P_n) = 2^n(n+2) - (2n+1+\varepsilon) \binom{n}{t}.$$

□

This result is equivalent to the elegant closed-form results of Myers [4], but here is expressed as a single formula by incorporating the “correction” mediated by ε when n is odd. In their Lemma 7 and Proposition 8, Lin and Zeng [3] recently used induction to prove identities equivalent to (10) and (11), and thence deduced expressions for the number of endomorphisms of P_n equivalent to those of Myers.

More generally, an explicit formula for $\pi(P_n, P_r)$ can be obtained by combining Theorem 3.2 with Theorem 7 of Sato and Cong [5], though the latter goes beyond integer arithmetic. For even $n > 0$ we have

$$\pi(P_n, P_r; 0, 0) = \frac{4}{r+2} \sum_{k=1}^s (2 \cos k\theta)^n \sin^2(r+1)k\theta$$

where $s = \lfloor (r+1)/2 \rfloor$ and $\theta = \pi/(r+2)$. Also, for $n \equiv r \pmod{2}$,

$$\pi(P_n, P_r; 0, r) = \frac{4}{r+2} \sum_{k=1}^s (2 \cos k\theta)^n \sin k\theta \sin(r+1)k\theta.$$

In all other cases $\pi(P_n, P_r; 0, 0)$ and $\pi(P_n, P_r; 0, r)$ are zero. Hence

Theorem 3.8 *The number of homomorphisms from P_n to P_r can be expressed in the form*

$$\pi(P_n, P_r) = 2^{n-1}(2r - n + 1) + S_1(n, r) + S_2(n, r),$$

where, for $s = \lfloor \frac{r+1}{2} \rfloor$ and $\theta = \frac{\pi}{r+2}$, the sums S_1 and S_2 are

$$S_1(n, r) = \frac{2^{n+1}}{r+2} \sum_{0 < 2j < n-1} \sum_{1 \leq k \leq s} (n - 2j - 1)(\cos k\theta)^{2j} \sin^2(r+1)k\theta,$$

$$S_2(n, r) = \frac{2^{n+1}}{r+2} \sum_{0 \leq 2j < n-r-1} \sum_{1 \leq k \leq s} (n - r - 2j - 1)(\cos k\theta)^{r+2j} \sin k\theta \sin(r+1)k\theta,$$

for $r < n$, and $S_2(n, r) = 0$ otherwise. $\square$

4 Generating Functions with End Effects

The main objective of this section is to obtain a generating function to enumerate the homomorphisms $f: P_n \rightarrow P_r$. If P is any path, the possible presence of endpoints in P complicates the problem of determining the cardinality of subsets of $\text{Hom}(P_n, P)$. We shall use finite Laurent series to treat all cases. (Finite Laurent series have a much wider applicability as generating functions which enumerate graph homomorphisms, but here we confine ourselves to the case of paths.)

When there are no endpoints, then $P = P_\infty$ and $\pi(P_n, P_\infty; a, b)$ is the coefficient of x^b in the generating function

$$\Pi[P_n, P_\infty; a](x) = x^a(x^{-1} + x)^n = \frac{(1 + x^2)^n}{x^{n-a}}.$$

Hence

$$\pi(P_n, P_\infty; a) = \Pi[P_n, P_\infty; a](1) = 2^n.$$

When there is one endpoint, then $P = P_\infty^+$. We discuss this case in detail before the case $P = P_r$, since the influence of one endpoint on the generating function is simpler than the interactive effects of two endpoints. Clarifying the details for P_∞^+ will obviate the necessity to explain most of the intricacies which arise for P_r .

Evidently

$$\Pi[P_0, P_\infty^+; a](x) = x^a.$$

For $n \geq 0$ we have the coefficient identities

$$\pi(P_{n+1}, P_\infty^+; a, 0) = \pi(P_n, P_\infty^+; a, 1)$$

and

$$\pi(P_{n+1}, P_{\infty}^+; a, b) = \pi(P_n, P_{\infty}^+; a, b-1) + \pi(P_n, P_{\infty}^+; a, b+1)$$

for $b > 0$, so the generating function satisfies the recurrence relation

$$\Pi[P_{n+1}, P_{\infty}^+; a](x) = (x^{-1} + x)\Pi[P_n, P_{\infty}^+; a](x) - x^{-1}\pi(P_n, P_{\infty}^+; a, 0).$$

When $n \leq a$ the fact that P_{∞}^+ has an endpoint at 0 is inconsequential, so

$$\Pi[P_n, P_{\infty}^+; a](x) = \Pi[P_n, P_{\infty}; a](x) = x^a(x^{-1} + x)^n$$

for $0 \leq n \leq a$. But, in general, $\Pi[P_n, P_{\infty}; a](x)$ exceeds $\Pi[P_n, P_{\infty}^+; a](x)$ by a finite Laurent series which expresses the boundary effect of the endpoint of P_{∞}^+ ; for brevity we shall refer to the latter series as the *endpoint correction*. The endpoint correction is conveniently expressed in terms of the following basic family of finite Laurent series, indexed by integers $r \geq 0$:

$$\Lambda_0(x) = 1 \quad \text{and} \quad \Lambda_r(x) = x^{-r} + x^r \quad \text{when } r > 0.$$

These series have several useful properties, including

$$\Lambda_r(x)\Lambda_s(x) = \Lambda_{r+s}(x) + \Lambda_{r-s}(x) \quad \text{when } r > s > 0,$$

and

$$\Lambda_r^n(x) = \sum_{2k \leq n} \binom{n}{k} \Lambda_{r(n-2k)}(x) \quad \text{when } r > 0,$$

where the sum formally ranges over all integers k such that $2k \leq n$, but effectively the constraint $k \geq 0$ also applies since the binomial coefficients are 0 when $k < 0$.

Theorem 4.1 *Let $m = n - (a + 1)$ and $s = \lfloor \frac{m}{2} \rfloor$. Then, for all integers $n \geq 0$,*

$$\Pi[P_n, P_{\infty}^+; a](x) = x^a(x^{-1} + x)^n - x^{-1} \sum_{k \leq s} \binom{n}{k} \Lambda_{m-2k}(x).$$

Proof When $0 \leq n \leq a$ the endpoint correction is 0, since the sum is empty when $s < 0$, and this agrees with what we already know is the generating function in this range. In particular,

$$\Pi[P_a, P_{\infty}^+; a](x) = \Pi[P_a, P_{\infty}; a](x) = x^a(x^{-1} + x)^a = (1 + x^2)^a$$

shows that $\pi(P_a, P_{\infty}^+; a, 0) = 1$, and then the recurrence relation for the generating function shows that the endpoint correction is x^{-1} when $n = a + 1$, in agreement with

our claim. So now suppose the claim holds as far as some particular integer $n > a$. It follows that

$$\pi(P_n, P_\infty^+; a, 0) = \binom{n}{u} - \binom{n}{u-1},$$

where $u = \frac{1}{2}(n - a)$ and $n \equiv a \pmod{2}$; otherwise, $\pi(P_n, P_\infty^+; a, 0) = 0$.

There are two cases depending on the parity of n . When $n \equiv a \pmod{2}$ the recurrence relation shows that $\Pi[P_{n+1}, P_\infty^+; a](x)$ has the claimed form provided

$$\sum_{k \leq s'} \binom{n+1}{k} \Lambda_{m'-2k}(x) = (x^{-1} + x) \sum_{k \leq s} \binom{n}{k} \Lambda_{m-2k}(x) + \pi(P_n, P_\infty^+; a, 0)$$

holds, where $m' = n - a = m + 1 = 2u$ and $s' = \lfloor \frac{m'}{2} \rfloor = s + 1 = u$. Thus, we require

$$\sum_{k \leq u} \binom{n+1}{k} \Lambda_{2(u-k)}(x) = (x^{-1} + x) \sum_{k \leq u-1} \binom{n}{k} \Lambda_{2(u-k)-1}(x) + \left[\binom{n}{u} - \binom{n}{u-1} \right].$$

The right side rearranges to

$$\begin{aligned} & \binom{n}{0} \Lambda_{2u}(x) + \sum_{1 \leq k \leq u-1} \left[\binom{n}{k-1} + \binom{n}{k} \right] \Lambda_{2(u-k)}(x) \\ & + 2 \binom{n}{u-1} \Lambda_0(x) + \left[\binom{n}{u} - \binom{n}{u-1} \right]. \end{aligned}$$

With Pascal's identity and $\Lambda_0(x) = 1$, this reduces to the left side. The case when $n - a$ is odd is similar but slightly easier, so we omit the details. Thus, the claim follows by induction. $\square$

Corollary 4.2 *If $n \equiv a + b \pmod{2}$ and $k = \frac{1}{2}(n + b - a)$, then*

$$\pi(P_n, P_\infty^+; a, b) = \binom{n}{k} - \binom{n}{k-b-1} \text{ when } 0 \leq b < n - a,$$

$$\pi(P_n, P_\infty^+; a, b) = \binom{n}{k} \text{ when } b \geq n - a.$$

If $n \not\equiv a + b \pmod{2}$, then $\pi(P_n, P_\infty^+; a, b) = 0$.

$\square$

Theorem 4.1 can also be written in a form which emphasizes the relationship between $\Pi[P_n, P_\infty^+; a](x)$ and $\Pi[P_n, P_\infty; a](x)$, as follows:

Corollary 4.3 *Let $m = n - (a + 1)$. Then*

$$\begin{aligned}\Pi[P_n, P_\infty; a](x) &= x^a \sum_{2k \leq n} \binom{n}{k} \Lambda_{n-2k}(x), \\ \Pi[P_n, P_\infty; a](x) - \Pi[P_n, P_\infty^+; a](x) &= x^{-1} \sum_{2k \leq m} \binom{n}{k} \Lambda_{m-2k}(x).\end{aligned}$$

□

We have just shown that the generating function $\Pi[P_n, P_\infty^+; a](x)$ is a polynomial in x expressed as the difference of two finite Laurent series: the first is the generating function enumerating homomorphisms of P_n into P_∞ , and the second, the endpoint correction, cancels out the coefficients of negative powers of x and reduces the coefficients of small positive powers of x , namely those with exponent $b < n - a$.

Now consider generating functions for enumerating homomorphisms of P_n into the finite path P_r . To avoid degeneracy we require $r \geq 1$. Here the relevant generating function is $\Pi[P_n, P_r; a](x)$, and $\pi(P_n, P_r; a, b)$ is the coefficient of x^b . Evidently

$$\Pi[P_0, P_r; a](x) = x^a.$$

For $n \geq 0$ we have

$$\pi(P_{n+1}, P_r; a, 0) = \pi(P_n, P_r; a, 1), \quad \pi(P_{n+1}, P_r; a, r) = \pi(P_n, P_r; a, r - 1),$$

and for $0 < b < r$

$$\pi(P_{n+1}, P_r; a, b) = \pi(P_n, P_r; a, b - 1) + \pi(P_n, P_r; a, b + 1),$$

so the generating function satisfies the recurrence relation

$$\begin{aligned}\Pi[P_{n+1}, P_r; a](x) &= (x^{-1} + x)\Pi[P_n, P_r; a](x) \\ &\quad - x^{-1}\pi(P_n, P_r; a, 0) - x^{r+1}\pi(P_n, P_r; a, r).\end{aligned}$$

When $n \leq \min\{a, r - a\}$, the fact that P_r has endpoints at 0 and r does not come into play, so

$$\Pi[P_n, P_r; a](x) = \Pi[P_n, P_\infty; a](x) = x^a(x^{-1} + x)^n \quad \text{for } 0 \leq n \leq \min\{a, r - a\}.$$

In general, $\Pi[P_n, P_r; a](x)$ falls short of $\Pi[P_n, P_\infty; a](x)$ by a finite Laurent series which expresses the effects of the two endpoints of P_r ; we call it the *double endpoint correction*.

Theorem 4.4 *For any integer $s \geq 0$ let*

$$L(s) = n - (a + 1) - s(r + 2) \quad \text{and} \quad R(s) = n - (r - a + 1) - s(r + 2).$$

Then for any integer $n \geq 0$ we have

$$\begin{aligned} \Pi[P_n, P_r; a](x) &= x^a(x^{-1} + x)^n \\ &- x^{-1} \sum_{s \geq 0} \sum_{2k \leq L(2s)} \binom{n}{k} \Lambda_{L(2s)-2k}(x) - x^{r+1} \sum_{s \geq 0} \sum_{2k \leq R(2s)} \binom{n}{k} \Lambda_{R(2s)-2k}(x) \\ &+ x^{-1} \sum_{s \geq 0} \sum_{2k \leq R(2s+1)} \binom{n}{k} \Lambda_{R(2s+1)-2k}(x) + x^{r+1} \sum_{s \geq 0} \sum_{2k \leq L(2s+1)} \binom{n}{k} \Lambda_{L(2s+1)-2k}(x). \end{aligned}$$

Proof As in the proof of Theorem 4.1, a routine but more complicated induction on n establishes the claim. The salient points for the argument are as follows. In the recurrence relation for the generating function, $\pi(P_n, P_r; a, r) = 0$ unless $n \geq r - a$ and $n \equiv a + r \pmod{2}$. For the inductive step, note that $L(s)$ and $R(s)$ both increase by 1 when n is replaced by $n + 1$. Then the multiplicative identities noted for $\Lambda_r(x)$ in the case of the one-way infinite path, together with Pascal's identity suffice to complete the argument, though the manipulation of the double sums is intricate. We omit the details. $\square$

We can also deduce an explicit formula for the coefficient $\pi(P_n, P_r; a, b)$. The same result obtained by different techniques appears as Theorem 3.1 of Arworn and Wojtylak [2].

Corollary 4.5 *Let $n \equiv a + b \pmod{2}$ and $k = \frac{1}{2}(n - a + b)$. Then*

$$\begin{aligned} \pi(P_n, P_r; a, b) &= \binom{n}{k} - \sum_{s \geq 0} \binom{n}{k - A(s)} - \sum_{s \geq 0} \binom{n}{k - B(s)} + \sum_{s \geq 0} \binom{n}{k - C(s)} \\ &+ \sum_{s \geq 0} \binom{n}{k - D(s)}, \end{aligned}$$

where

$$\begin{aligned} A(s) &= s(r + 2) + b + 1, \quad B(s) = (s + 1)(r + 2) - a - 1, \\ C(s) &= (s + 1)(r + 2) - a + b, \quad D(s) = (s + 1)(r + 2). \end{aligned}$$

If $n \not\equiv a + b \pmod{2}$, then $\pi(P_n, P_r; a, b) = 0$. $\square$

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