A Note on Turán Numbers for Even Wheels

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Abstract The Turán number $ex(n, G)$ is the maximum number of edges in any n -vertex graph that does not contain a subgraph isomorphic to G . We consider a very special case of the Simonovits's theorem (Simonovits in: Theory of graphs, Academic Press, New York, 1968) which determine an asymptotic result for Turán numbers for graphs with some properties. In the paper we present a more precise result for even wheels. We provide the exact value for Turán number $ex(n, W_{2k})$ for $n \geq 6k - 10$ and $k \geq 3$. In addition, we show that $ex(n, W_6) = \lfloor \frac{n^2}{3} \rfloor$ for all $n \geq 6$. These numbers can be useful to calculate some Ramsey numbers.

Keywords Turán problem · Cycles

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1 Introduction

In this paper, all graphs considered are undirected, finite and contain neither loops nor multiple edges. Let G be such a graph. The vertex set of G is denoted by $V(G)$, the edge set of G by $E(G)$, and the number of edges in G by $e(G)$. Let $d(v)$ be the degree of vertex v , $\Delta(G)$ be the maximum degree of vertices of G and $\chi(G)$ be the chromatic number of graph G . C_m denotes the cycle of length m . A *wheel* W_n is a graph on n vertices obtained from a C_{n-1} by adding one vertex w and making w adjacent to all vertices of the C_{n-1} . Next notation follow that of Simonovits [4]. Let us consider the following graph: n vertices are divided into d classes each of which contains almost

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the same number of vertices: they contain $\lfloor \frac{n}{d} \rfloor$ or $\lfloor \frac{n}{d} \rfloor + 1$ vertices. Join two vertices by an edge if and only if they belong to different classes. The graph obtained thus is denoted by $T^{n,d}$.

The Turán number $ex(n, G)$ is the maximum number of edges in any n -vertex graph that does not contain a subgraph isomorphic to G . A graph on n vertices is said to be *extremal with respect to G* if it does not contain a subgraph isomorphic to G and has exactly $ex(n, G)$ edges.

The aim of this paper is to determine the Turán numbers of wheels W_n for even n .

2 Known Results

First, we recall a result, which was proved by Mantel in 1907.

Theorem 1 (Mantel, [3]) *The maximum number of edges in an n -vertex triangle-free graph is $\lfloor \frac{n^2}{4} \rfloor$*

By Theorem 1 and since $W_3 = C_3$, it is easy to have the property that for all integers $n, n \geq 3$, $ex(n, W_3) = \lfloor \frac{n^2}{4} \rfloor$.

The famous Turán's theorem may be stated as follows.

Theorem 2 (Turán, [5]) *Let G be any subgraph of K_n such that G is K_{r+1} -free. Then the number of edges in G is $e(G) = e(T^{n,r}) = \lfloor \frac{(r-1)n^2}{2r} \rfloor$. In particular, $ex(n, K_4) = \lfloor \frac{n^2}{3} \rfloor$.*

As a special case, for $r = 2$, one obtains Mantel's theorem. Since $W_4 = K_4$, we obtain that for all integers $n, n \geq 3$, $ex(n, W_4) = \lfloor \frac{n^2}{3} \rfloor$.

The remaining part of this paper is connected with the result of M. Simonovits.

Theorem 3 (Simonovits, [4]) *Let G be a given graph such that $\chi(G) \geq d + 1$ but there is an edge e in it such that $\chi(G - \{e\}) = d$ (G is color critical). Then there exists an n_0 such that if $n > n_0$ then $T^{n,d}$ is the only extremal graph with respect to G .*

One can observe that the wheel W_{2k} is color critical, so it has Turán number equal to the Turán number of K_4 for sufficiently large n . In this paper we determine how large n is.

The following theorems and corollaries will be used to prove the main results of this paper.

Theorem 4 (Woodall, [6]) *Let G be a graph on n ($n \geq 3$) vertices with more than $\frac{n^2}{4}$ edges. Then G contains a cycle of length k for each k ($3 \leq k \leq \lfloor \frac{1}{2}(n + 3) \rfloor$).*

Now, it is a simple observation that if graph G has more than $\frac{n^2}{4}$ edges and $2k - 1 \leq \lfloor \frac{1}{2}(n + 3) \rfloor$, then G contains a cycle of length $2k - 1$. Since $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ contains no odd cycles, the maximum number of edges in a graph containing no C_{2k-1} (the Turán number for C_{2k-1}) is $\lfloor \frac{n^2}{4} \rfloor$ for sufficiently large n . By these considerations we easily obtain the following

Corollary 5 For all integers $k, n, n \geq 3, 2k - 1 \leq \lfloor \frac{1}{2}(n + 3) \rfloor$

$$ex(n, C_{2k-1}) = \lfloor \frac{n^2}{4} \rfloor$$

Moreover, another formula is known for $ex(n, C_{2k-1})$. We give this, as the following theorem which can be found in the appendix IV of [1].

Theorem 6 ([1]) Assume that $2k - 1 \geq \frac{1}{2}(n + 3)$. Then

$$ex(n, C_{2k-1}) = \binom{n - (2k - 1) + 2}{2} + \binom{(2k - 1) - 1}{2}$$

One can see that the above theorem holds for $n \leq 4k - 5$. Let us consider the situation where $n = 4k - 6$. By using Theorem 6 we obtain the property that $ex(4k - 6, C_{2k-1}) = \binom{2k-3}{2} + \binom{2k-2}{2} = (2k - 3)^2 = \frac{(4k-6)^2}{4} = \frac{n^2}{4}$. Since Corollary 5 holds for $n \geq 4k - 5$, then one can observe that for all $n \geq 4k - 6$ we have the following

Theorem 7 For all integers $k, n, n \geq 3, n \geq 4k - 6$

$$ex(n, C_{2k-1}) = \lfloor \frac{n^2}{4} \rfloor$$

3 Main Result

Now, we prove the following main result

Theorem 8 For all $k \geq 3$ and $n \geq 6k - 10$,

$$ex(n, W_{2k}) = \lfloor \frac{n^2}{3} \rfloor$$

Proof Let G be a graph which does not contain a W_{2k} and assume that G has $n \geq 6k - 10$ vertices and $e(G) = \lfloor \frac{n^2}{3} \rfloor + 1$ edges. One can observe that $\Delta(G) \geq n - \lfloor \frac{n}{3} \rfloor$ since $\frac{(n - \lfloor \frac{n}{3} \rfloor - 1)n}{2} < \lfloor \frac{n^2}{3} \rfloor + 1$.

Suppose that there is a vertex $v \in V(G)$ such that $d(v) = n - 1$. Then $G - v$ has exactly $\lfloor \frac{n^2}{3} \rfloor + 1 - (n - 1)$ edges. By Theorem 7 we know that for all $n \geq 4k - 6$ we have that $ex(n, C_{2k-1}) = \lfloor \frac{n^2}{4} \rfloor$ and for all $k \geq 3$ we have $6k - 11 > 4k - 6$. We immediately obtain a contradiction with $ex(n - 1, C_{2k-1}) = \lfloor \frac{(n-1)^2}{4} \rfloor$ since $\lfloor \frac{n^2}{3} \rfloor + 1 - (n - 1) > \lfloor \frac{(n-1)^2}{4} \rfloor$ for all $n \geq 6$. In general case, let us consider the situation when $\Delta(G) = n - p$ where $1 \leq p \leq \lfloor \frac{n}{3} \rfloor$ and there exists a vertex v of degree $n - p$ in graph G . By Theorem 7 we know that for all $n \geq 4k - 6$ we have that $ex(n, C_{2k-1}) = \lfloor \frac{n^2}{4} \rfloor$ and for all $n \geq 6k - 10$ we have that $\Delta(G) \geq n - \lfloor \frac{n}{3} \rfloor \geq 4k - 6$. Let us consider a subgraph G' with $n - p$ vertices obtained by removing a vertex v and $p - 1$ vertices not adjacent

to v from the set of vertices of G . In this situation, the minimum number of edges in subgraph G' is $e(G') \geq e(G) - p(n - p) = \lfloor \frac{n^2}{3} \rfloor + 1 - p(n - p)$. We prove that $\lfloor \frac{n^2}{3} \rfloor + 1 - p(n - p) > \lfloor \frac{(n-p)^2}{4} \rfloor = ex(n - p, C_{2k-1})$. One can calculate that formula $p(n - p) + \lfloor \frac{(n-p)^2}{4} \rfloor$ achieves the maximum value for $p = \lfloor \frac{n}{3} \rfloor$. The maximum value of this formula is equal to $\lfloor \frac{n^2}{3} \rfloor$. We obtain that $p(n - p) + \lfloor \frac{(n-p)^2}{4} \rfloor \leq \lfloor \frac{n^2}{3} \rfloor$, so we have a contradiction with our assumption about the number of edges in graph G and $ex(n, W_{2k}) \leq \lfloor \frac{n^2}{3} \rfloor$.

Now, we show a graph G on n vertices and $\lfloor \frac{n^2}{3} \rfloor$ edges which does not contain a W_{2k} . Let graph G be $T^{n,3}$. One can observe that G is tripartite graph and does not contain a wheel of order $2k$. The number of edges of this graph was calculated above and it is equal to $\lfloor \frac{n^2}{3} \rfloor$, so $ex(n, W_{2k}) \geq \lfloor \frac{n^2}{3} \rfloor$ and the proof is complete.

All values of $ex(n, C_5)$, where $5 \leq n \leq 23$, are determined in [7], so by Theorem 7 and [7] we obtain the following

Corollary 9 *Let $n \geq 5$. Then*

$$ex(n, C_5) = \begin{cases} 7 & \text{if } n = 5, \\ \lfloor \frac{n^2}{4} \rfloor & \text{otherwise.} \end{cases}$$

Now, we prove the following result.

Theorem 10 *For all $n \geq 6$,*

$$ex(n, W_6) = \lfloor \frac{n^2}{3} \rfloor$$

Proof Let G be a graph of order 6. If G has at least 13 edges, then it contains a vertex $v \in V(G)$ such that $d(v) = 5$. $G - v$ has at least 8 edges and we immediately obtain a contradiction with $ex(5, C_5) = 7$. We have that $ex(6, W_6) = 12 = \frac{6^2}{3}$.

Let us consider a graph G on 7 vertices which have $\lfloor \frac{7^2}{3} \rfloor + 1 = 17$ edges. It is easy to see that $\Delta(G) \geq 5$. Suppose that there is a vertex $v \in V(G)$ such that $d(v) = 6$. Then $G - v$ has exactly 11 edges and we immediately obtain a contradiction with $ex(6, C_5) = 9$. Now, suppose that there is a vertex $v \in V(G)$ such that $d(v) = 5$. There is a vertex w which not adjacent to v . Since $ex(5, C_5) = 7$, if the number of edges in a subgraph of G on 5 vertices obtained by removing the vertices v and w from G is at least 8, we have a W_6 . Consider the last case when $d(v) = d(w) = 5$. In [2] we find a characterization of the extremal graphs of order 5 without a C_5 . There are two such graphs: $(K_3 \cup K_1) + K_1$ and $K_2 + \overline{K_3}$. These graphs contain a vertex of degree 6, a contradiction. We obtain a W_6 , so $ex(7, W_6) = 16 = \lfloor \frac{7^2}{3} \rfloor$ and an extremal graph is a complete tripartite graph $K_{2,2,3}$.

By using Theorem 8 we obtain the remaining part of proof.

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