



# On Bulk Singularities in the Random Normal Matrix Model

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**Abstract** We extend the method of rescaled Ward identities of Ameur, Kang, and Makarov to study the distribution of eigenvalues close to a bulk singularity, i.e., a point in the interior of the droplet where the density of the classical equilibrium measure vanishes. We prove results to the effect that a certain “dominant part” of the Taylor expansion determines the microscopic properties near a bulk singularity. A description of the distribution is given in terms of the Bergman kernel of a certain Fock-type space of entire functions.

**Keywords** Two-dimensional eigenvalue ensemble · Bulk singularity · Microscopic density · Ward’s equation · Fock space

**Mathematics Subject Classification** 60G55 · 30C40 · 30D15 · 35R09

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## 1 Introduction and Main Results

**1.1.** Consider a system  $\{\zeta_j\}_1^n$  of identical point-charges in the complex plane in the presence of an external field  $nQ$ , where  $Q$  is a suitable function. The system is assumed to be picked randomly with respect to the Boltzmann–Gibbs probability law at inverse temperature  $\beta = 1$ ,

$$d\mathbf{P}_n(\zeta) = \frac{1}{Z_n} e^{-H_n(\zeta)} d^{2n}\zeta, \quad (1.1)$$

where  $H_n$  is the weighted energy of the system,

$$H_n(\zeta_1, \dots, \zeta_n) = \sum_{j \neq k} \log \frac{1}{|\zeta_j - \zeta_k|} + n \sum_{j=1}^n Q(\zeta_j).$$

The constant  $Z_n$  in (1.1) is chosen so that the total mass is 1.

It is well known that (with natural conditions on  $Q$ ) the normalized counting measures  $\mu_n = \frac{1}{n} \sum_{j=1}^n \delta_{\zeta_j}$  converge to *Frostman's equilibrium measure* as  $n \rightarrow \infty$ . This is a probability measure of the form

$$d\sigma(\zeta) = \chi_S(\zeta) \Delta Q(\zeta) dA(\zeta), \quad (1.2)$$

where  $\chi_S$  is the indicator function of a certain compact set  $S$  called the *droplet*.

We necessarily have  $\Delta Q \geq 0$  on  $S$ . In the papers [4, 5], the method of rescaled Ward identities was introduced and applied to study microscopic properties of the system  $\{\zeta_j\}_1^n$  close to a (moving) point  $p \in S$ . The situation in those papers is however restricted by the condition that the point  $p$  be “regular” in the sense that  $\Delta Q(p) \geq \text{const.} > 0$ . In this paper, we extend the method to allow for a “bulk singularity,” i.e., an isolated point  $p$  in the interior of  $S$  at which  $\Delta Q = 0$ .

A bulk singularity tends to repel particles away, which means that one must use a relatively coarse scale in order to capture the relevant structure. We prove results to the effect that (in many cases) the dominant terms in the Taylor expansion of  $\Delta Q$  about  $p$  determine the microscopic properties of the system in the vicinity of  $p$ . Our characterization uses the Bergman kernel for a certain space of entire functions, associated with these dominant terms. In particular, we obtain quite different distributions depending on the degree to which  $\Delta Q$  vanishes at  $p$ .

*Remark* It is well known that the particles  $\{\zeta_j\}_1^n$  can be identified with eigenvalues of random normal matrices with a suitable weighted distribution. The details of this identification are not important for the present investigation. However, following tradition, we shall sometimes speak of a “configuration of random eigenvalues” instead of a “particle-system.”

*Remark* The meaning of the convergence  $\mu_n \rightarrow \sigma$  is that  $\mathbf{E}_n[\mu_n(f)] \rightarrow \sigma(f)$  as  $n \rightarrow \infty$ , where  $f$  is a suitable test-function, say, continuous and bounded, and  $\mathbf{E}_n$  is expectation with respect to (1.1). In fact, more can be said, see [2].

**Notation** We write  $\Delta = \partial \bar{\partial}$  for 1/4 of the usual Laplacian and  $dA$  for  $1/\pi$  times Lebesgue measure on the plane  $\mathbb{C}$ . Here  $\partial = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$  and  $\bar{\partial} =$

$\frac{1}{2}(\partial/\partial x + i\partial/\partial y)$  are the usual complex derivatives. We write  $\bar{z}$  (or occasionally  $z^*$ ) for the complex conjugate of a number  $z$ . A continuous function  $h(z, w)$  will be called *Hermitian* if  $h(z, w) = h(w, z)^*$ .  $h$  is called *Hermitian-analytic (Hermitian-entire)* if  $h$  is Hermitian and analytic (entire) in  $z$  and  $\bar{w}$ . A Hermitian function  $c(z, w)$  is called a *cocycle* if there is a unimodular function  $g$  such that  $c(z, w) = g(z)\bar{g}(w)$ , where for functions we use the notation  $\tilde{f}(z) = f(z)^*$ . We write  $D(p, r)$  for the open disk with center  $p$  and radius  $r$ .

### 1.2 Potential and Equilibrium Measure

The function  $Q$  is usually called the “external potential.” This function is assumed to be lower semi-continuous and real-valued, except that it may assume the value  $+\infty$  in portions of the plane. We also assume: (i) the set  $\Sigma_0 = \{Q < \infty\}$  has dense interior, (ii)  $Q$  is real-analytic in  $\text{Int } \Sigma_0$ , and (iii)  $Q$  satisfies the growth condition

$$\liminf_{\zeta \rightarrow \infty} \frac{Q(\zeta)}{\log |\zeta|^2} > 1. \tag{1.3}$$

For a suitable measure on  $\mathbb{C}$ , we define its  $Q$ -energy by

$$I_Q[\mu] = \iint_{\mathbb{C}^2} \log \frac{1}{|\zeta - \eta|} d\mu(\zeta)d\mu(\eta) + \int_{\mathbb{C}} Q d\mu.$$

The *equilibrium measure*  $\sigma = \sigma_Q$  is defined as the probability measure that minimizes  $I_Q[\mu]$  over all compactly supported Borel probability measures  $\mu$ . Existence and uniqueness of such a minimizer is well known, see, e.g., [28], where also the explicit expression (1.2) is derived, with  $S = \text{supp } \sigma$ .

### 1.3 Rescaling

Recall that  $\Delta Q \geq 0$  on  $S$ . The purpose of the present investigation is to study (isolated) points  $p \in \text{Int } S$  at which  $\Delta Q(p) = 0$ . We refer to such points as *bulk singularities*. Without loss of generality, we can assume that  $p = 0$  is such a point, and we study the microscopic behavior of the system  $\{\zeta_j\}_1^n$  near 0.

By the *microscopic scale* at  $p = 0$ , we mean the positive number  $r_n = r_n(p)$  having the property

$$n \int_{D(p, r_n)} \Delta Q dA = 1.$$

Intuitively,  $r_n(p)$  means the expected distance from a particle at  $p$  to its closest neighbor. If  $p$  is a regular bulk point, then, as is easily seen,

$$r_n(p) = 1/\sqrt{n\Delta Q(p)} + O(1/n), \quad (n \rightarrow \infty),$$

which gives the familiar scaling factor used in papers such as [1,4].

Since the Laplacian  $\Delta Q$  vanishes at 0 and is real-analytic and nonnegative in a neighborhood, there is an integer  $k \geq 1$  such that the Taylor expansion of  $\Delta Q$  about 0 takes the form  $\Delta Q(\zeta) = \tilde{P}(\zeta) + O(|\zeta|^{2k-1})$ , where  $\tilde{P}(x + iy) = \sum_{j=0}^{2k-2} a_j x^j y^{2k-2-j}$  is a positive semi-definite polynomial, homogeneous of degree  $2k - 2$ .

We refer to the number  $2k - 2 = \text{degree } \tilde{P}$  as the *type* of the bulk-singularity at the origin. We shall say that the singularity is *non-degenerate* if  $\tilde{P}$  is positive definite, i.e., if there is a positive constant  $c$  such that  $\tilde{P}(\zeta) \geq c |\zeta|^{2k-2}$ . Hereafter, we tacitly assume that this condition is satisfied.

It will be important to have a good grasp of the size of  $r_n = r_n(0)$  as  $n \rightarrow \infty$ . For this, we note that

$$\begin{aligned} 1 &= n \int_{|\zeta| < r_n} \Delta Q(\zeta) \, dA(\zeta) \\ &= n \int_0^{r_n} r^{2k-1} \, dr \frac{1}{\pi} \int_0^{2\pi} \tilde{P}(e^{i\theta}) \, d\theta + O(n r_n^{2k+1}) \\ &= \tau_0^{-2k} n r_n^{2k} + O(n r_n^{2k+1}), \end{aligned}$$

where  $\tau_0 = \tau_0[Q, 0]$  is the positive constant satisfying

$$\tau_0^{-2k} = \frac{1}{2\pi k} \int_0^{2\pi} \tilde{P}(e^{i\theta}) \, d\theta. \quad (1.4)$$

We will call  $\tau_0$  the *modulus* of the bulk singularity at 0. We have the following lemma; the simple verification is omitted here.

**Lemma 1.1** *For the microscopic scale  $r_n$  at 0, we have  $r_n = \tau_0 n^{-1/2k} (1 + O(n^{-1/2k}))$  as  $n \rightarrow \infty$ , where  $\tau_0$  is the modulus (1.4).*

*Example* For the Mittag–Leffler potential  $Q = |\zeta|^{2k}$ , the droplet is the disk  $|\zeta| \leq k^{-1/2k}$ . For  $k = 1$ , we have the well-known Ginibre potential, cf., e.g., [16, Chapter 15]. For  $k \geq 2$ , the Mittag–Leffler potential has a bulk singularity at the origin of type  $2k - 2$ . It is easy to check that the modulus equals  $\tau_0 = k^{-1/2k}$ .

Let  $p$  be an integer,  $1 \leq p \leq n$ . The  $p$ -point function of the point-process  $\{\zeta_j\}_1^n$  is the function of  $p$  complex variables  $\eta_1, \dots, \eta_p$  defined by

$$\mathbf{R}_{n,p}(\eta_1, \dots, \eta_p) = \lim_{\delta \rightarrow 0} \frac{\mathbf{P}_n(\{\zeta_j\}_1^n \cap D(\eta_\ell, \delta) \neq \emptyset, \ell = 1, \dots, p)}{\delta^{2p}}.$$

The  $p$ -point function  $\mathbf{R}_{n,p}$  should really be understood as the density in the measure  $\mathbf{R}_{n,p}(\eta_1, \dots, \eta_p) \, dA(\eta_1) \cdots dA(\eta_p)$ . This should be kept in mind when we subject the  $\eta_j$  to various transformations.

A well-known algebraic fact (“Dyson’s determinant formula,” see, e.g., [26] or [28, p. 249]) states that the  $p$ -point function takes the form of a determinant,

$$\mathbf{R}_{n,p}(\eta_1, \dots, \eta_p) = \det (\mathbf{K}_n(\eta_i, \eta_j))_{i,j=1}^p,$$

where  $\mathbf{K}_n$  is a certain Hermitian function called a *correlation kernel* of the process (Cf. Sect. 2). Of particular importance is the *one-point function*  $\mathbf{R}_n = \mathbf{R}_{n,1}$ .

We now rescale about the origin on the microscopic scale  $r_n$  about the bulk singularity at 0. The *rescaled system*  $\{z_j\}_1^n$  is taken to be

$$z_j = r_n^{-1} \zeta_j, \quad j = 1, \dots, n, \tag{1.5}$$

with the law given by the image of the Boltzmann–Gibbs distribution (1.1) under the scaling (1.5).

It follows that the rescaled system  $\{z_j\}_1^n$  is determinantal with  $p$ -point function

$$R_{n,p}(z_1, \dots, z_p) = r_n^{2p} \mathbf{R}_{n,p}(\zeta_1, \dots, \zeta_p) = \det(K_n(z_i, z_j))_{i,j=1}^p,$$

where the correlation kernel  $K_n$  for the rescaled system is given by

$$K_n(z, w) = r_n^2 \mathbf{K}_n(\zeta, \eta), \quad (z = r_n^{-1} \zeta, w = r_n^{-1} \eta).$$

In particular, the one-point function of the process  $\{z_j\}_1^n$  is  $R_n(z) = K_n(z, z)$ .

Clearly, a correlation kernel  $K_n(z, w)$  is only determined up to multiplication by a cocycle  $c_n(z, w)$ .

### 1.4 Main Structural Lemma

Now suppose that  $Q$  has a bulk-singularity of type  $2k - 2$  at the origin. It will be useful to single out a canonical “dominant part” of  $Q$  near 0. To this end, let  $P(x + iy)$  be the Taylor polynomial of  $Q$  of degree  $2k$  about the origin. Let  $H$  be the holomorphic polynomial

$$H(\zeta) = Q(0) + 2\partial Q(0) \cdot \zeta + \partial^2 Q(0) \cdot \zeta^2 + \dots + \frac{2}{(2k)!} \partial^{2k} Q(0) \cdot \zeta^{2k}.$$

We will write

$$Q_0 = P - Re H.$$

We then have the basic decomposition

$$Q = Q_0 + Re H + Q_1, \tag{1.6}$$

where  $Q_1(\zeta) = O(|\zeta|^{2k+1})$  as  $\zeta \rightarrow 0$ .

The following lemma gives the basic structure of limiting kernels at a singular point (not necessarily in the bulk).

**Lemma 1.2** *There exists a sequence  $c_n$  of cocycles such that every subsequence of the sequence  $c_n K_n$  has a subsequence converging uniformly on compact subsets to some Hermitian function  $K$ . Every limit point  $K$  has the structure*

$$K(z, w) = L(z, w)e^{-Q_0(\tau_0 z)/2 - Q_0(\tau_0 w)/2},$$

where  $L$  is a Hermitian-entire function.

Following [4], we refer to a limit point  $K$  in Lemma 1.2 as a *limiting kernel*, whereas  $L$  is a *limiting holomorphic kernel*. We also speak of the *limiting 1-point function*

$$R(z) = K(z, z) = L(z, z)e^{-Q_0(\tau_0 z)}.$$

Note that  $R$  determines  $K$  and  $L$  by polarization.

*Remark* Each limiting one-point function gives rise to a unique *limiting point field* (or “infinite particle system”)  $\{z_j\}_1^\infty$  with intensity functions

$$R_k(z_1, \dots, z_p) = \det(K(z_i, z_j))_{i,j=1}^p.$$

(This follows from Lenard’s theory, see [29] or [4].) It is possible that a limiting point field is *trivial* in the sense that  $K = 0$ .

## 1.5 Universality Results

We will prove universality for two kinds of bulk singularities. Referring to the canonical decomposition  $Q = Q_0 + Re H + Q_1$  with  $Q_0$  of degree  $2k$ , we say a singularity at 0 is:

- (i) *homogeneous* if  $Q_1 = 0$  and  $H(z) = c z^{2k}$  for some constant  $c$ ,
- (ii) *dominant radial* if  $Q_0$  is radially symmetric, i.e.,  $Q_0(z) = Q_0(|z|)$ .

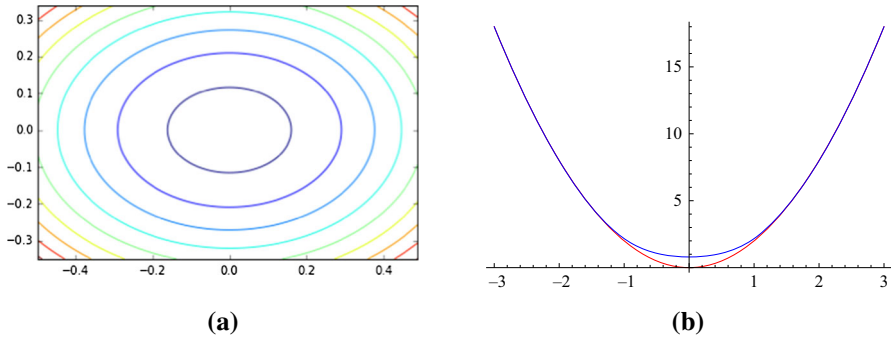
We remark that a homogeneous singularity is necessarily located in the bulk of the droplet; for other types of singularities this must be postulated.

In the following, we denote by  $L_0$  the Bergman kernel of the space of entire functions  $L_a^2(\mu_0)$  associated with the measure

$$d\mu_0(z) = e^{-Q_0(\tau_0 z)} dA(z).$$

**Theorem 1.3** *If there is a homogeneous singularity at 0, we have  $L = L_0$  for each limiting holomorphic kernel  $L$ .*

The next result concerns limiting holomorphic kernels  $L(z, w)$  which are *rotationally symmetric* in the sense that  $L(z, w) = L(ze^{it}, we^{it})$  for all real  $t$ . Equivalently,  $L$  is rotationally symmetric if there is an entire function  $E$  such that  $L(z, w) = E(z\bar{w})$ . (We leave the simple verification of this to the reader.)



**Fig. 1** Some level curves of  $R_0(z) = L_0(z, z)e^{-Q_0(\tau_0 z)}$  for  $Q_0(z) = |z|^4 - |z|^2 \operatorname{Re}(z^2)/2$  and the graph of  $R_0(x) = M_2(x^2)e^{-Q_0(\tau_0 x)}$  for the Mittag–Leffler potential  $Q_0(z) = |z|^4$ . The lower (red) curve shows the Laplacian  $\Delta_z[Q_0(\tau_0 z)]$ . (color figure online)

**Theorem 1.4** *If a bulk singularity at 0 is dominant radial, then  $L = L_0$  for each rotationally symmetric limiting kernel.*

The result was conjectured in [4, Section 7.3].

We do not know whether or not each limiting kernel at a dominant radial bulk singularity is rotationally symmetric. This question has some similarity to that of deciding the *translation invariance* of limiting kernels at regular boundary points. See [4] for several comments about this, notably the interpretation in terms of a twisted convolution equation in Section 7.1.

It is natural to conjecture that the kernel in Theorem 1.3 be equal to the limiting kernel in general, regardless of the nature of a (nondegenerate) bulk singularity.

*Remark* Note that, as a consequence of the reproducing property of the kernel  $L_0$ , we have in the situation of the above theorems the *mass-one equation* for a limiting kernel  $K$ ,  $\int_{\mathbb{C}} |K(z, w)|^2 dA(w) = R(z)$ .

*Example* For the Mittag–Leffler potential  $Q = |\zeta|^{2k}$ , it is possible to calculate the limiting kernel  $L$  explicitly, using orthogonal polynomials (see [4, Section 7.3] or [30]). The result is that

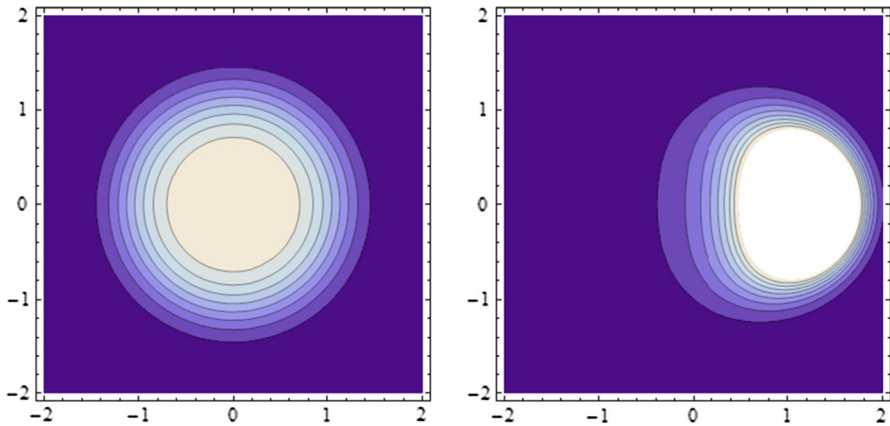
$$L(z, w) = M_k(z\bar{w}), \tag{1.7}$$

where

$$M_k(z) = \tau_0^2 k \sum_0^\infty \frac{(\tau_0^2 z)^j}{\Gamma\left(\frac{1+j}{k}\right)}.$$

The function  $M_k$  can be expressed as  $M_k(z) = \tau_0^2 k E_{1/k, 1/k}(\tau_0^2 z)$ , where  $E_{a,b}$  is the Mittag–Leffler function (see [18])

$$E_{a,b}(z) = \sum_0^\infty \frac{z^j}{\Gamma(aj + b)}.$$



**Fig. 2** Some level curves of the Berezin kernel  $B(z, w)$  rooted at  $z = 0$  and  $z = 1$  pertaining to  $Q_0(z) = |z|^4$

Using Theorem 1.3, we can now see that the kernel in (1.7) is universal for potentials of the form  $Q = |\zeta|^{2k} + Re(c \zeta^{2k})$ . (We must insist that  $|c| < 1$  to insure that the growth assumption of  $Q$  at infinity is satisfied, see (1.3).) By Theorem 1.4, the universality holds also for all rotationally symmetric limiting kernels  $L(z, w) = E(z\bar{w})$  for more general potentials of the form  $Q(\zeta) = |\zeta|^{2k} + ReH(\zeta) + Q_1(\zeta)$ .

*Remark* For  $k = 1$  (i.e., when 0 is a “regular” bulk point) the space  $L_a^2(\mu_0)$  becomes the standard Fock space, normed by  $\|f\|^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dA(z)$ . In this case, we have  $R = 1$  for the limiting 1-point function, by the well-known Ginibre( $\infty$ )-limit. (See, e.g., [4].)

### 1.6 Further Results

In the following, we consider a potential with canonical decomposition  $Q = Q_0 + ReH + Q_1$ . Following [4], we shall prove auxiliary results that fall into three categories.

*Ward’s Equation* Let  $R(z) = K(z, z)$  be a limiting kernel in Lemma 1.2. At a point  $z$  where  $R > 0$ , we write

$$B(z, w) = \frac{|K(z, w)|^2}{K(z, z)} = \frac{|L(z, w)|^2}{L(z, z)} e^{-Q_0(\tau_0 w)}, \tag{1.8}$$

$$C(z) = \int_{\mathbb{C}} \frac{B(z, w)}{z - w} dA(w). \tag{1.9}$$

We call  $B(z, w)$  a *limiting Berezin kernel* rooted at  $z$ ;  $C(z)$  is its *Cauchy transform*.



**Theorem 1.5** *Let  $R$  be a limiting 1-point function.*

- (i) *Zero-one law: Either  $R = 0$  identically, or else  $R > 0$  everywhere.*
- (ii) *Ward’s equation: If  $R$  is nontrivial, we have that*

$$\bar{\delta}C(z) = R(z) - \Delta_z [Q_0(\tau_0 z)] - \Delta_z \log R(z). \tag{1.10}$$

As  $n \rightarrow \infty$ , it may well happen that  $R_n \rightarrow 0$  locally uniformly (if the singularity at 0 is in the exterior of the droplet).

*A priori Estimates* To rule out the possibility of trivial limiting kernels, we shall use the following result.

**Theorem 1.6** *Let  $R$  be any limiting kernel, and let  $R_0(z) = L_0(z, z)e^{-Q_0(\tau_0 z)}$ , where  $L_0$  is the Bergman kernel of the space  $L_a^2(\mu_0)$ . Then*

- (i)  $R_0(z) = \Delta_z [Q_0(\tau_0 z)] \cdot (1 + O(z^{1-k}))$ , as  $z \rightarrow \infty$ ,
- (ii)  $R(z) = \Delta_z [Q_0(\tau_0 z)] \cdot (1 + O(z^{1-k}))$ , as  $z \rightarrow \infty$ .

Part (i) depends on an estimate of the Bergman kernel for the space  $L_a^2(\mu_0)$ . Related estimates valid when  $Q_0$  is a function satisfying uniform estimates of the type  $0 < c \leq \Delta Q_0 \leq C$  are found in Lindholm’s paper [24].

In our situation, the function  $\Delta Q_0$  takes on all values between 0 and  $+\infty$ , which means that the results from [24] are not directly applicable. It is convenient to include an elementary discussion for the case at hand, following the method of “approximate Bergman projections” in the spirit of [4, Section 5]. This has the advantage that proof of part (ii) follows after relatively simple modifications.

*Remark* Part (i) of Theorem 1.6 seems to be of some relevance for the investigation of density conditions for sampling and interpolation in Fock-type spaces  $L_a^2(\mu_0)$ ; see the recent paper [17, Remark 5.6], cf. [31] for the classical Fock space case. (A very general result of this sort was obtained by different methods in the paper [25], where the hypothesis on the “weight”  $Q_0$  is merely that the Laplacian  $\Delta Q_0$  be a doubling measure.)

*Remark* In the case  $Q = |z|^{2\lambda}$ , the asymptotic formula in Theorem 1.6 (i) has an alternative proof by more classical methods, using an asymptotic expansion for the function  $M_\lambda(z)$  as  $z \rightarrow \infty$  ([18, Section 4.7]). The formula (i) can be recognized as giving the leading term in that expansion.

*Positivity* Recall that a Hermitian function  $K$  is called a *positive matrix* if

$$\sum_{i,j=1}^N \alpha_i \bar{\alpha}_j K(z_i, z_j) \geq 0$$

for all points  $z_j \in \mathbb{C}$  and all complex scalars  $\alpha_j$ . It is clear that each limiting (holomorphic) kernel is a positive matrix.

**Theorem 1.7** *Let  $L$  be a limiting holomorphic kernel. Then  $L$  is the Bergman kernel for a Hilbert space  $\mathcal{H}_*$  of entire functions that sits contractively in  $L_a^2(\mu_0)$ . Moreover,  $L_0 - L$  is a positive matrix.*

Here  $L_0$  is the Bergman kernel of  $L_a^2(\mu_0)$ .

It may well happen that the space  $\mathcal{H}_*$  degenerates to  $\{0\}$ . This is the case when the singularity at 0 is located in the exterior of the droplet.

*Comments* An interesting generalization of our situation is obtained by allowing for a suitably scaled logarithmic singularity at a (regular or singular) bulk point. More precisely, if  $\tilde{Q}$  is smooth in a neighborhood of 0, we consider a potential of the form  $Q(\zeta) = \tilde{Q}(\zeta) + 2(c/n) \log |\zeta|$ , where  $c < 1$  is a constant. Rescaling by  $z = r_n^{-1} \zeta$ , where  $c + n \int_{D(0, r_n)} \Delta \tilde{Q} dA = 1$ , we find  $r_n \sim (1 - c)^{1/2k} \tau_0 n^{-1/2k}$  as  $n \rightarrow \infty$ , where  $2k - 2$  is the type of  $\tilde{Q}$  and  $\tau_0 = \tau_0[\tilde{Q}, 0]$ . It is hence natural to define the dominant part by  $Q_0(z) := c' \tilde{Q}_0(z) + 2c \log |z|$ , where  $\tilde{Q}_0$  is the dominant part of  $\tilde{Q}$  and  $c'$  a suitable constant depending on  $c$ . In particular, if  $Q(\zeta) = c_1 |\zeta|^{2\lambda} + (c_2/n) \log |\zeta|$  with suitable  $c_1, c_2 > 0$ , the dominant part becomes of the type

$$Q_0(z) = r^{2\lambda} + 2 \left( 1 - \frac{\lambda}{\mu} \right) \log r, \quad r = |z|, \tag{1.11}$$

for suitable constants  $\lambda$  and  $\mu$ . The potential (1.11) was introduced in the paper [3], where all rotationally symmetric solutions to the corresponding Ward equation (1.10) were found. Recently, certain potentials of this form were studied in a context of Riemann surfaces, in a scaling limit about certain types of singular points (conical singularities and branch points), see [22]. We will return to this issue in a forthcoming paper [6].

An interesting occurrence of Mittag–Leffler potentials with logarithmic singularities is found in the paper [11]; the dominant part  $Q_0 = |z|^2 - (2/M) \log |z|$  is there connected to the eigenvalue density of the product of  $M$  complex Gaussian matrices. We are grateful to one of the referees for bringing this to our attention.

As in [4, Section 7.7], we note that it is possible to introduce an “inverse temperature”  $\beta$  into the setting; the case at hand then corresponds to  $\beta = 1$ . For general  $\beta$ , the rescaled process  $\{z_j\}_1^n$  is no longer determinantal, but the rescaled intensity functions  $R_{n,p}^\beta$  make perfect sense. As  $n \rightarrow \infty$ , we formally obtain a “Ward’s equation at a bulk singularity” of the form

$$\bar{\partial} C^\beta(z) = R^\beta(z) - \Delta_z [Q_0(\tau_0 z)] - \frac{1}{\beta} \Delta_z \log R^\beta(z). \tag{1.12}$$

Here  $C^\beta(z)$  should be understood as the Cauchy transform of the  $\beta$ -Berezin kernel  $B^\beta(z, w) = (R_1^\beta(z)R_1^\beta(w) - R_2^\beta(z, w))/R_1^\beta(z)$ . The objects in (1.12) are so far understood mostly on a physical level. We now give a few remarks in this spirit.

First, if 0 is a regular bulk-point, i.e., if  $\Delta Q(0) > 0$ , then it is believed that  $R^\beta = 1$  identically, i.e., the right-hand side in (1.12) should vanish. The equation (1.12) then reflects the fact that the Berezin kernel  $B^\beta(z, w) = b^\beta(r)$  depends only on the distance  $r = |z - w|$ . When  $\beta = 1$ , one has the well-known identity  $b^1(r) = e^{-r^2}$ . For other  $\beta$  we do not know of an explicit expression, but it was shown by Jancovici in [20] that

$$b^\beta(r) = b^1(r) + (\beta - 1)f(r) + O((\beta - 1)^2), \quad (\beta \rightarrow 1),$$

where  $f$  is a certain explicit function. In the bulk-singular case, the kernel  $B^\beta(z, w)$  will not just depend on  $|z - w|$ , but still it seems natural to expect that we have an expansion of the form

$$B^\beta(z, w) = b^1(z, w) + (\beta - 1)f(z, w) + O((\beta - 1)^2), \quad (\beta \rightarrow 1), \quad (1.13)$$

where  $b^1(z, w) = |L_0(z, w)|^2 e^{-Q_0(\tau_0 w)} / L_0(z, z)$ ,  $L_0$  being the Bergman kernel of the space  $L_a^2(\mu_0)$ . A natural problem, which will not be taken up here, is to determine the function  $f(z, w)$  in (1.13). (A similar investigation at regular boundary points was made recently in the paper [12].)

For boundary points, the term “singular” has a different meaning than for bulk points. Indeed, the singular points  $p$  (cusps or double points) studied in the paper [5] all satisfy  $\Delta Q(p) > 0$ . An example of a situation at which  $\Delta Q = 0$  at a boundary point (at 0) is provided by the potential  $Q = |\zeta|^4 - \sqrt{2}Re(\zeta^2)$ . (The boundary of  $S$  is here a “figure 8” with 0 at the point of self-intersection, see [9].) A natural question is whether it is possible to define nontrivial scaling limits at (or near) these kinds of singular points, in the spirit of [5].

There is a parallel theory for scaling limits for Hermitian random matrix ensembles. In this situation, the droplet is a union of compact intervals. It is well known that the sine-kernel appears in the scaling limit about a “regular bulk point,” i.e., an interior point where the density of the equilibrium measure is strictly positive. In a generic case, all points are regular, see [21]. Special bulk points where the equilibrium density vanishes may be called “singular”; at such points other types of universality classes appear, see [10, 14, 27]; cf. [15] for a corresponding nonbulk situation.

Finally, we wish to mention that the investigations in this paper were partly motivated by applications to the distribution of Fekete points close to a bulk singularity (see [1]). This issue will be taken up in a later publication.

*Addendum* After the completion of this paper, at least two relevant papers have appeared. The note [7] generalizes and improves some of our above results, notably Theorem 1.6, in a context allowing for the logarithmic singularities discussed above. The paper [23] investigates logarithmic singularities with respect to properties of associated orthogonal polynomials.

### 1.7 Plan of the Paper

In Sect. 2, we prove the general structure formula for limiting kernels (Lemma 1.2). We also prove the positivity theorem (Theorem 1.7). In Sect. 3, we prove Ward’s equation and the zero-one law (Theorem 1.5). In Sect. 4, we prove the universality results (Theorems 1.3 and 1.4). Our proof of Theorem 1.4 depends on the apriori estimate from Theorem 1.6, part (ii). In the last two sections, we prove the asymptotics for the functions  $R_0$  and  $R$  in Theorem 1.6. For  $R_0$  (part (i)), see Sect. 5; for  $R$  (part (ii)), see Sect. 6.

## 1.8 Convention

Multiplying the potential  $Q$  by a suitable constant, we can in the following assume that the modulus  $\tau_0 = 1$ . In fact, the slightly more general assumption that  $\tau_0 = 1 + O(n^{-1/2k})$  as  $n \rightarrow \infty$  will do equally well. This means the microscopic scale about 0 can be taken as  $r_n = n^{-1/2k}$ , where  $2k - 2$  is the type of the singularity. This will be assumed *throughout* the rest of this paper.

## 2 Structure of Limiting Kernels

In this section, we prove Lemma 1.2 on the general structure of limiting kernels and the positivity Theorem 1.7. We shall actually prove a little more: a limiting holomorphic kernel can be written as a subsequential limit of kernels for certain specific Hilbert spaces of entire functions. In later sections, we will use this additional information for our analysis of homogeneous bulk singularities.

### 2.1 Spaces of Weighted Polynomials

It is well known that we can take for correlation kernel for the process  $\{\zeta_j\}_1^n$  the reproducing kernel for a suitable space of weighted polynomials. Here the “weight” can either be incorporated into the polynomials themselves or into the norm of the polynomials. We will use both these possibilities. In the following, we shall use the symbol “ $\text{Pol}(n)$ ” for the linear space of holomorphic polynomials of degree at most  $n - 1$  (without any topology). We write  $\mu_n$  for the measure  $d\mu_n = e^{-nQ} dA$ .

We let  $\mathcal{P}_n$  denote the space  $\text{Pol}(n)$  regarded as a subspace of  $L^2(\mu_n)$ . The symbol  $\mathcal{W}_n$  will denote the set of weighted polynomials  $f = p e^{-nQ/2}$ , ( $p \in \text{Pol}(n)$ ) regarded as a subspace of  $L^2 = L^2(dA)$ . We write  $\mathbf{k}_n$  and  $\mathbf{K}_n$  for the reproducing kernels of  $\mathcal{P}_n$  and  $\mathcal{W}_n$ , respectively, and we note that

$$\mathbf{K}_n(\zeta, \eta) = \mathbf{k}_n(\zeta, \eta) e^{-nQ(\zeta)/2 - nQ(\eta)/2}.$$

Now suppose that  $Q$  has a bulk singularity at the origin, of type  $2k - 2$  and rescale at the microscopic scale by

$$k_n(z, w) = r_n^2 \mathbf{k}_n(\zeta, \eta), \quad K_n(z, w) = r_n^2 \mathbf{K}_n(\zeta, \eta), \quad (z = r_n^{-1} \zeta, \quad w = r_n^{-1} \eta).$$

### 2.2 Limiting Holomorphic Kernels

Suppose that there is a bulk singularity of type  $2k - 2$  at the origin. Consider the canonical decomposition  $Q = Q_0 + Re H + Q_1$  and write  $h = Re H$ . Thus  $h$  is of degree at most  $2k$ ,  $Q_0$  is a positive definite homogeneous polynomial of degree  $2k$ , and  $Q_1(\zeta) = O(|\zeta|^{2k+1})$  as  $\zeta \rightarrow 0$ .

**Lemma 2.1** *For each compact subset  $V$  of  $\mathbb{C}$ , there is a constant  $C = C(V)$  such that  $K_n(z, z) \leq C$  for  $z \in V$ .*

*Proof* Let  $\tilde{\mathcal{W}}_n$  denote the space of all “rescaled” weighted polynomials  $p \cdot e^{-\tilde{Q}_n/2}$ , where  $p \in \text{Pol}(n)$  and  $\tilde{Q}_n(z) = nQ(r_n z)$ . Regarding  $\tilde{\mathcal{W}}_n$  as a subspace of  $L^2$ , we recognize that  $K_n$  is the reproducing kernel of  $\tilde{\mathcal{W}}_n$ . Hence

$$K_n(z, z) = \sup\{|f(z)|^2; f \in \tilde{\mathcal{W}}_n, \|f\| \leq 1\}. \tag{2.1}$$

Fix a number  $\delta > 0$ , and let  $V_\delta = \{z \in \mathbb{C}; \text{dist}(z, V) \leq \delta\}$ . We also pick a number  $\alpha > \sup\{\Delta Q_0(z); z \in V_\delta\}$ . Now let  $u$  be an analytic function in a neighborhood of  $V_\delta$ , and consider the function  $g_n(z) = u(z) e^{-\tilde{Q}_n(z)/2 + \alpha|z|^2/2}$ . Note that

$$\begin{aligned} \Delta \tilde{Q}_n(z) &= n r_n^2 (\Delta Q_0(r_n z) + \Delta Q_1(r_n z)) \\ &= n r_n^{2k} \Delta Q_0(z) + O(n r_n^{2k+1}), \quad (n r_n^{2k} = 1). \end{aligned}$$

Hence  $\Delta \log |g_n(z)|^2 \geq -\Delta \tilde{Q}_n(z) + \alpha > 0$  for all sufficiently large  $n$  and all  $z \in V_\delta$ . Thus  $|g_n|^2$  is subharmonic in  $V_\delta$ , so for  $z \in V$ ,

$$\begin{aligned} |g_n(z)|^2 &\leq \delta^{-2} \int_{D(z, \delta)} |g_n(w)|^2 dA(w) \\ &= \delta^{-2} e^{\alpha(|z|+\delta)^2} \int_{D(z, \delta)} |u(w)|^2 e^{-\tilde{Q}_n(w)} dA(w). \end{aligned}$$

We obtain

$$|u(z)|^2 e^{-\tilde{Q}_n(z)} \leq \delta^{-2} e^{\alpha(2M_V \delta + \delta^2)} \int_{D(z, \delta)} |u|^2 e^{-\tilde{Q}_n} dA, \tag{2.2}$$

where  $M_V = \sup_{z \in V} |z|$ . By (2.1) and (2.2),  $K_n(z, z)$  is bounded for  $z \in V$ . □

We now use the holomorphic polynomial  $H$  in the decomposition  $Q = Q_0 + Re H + Q_1$  to define a Hermitian-entire function (“rescaled holomorphic kernel”) by

$$L_n(z, w) = r_n^2 \mathbf{k}_n(\zeta, \eta) e^{-n(H(\zeta) + \bar{H}(\eta))/2}, \quad z = r_n^{-1} \zeta, w = r_n^{-1} \eta. \tag{2.3}$$

Let us write

$$H_n(z) = n H(r_n z), \quad Q_{1,n}(z) = n Q_1(r_n z),$$

so that  $nQ(r_n z) = Q_0(z) + Re H_n(z) + Q_{1,n}(z)$  and

$$L_n(z, w) = k_n(z, w) e^{-H_n(z)/2 - \bar{H}_n(w)/2}.$$

Define a Hilbert space of entire functions by

$$\mathcal{H}_n = \{f = q \cdot e^{-H_n/2}; q \in \text{Pol}(n)\}$$

equipped with the norm of  $L^2(\tilde{\mu}_n)$ , where

$$d\tilde{\mu}_n(z) = e^{-Q_0(z) - Q_{1,n}(z)} dA(z). \tag{2.4}$$

Observe that  $Q_{1,n} = O(r_n)$  as  $n \rightarrow \infty$ , where the  $O$ -constant is uniform on each given compact subset of  $\mathbb{C}$ . In particular,  $\tilde{\mu}_n \rightarrow \mu_0$  vaguely where  $d\mu_0 = e^{-Q_0} dA$ .

The following result implies Lemma 1.2; it also generalizes [4, Lemma 4.9].

**Lemma 2.2** *Each subsequence of the kernels  $L_n$  has a further subsequence converging locally uniformly to a Hermitian-entire limit  $L$ . Furthermore,  $L_n$  is the reproducing kernel of the space  $\mathcal{H}_n$ , and  $L$  satisfies the “mass-one inequality,”*

$$\int |L(z, w)|^2 d\mu_0(w) \leq L(z, z). \tag{2.5}$$

Finally, there exists a sequence of cocycles  $c_n$  such that each subsequence of  $c_n K_n$  converges locally uniformly to a Hermitian function  $K$  of the type  $K(z, w) = L(z, w)e^{-Q_0(z)/2 - Q_0(w)/2}$ .

*Proof* Define a function  $E_n(z, w)$  by

$$\begin{aligned} E_n(z, w) &= e^{n(H(\zeta)/2 + \bar{H}(\eta)/2 - Q(\zeta)/2 - Q(\eta)/2)} \\ &= e^{-Q_0(z)/2 - Q_0(w)/2 - Q_{1,n}(z)/2 - Q_{1,n}(w)/2 + i \operatorname{Im}(H_n(z) - H_n(w))/2}. \end{aligned}$$

Note that  $K_n = L_n E_n$ , where  $L_n$  is the Hermitian-entire kernel (2.3). Now, if  $h = \operatorname{Re} H$ , then

$$\operatorname{Im}(H_n(z) - H_n(w))/2 = \sum_{j=1}^{2k} n r_n^j \operatorname{Im} \left( \frac{\partial^j h(0)}{j!} (z^j - w^j) \right).$$

We have shown that

$$E_n(z, w) = c_n(z, w) e^{-Q_0(z)/2 - Q_0(w)/2} (1 + o(1)), \quad (n \rightarrow \infty),$$

where  $o(1) \rightarrow 0$  locally uniformly on  $\mathbb{C}^2$  and  $c_n$  is a cocycle:

$$c_n(z, w) = \prod_{j=1}^{2k} \exp \left[ i n r_n^j \operatorname{Im} \left( \frac{\partial^j h(0)}{j!} (z^j - w^j) \right) \right].$$

On the other hand, for each compact subset  $V$  of  $\mathbb{C}^2$ , there is a constant  $C$  such that

$$|L_n(z, w)|^2 = \left| \frac{K_n(z, w)}{E_n(z, w)} \right|^2 \leq C K_n(z, z) K_n(w, w) e^{Q_0(z) + Q_0(w)}$$

for sufficiently large  $n$ . By Lemma 2.1, the functions  $L_n$  have a uniform bound on  $V$ . We have shown that  $\{L_n\}$  is a normal family. We can hence extract a subsequence  $\{L_{n_\ell}\}$ , converging locally uniformly to a Hermitian-entire function  $L(z, w)$ .

Choosing cocycles  $c_n$  such that  $c_n E_n \rightarrow e^{-Q_0(z)/2 - Q_0(w)/2}$  uniformly on compact subsets as  $n \rightarrow \infty$ , we now obtain that

$$c_{n_\ell} K_{n_\ell} = c_{n_\ell} E_{n_\ell} L_{n_\ell} \rightarrow e^{-Q_0(z)/2 - Q_0(w)/2} L(z, w) = K(z, w).$$

The reproducing property  $\int |K_{n_\ell}(z, w)|^2 dA(w) = K_{n_\ell}(z, z)$  means that

$$\int |E_{n_\ell}(z, w) L_{n_\ell}(z, w)|^2 dA(w) = E_{n_\ell}(z, z) L_{n_\ell}(z, z).$$

Letting  $\ell \rightarrow \infty$ , we obtain the mass-one inequality (2.5) by Fatou’s lemma.

There remains to prove that  $L_n$  is the reproducing kernel for the space  $\mathcal{H}_n$ . For this, we write  $L_{n,w}(z) = L_n(z, w)$  and note that for an element  $f = q \cdot e^{-H_n/2}$  of  $\mathcal{H}_n$ , we have

$$\begin{aligned} \langle f, L_{n,w} \rangle_{L^2(\tilde{\mu}_n)} &= \int_{\mathbb{C}} q(z) e^{-H_n(z)/2} \bar{L}_n(z, w) e^{-Q_0(z) - Q_{1,n}(z)} dA(z) \\ &= e^{-H_n(w)/2} \int_{\mathbb{C}} q(z) \bar{k}_n(z, w) e^{-nQ(r_n z)} dA(z). \end{aligned}$$

Noting that  $k_n$  is the reproducing kernel for the space  $\tilde{\mathcal{P}}_n$  of polynomials of degree at most  $n - 1$  normed by  $\|p\|^2 = \int_{\mathbb{C}} |p(z)|^2 e^{-nQ(r_n z)} dA(z)$ , we now see that

$$\langle f, L_{n,w} \rangle_{L^2(\tilde{\mu}_n)} = e^{-H_n(w)/2} q(w) = f(w).$$

The proof of the lemma is complete. □

### 2.3 The Positivity Theorem

Let  $\mu_0$  be the measure  $d\mu_0 = e^{-Q_0} dA$ , and define  $L_0(z, w)$  to be the Bergman kernel for the Bergman space  $L^2_a(\mu_0)$ . Let  $L = \lim L_{n_\ell}$  be a limiting holomorphic kernel at 0.

Recall that the kernel  $L_n$  is the reproducing kernel for a certain subspace  $\mathcal{H}_n$  of  $L^2_a(\tilde{\mu}_n)$ , where  $\tilde{\mu}_n \rightarrow \mu_0$  in the sense that the densities converge uniformly on compact sets, as  $n \rightarrow \infty$ . See Lemma 2.2.

For  $L = \lim L_{n_\ell}$ , the assignment  $\langle L_z, L_w \rangle_* = L(w, z)$  defines a positive semi-definite inner product on the linear span  $\mathcal{M}$  of the  $L_z$ ’s. In fact, the inner product is either trivial ( $L(z, z) = 0$  for all  $z$ ), or else it is positive definite: this holds by the zero-one law in Theorem 1.5, which will be proved in the next section.

By Fatou’s lemma, we now see that, for all choices of points  $z_j$  and scalars  $\alpha_j$ ,

$$\begin{aligned} \left\| \sum_{j=1}^N \alpha_j L_{z_j} \right\|_{L^2(\mu_0)}^2 &\leq \liminf_{\ell \rightarrow \infty} \sum_{i,j=1}^N \alpha_i \bar{\alpha}_j \int_{\mathbb{C}} L_{n_\ell}(w, z_i) \bar{L}_{n_\ell}(w, z_j) d\bar{\mu}_{n_\ell}(w) \\ &= \liminf_{\ell \rightarrow \infty} \sum_{i,j=1}^N \alpha_i \bar{\alpha}_j L_{n_\ell}(z_i, z_j) = \sum_{i,j=1}^N \alpha_i \bar{\alpha}_j L(z_i, z_j) \\ &= \left\| \sum_{j=1}^N \alpha_j L_{z_j} \right\|_*^2. \end{aligned}$$

This shows that  $\mathcal{M}$  is contained in  $L^2(\mu_0)$  and that the inclusion  $I : \mathcal{M} \rightarrow L^2(\mu_0)$  is a contraction. Hence the completion  $\mathcal{H}_*$  of  $\mathcal{M}$  can be regarded as a contractively embedded subspace of  $L_a^2(\mu_0)$ .

Since the space  $L_a^2(\mu_0)$  has reproducing kernel  $L_0(z, w)$ , it follows from a theorem of Aronszajn ([8, p. 355]) that the difference  $L_0 - L$  is a positive matrix. The proof of Theorem 1.7 is complete. □

### 3 Ward’s Equation and the Zero-One Law

#### 3.1 Ward’s Equation

Given a limiting kernel  $K$  in Lemma 2.2, we recall the definitions

$$R(z) = K(z, z), \quad B(z, w) = \frac{|K(z, w)|^2}{K(z, z)}, \quad C(z) = \int_{\mathbb{C}} \frac{B(z, w)}{z - w} dA(w).$$

The goal of this section is to prove Theorem 1.5, which we here restate in the following form (the case  $\tau_0 = 1$ ).

**Lemma 3.1** *If  $R$  does not vanish identically, then  $R > 0$  everywhere and we have*

$$\bar{\partial}C(z) = R(z) - \Delta Q_0(z) - \Delta \log R(z).$$

For the proof of Lemma 3.1, we recall the setting of Ward’s identity from [4].

For a test function  $\psi \in C_0^\infty(\mathbb{C})$ , we define a function  $W_n^+[\psi]$  of  $n$  variables by

$$W_n^+[\psi] = I_n[\psi] - II_n[\psi] + III_n[\psi],$$



where

$$I_n[\psi](\zeta) = \frac{1}{2} \sum_{j \neq k}^n \frac{\psi(\zeta_j) - \psi(\zeta_k)}{\zeta_j - \zeta_k}, \quad II_n[\psi](\zeta) = n \sum_{j=1}^n \partial Q(\zeta_j) \cdot \psi(\zeta_j), \quad \text{and}$$

$$III_n[\psi](\zeta) = \sum_{j=1}^n \partial \psi(\zeta_j) \quad \text{for } \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n.$$

We now regard  $\zeta$  as picked randomly with respect to the Boltzmann–Gibbs distribution (1.1).  $W_n^+[\psi]$  is then a random variable; the Ward identity proved in [4, Section 4.1] states that its expectation vanishes:

$$\mathbf{E}_n W_n^+[\psi] = 0. \tag{3.1}$$

We shall now rescale in Ward’s identity about 0 at the microscopic scale  $r_n = n^{-1/2k}$ , given that the basic decomposition  $Q = Q_0 + ReH + Q_1$  in (1.6) holds. (We do not need to assume that 0 is in the bulk at this stage.)

To facilitate the calculations, it is convenient to recall a simple algebraic fact (see, e.g., [26]): if  $f$  is a function of  $p$  complex variables, and if  $f(\zeta_1, \dots, \zeta_p)$  is regarded as a random variable on the sample space  $\{\zeta_j\}_1^n$  with respect to the Boltzmann–Gibbs law, then the expectation is

$$\mathbf{E}_n [f(\zeta_1, \dots, \zeta_p)] = \frac{(n - p)!}{n!} \int_{\mathbb{C}^p} f \cdot \mathbf{R}_{n,p} \, dV_p,$$

where  $dV_p(\zeta_1, \dots, \zeta_p) = dA(\zeta_1) \cdots dA(\zeta_p)$ .

We rescale about 0 via  $z = r_n^{-1}\zeta$ ,  $w = r_n^{-1}\eta$ , recalling that the  $p$ -point functions transform as densities. We recall that  $R_{n,p}(z) = r_n^{2p} \mathbf{R}_{n,p}(\zeta)$  denotes the rescaled  $p$ -point function and use the abbreviation  $R_n = R_{n,1}$  for the one-point function. We also write

$$B_n(z, w) = \frac{R_n(z)R_n(w) - R_{n,2}(z, w)}{R_n(z)} = \frac{|K_n(z, w)|^2}{R_n(z)}, \tag{3.2}$$

$$C_n(z) = \int \frac{B_n(z, w)}{z - w} \, dA(w). \tag{3.3}$$

**Lemma 3.2** *We have that*

$$\bar{\partial} C_n(z) = R_n(z) - \Delta Q_0(z) - \Delta \log R_n(z) + o(1),$$

where  $o(1) \rightarrow 0$  uniformly on compact subsets of  $\mathbb{C}$  as  $n \rightarrow \infty$ .

*Proof* We fix a test function  $\psi \in C_0^\infty(\mathbb{C})$  and let  $\psi_n(\zeta) = \psi(r_n^{-1}\zeta)$ . The change of variables  $z = r_n^{-1}\zeta$  and  $w = r_n^{-1}\eta$  gives that

$$\begin{aligned} \mathbf{E}_n I_n[\psi_n] &= \int_{\mathbb{C}} \psi_n(\zeta) \, d(\zeta) \int_{\mathbb{C}} \frac{\mathbf{R}_{n,2}(\zeta, \eta)}{\zeta - \eta} \, dA(\eta) \\ &= r_n^{-1} \int_{\mathbb{C}} \psi(z) \, dA(z) \int_{\mathbb{C}} \frac{R_{n,2}(z, w)}{z - w} \, dA(w) \end{aligned}$$

and

$$\begin{aligned} \mathbf{E}_n II_n[\psi_n] &= n \int_{\mathbb{C}} \partial Q(\zeta) \psi_n(\zeta) \mathbf{R}_{n,1}(\zeta) \, dA(\zeta) \\ &= n \int_{\mathbb{C}} \partial Q(r_n z) \psi(z) R_{n,1}(z) \, dA(z). \end{aligned}$$

Likewise, changing variables and integrating by parts, we obtain

$$\begin{aligned} \mathbf{E}_n III_n[\psi_n] &= \int_{\mathbb{C}} \partial \psi_n(\zeta) \mathbf{R}_{n,1}(\zeta) \, dA(\zeta) = r_n^{-1} \int_{\mathbb{C}} \partial \psi(z) R_{n,1}(z) \, dA(z) \\ &= -r_n^{-1} \int_{\mathbb{C}} \psi(z) \partial R_{n,1}(z) \, dA(z). \end{aligned}$$

Hence, by the Ward identity in (3.1), we have

$$\begin{aligned} &\int_{\mathbb{C}} \psi(z) \, dA(z) \int_{\mathbb{C}} \frac{R_{n,2}(z, w)}{z - w} \, dA(w) \\ &= n r_n \int_{\mathbb{C}} \partial Q(r_n z) \psi(z) R_{n,1}(z) \, dA(z) + \int_{\mathbb{C}} \psi(z) \partial R_{n,1}(z) \, dA(z). \end{aligned}$$

Since  $\psi$  is an arbitrary test function, we have in the sense of distributions,

$$\int_{\mathbb{C}} \frac{R_{n,2}(z, w)}{z - w} \, dA(w) = n r_n \partial Q(r_n z) R_{n,1}(z) + \partial R_{n,1}(z).$$

Dividing through by  $R_{n,1}(z)$  and using the fact that

$$R_{n,2}(z, w) = R_{n,1}(z) (R_{n,1}(w) - B_n(z, w)),$$

we obtain

$$\int_{\mathbb{C}} \frac{R_{n,1}(w)}{z - w} \, dA(w) - \int_{\mathbb{C}} \frac{B_n(z, w)}{z - w} \, dA(w) = n r_n \partial Q(r_n z) + \partial \log R_{n,1}(z).$$

Differentiating with respect to  $\bar{z}$ , we get

$$R_{n,1}(z) - \bar{\partial} C_n(z) = n r_n^2 \Delta Q(r_n z) + \Delta \log R_{n,1}(z).$$

Since  $\Delta Q(r_n z) = r_n^{2(k-1)} \Delta Q_0(z) + O(r_n^{2k-1})$  uniformly on compact subsets of  $\mathbb{C}$  as  $n \rightarrow \infty$  and  $r_n = n^{-1/2k}$ , we obtain

$$\bar{\partial} C_n(z) = R_{n,1}(z) - \Delta Q_0(z) - \Delta \log R_{n,1}(z) + o(1),$$

where  $o(1) \rightarrow 0$  uniformly on compact subsets of  $\mathbb{C}$  as  $n \rightarrow \infty$ . □

### 3.2 The Proof of Theorem 1.5

We will need a few lemmas.

**Lemma 3.3** *If  $R(z_0) = 0$ , then there is a real analytic function  $\tilde{R}$  such that*

$$R(z) = |z - z_0|^2 \tilde{R}(z).$$

*If  $R$  does not vanish identically, then all zeros of  $R$  are isolated.*

*Proof* The assumption gives that the holomorphic kernel  $L$  corresponding to  $R$  satisfies  $L(z_0, z_0) = 0$ . Hence  $\int e^{-Q_0(w)} |L(z_0, w)|^2 dA(w) \leq 0$  by the mass-one inequality (2.5). Thus  $L(z_0, w) = 0$  for all  $w \in \mathbb{C}$ . Since  $L$  is Hermitian-entire, we can thus write

$$L(z, w) = (z - z_0)(w - z_0)^* \tilde{L}(z, w)$$

for some Hermitian-entire function  $\tilde{L}$ . We now have  $R(z) = |z - z_0|^2 \tilde{L}(z, z)e^{-Q_0(z)}$ .

For the second statement, we assume that  $R$  does not vanish identically and there exists a zero  $z_0$  of  $R$  which is not isolated. Then, we can take a sequence  $\{z_j\}_1^\infty$  of distinct zeros of  $R$  which converges to  $z_0$ , whence by the above argument, for each  $j$  we obtain  $L(z_j, w) = 0$  for all  $w \in \mathbb{C}$ . If we fix  $w$ , then  $L(z, w) = 0$  for all  $z \in \mathbb{C}$  since  $L(z, w)$  is holomorphic in  $z$ . Hence  $L = 0$  identically. □

**Lemma 3.4**  *$L(z, w)$  is a positive matrix and  $z \mapsto L(z, z)$  is logarithmically subharmonic.*

*Proof* It is clear that  $L$  is a positive matrix. Now write  $L_z(w) := L(w, z)$  and define a semi-definite inner product by  $\langle L_z, L_w \rangle_* := L(w, z)$  on the linear span of the functions  $L_z$  for  $z \in \mathbb{C}$ . The completion of this span forms a (perhaps semi-normed) Hilbert space  $\mathcal{H}_*$ , and  $L$  is a reproducing kernel of the space. Now when  $L(z, z) > 0$ ,

$$\Delta_z \log L(z, z) = \frac{L(z, z)\Delta L(z, z) - \partial_z L(z, z)\bar{\partial}_z L(z, z)}{L(z, z)^2}. \tag{3.4}$$

Since  $L(z, w)$  is Hermitian-entire, we have  $\bar{\partial}_z L_z \in \mathcal{H}_*$ ,  $\langle \bar{\partial}_z L_z, L_z \rangle_* = \bar{\partial}_z L(z, z)$ , and  $\langle \bar{\partial}_z L_z, \bar{\partial}_z L_z \rangle_* = \Delta L(z, z)$ . Hence, the numerator of (3.4) can be written as

$$\|L_z\|_*^2 \cdot \|\bar{\partial}L_z\|_*^2 - |\langle \bar{\partial}_z L_z, L_z \rangle_*|^2,$$

which is nonnegative by the Cauchy–Schwarz inequality.

At points where  $L(z, z) = 0$ ,  $\log L(z, z)$  satisfies the sub-mean value property since  $\log L(z, z) = -\infty$ . Hence the function  $\log L(z, z)$  is subharmonic on  $\mathbb{C}$ .  $\square$

**Lemma 3.5** *If  $R(z_0) = 0$  and  $R(z) = |z - z_0|^2 \tilde{R}(z)$ , then  $\Delta Q_0 + \Delta \log \tilde{R} \geq 0$  in a neighborhood of  $z_0$ .*

*Proof* We choose a small disc  $D = D(z_0, \epsilon)$  and consider the function

$$S(z) = \log \left( e^{Q_0(z)} \tilde{R}(z) \right).$$

Observing that  $\Delta_z \log L(z, z) = \Delta Q_0(z) + \Delta \log \tilde{R}(z) + \delta_{z_0}$  in the sense of distributions, Lemma 3.4 gives us that  $\Delta S \geq 0$  in the sense of distributions on  $D \setminus \{z_0\}$ . If  $\tilde{R}(z_0) > 0$ , we extend  $S$  analytically to  $z_0$ . On the other hand, if  $\tilde{R}(z_0) = 0$ , we define  $S(z_0) = -\infty$ . In both cases, the extended function  $S$  is subharmonic on  $D$ .  $\square$

We now turn to the left-hand side in the rescaled version of Ward’s identity, namely the function  $\bar{\partial}C_n$ , where  $C_n$  is the Cauchy transform of  $B_n$  (see (3.3)).

**Lemma 3.6** *Suppose that  $R = \lim R_{n_\ell}$  is a limiting 1-point function that does not vanish identically. Let  $Z$  be the set of isolated zeros of  $R$ , and let  $B(z, w) = \lim B_{n_\ell}(z, w)$  be the corresponding Berezin kernel for  $z \notin Z$ . Then  $C_{n_\ell} \rightarrow C$  locally uniformly on the complement  $Z^c = \mathbb{C} \setminus Z$  as  $\ell \rightarrow \infty$ , where the function*

$$C(z) = \int \frac{B(z, w)}{z - w} dA(w)$$

*is bounded on  $Z^c \cap V$  for each compact subset  $V$  of  $\mathbb{C}$ .*

*Proof* We have that  $c_{n_\ell} K_{n_\ell} \rightarrow K$  locally uniformly on  $\mathbb{C}^2$ , where  $K(z, z) = R(z) > 0$  when  $z \notin Z$ . Hence, for fixed  $\epsilon$  with  $0 < \epsilon < 1$ , we can choose  $N$  such that if  $\ell \geq N$ , then

$$|B_{n_\ell}(z, w) - B(z, w)| < \epsilon^2$$

for all  $z, w$  with  $|z| \leq 1/\epsilon, |w| \leq 2/\epsilon$ , and  $\text{dist}(z, Z) \geq \epsilon$ . Then, for  $z$  with  $|z| \leq 1/\epsilon$  and  $\text{dist}(z, Z) \geq \epsilon$ ,

$$\begin{aligned} |C_{n_\ell}(z) - C(z)| &\leq \left( \int_{|z-w|<1/\epsilon} + \int_{|z-w|>1/\epsilon} \right) \left| \frac{B_{n_\ell}(z, w) - B(z, w)}{z - w} \right| dA(w) \\ &\leq \epsilon^2 \int_{|z-w|<1/\epsilon} \frac{1}{|z - w|} dA(w) \\ &\quad + \epsilon \int |B_{n_\ell}(z, w) - B(z, w)| dA(w) \\ &\leq 4\epsilon. \end{aligned}$$

Here, we have used the mass-one inequality for the third inequality. Thus  $C_{n_\ell} \rightarrow C$  uniformly on compact subsets of  $Z^c$ .

Now fix a compact subset  $V$  of  $\mathbb{C}$ . Then, for all  $z, w$  with  $z \in V \setminus Z$  and  $\text{dist}(w, V) \leq 1$ ,

$$B_{n_\ell}(z, w) = \frac{|K_{n_\ell}(z, w)|^2}{K_{n_\ell}(z, z)} \leq K_{n_\ell}(w, w) \leq M$$

for some  $M = M_V$  that depends only on  $V$  by Lemma 2.1. Thus, for  $z \in V \setminus Z$ ,

$$\begin{aligned} |C_{n_\ell}| &\leq \left( \int_{|z-w|<1} + \int_{|z-w|>1} \right) \left| \frac{B_{n_\ell}(z, w)}{z-w} \right| dA(w) \\ &\leq M \int_{|z-w|<1} \frac{1}{|z-w|} dA(w) + \int B_{n_\ell}(z, w) dA(w) \leq 2M + 1. \end{aligned}$$

Hence we obtain  $|C(z)| \leq 2M + 1$  for  $z \in V \setminus Z$ . □

**Lemma 3.7** *If  $R$  does not vanish identically, the Ward’s equation*

$$\bar{\partial}C = R - \Delta Q_0 - \Delta \log R$$

*holds in the sense of distributions.*

*Proof* The preceding lemmas show that

$$\bar{\partial}C_n = R_n - \Delta Q_0 - \Delta \log R_n + o(1) \tag{3.5}$$

and that a subsequence  $C_{n_\ell}$  converges to  $C$  boundedly and locally uniformly on  $\mathbb{C} \setminus Z$ . Since  $Z \cap V$  is a finite set for each compact set  $V$ , it follows that  $C_{n_\ell} \rightarrow C$  in the sense of distributions, and hence  $\bar{\partial}C_{n_\ell} \rightarrow \bar{\partial}C$ . By Ward’s equation and the locally uniform convergence  $R_{n_\ell} \rightarrow R$ , it then follows that  $\Delta \log R_{n_\ell} \rightarrow \Delta \log R$  in the sense of distributions. We can thus pass to the limit as  $n_\ell \rightarrow \infty$  in the rescaled Ward identity (3.5). □

*Proof of Theorem 1.5* We follow the strategy in [4, Theorem 4.8]. Suppose that  $R(z_0) = 0$ . We must prove that  $R = 0$  identically.

Let  $D$  be a small disk centered at  $z_0$ , and write  $\chi = \chi_D$  for the characteristic function. Also write  $R(z) = |z - z_0|^2 \tilde{R}(z)$ .

Consider the measures  $\mu = \chi \cdot (\Delta Q_0 + \Delta \log R)$  and  $\nu = \chi \cdot (\Delta Q_0 + \Delta \log \tilde{R})$ . By lemmas 3.4 and 3.5, these measures are positive, and  $\mu = \delta_{z_0} + \nu$ . Write  $C^\mu(z) = \int_{\mathbb{C}} \frac{1}{z-w} d\mu(w)$  for the Cauchy transform of  $\mu$ . Clearly,

$$C^\mu(z) = \frac{1}{z - z_0} + C^\nu(z), \quad z \in D.$$

Also  $\bar{\partial}C^\nu = \nu \geq 0$ . When  $z \in D$ , the right-hand side in Ward’s equation equals  $R(z) - \Delta(Q_0 + \log R)(z) = R(z) - \bar{\partial}C^\mu(z)$ . If  $C(z) = \int \frac{B(z,w)}{z-w} dA(w)$ , we have, by Ward’s equation, that

$$\bar{\partial}(C + C^\mu)(z) = R(z).$$

Hence, by Weyl’s lemma,  $C(z) = -1/(z - z_0) - C^\nu(z) + v(z)$ , where  $v$  is smooth near  $z_0$ . If  $C^\mu(z)$  were bounded as  $z \rightarrow z_0$ , then the measure  $\mu = \nu + \delta_{z_0}$  would place no mass at  $\{z_0\}$ , so  $\nu = -\delta_{z_0} + \rho$ , where  $\rho(\{z_0\}) = 0$ . This contradicts that  $\nu \geq 0$ . The contradiction shows that  $|C(z)| \rightarrow \infty$  as  $z \rightarrow z_0$ . This in turn contradicts that  $C$  is bounded (Lemma 3.6), and hence  $R(z_0) = 0$  is impossible. Hence  $\Delta \log R$  is a smooth function on  $\mathbb{C}$ . Applying Weyl’s lemma to the distributional Ward equation  $\bar{\partial}C = R - \Delta Q_0 - \Delta \log R$  now shows that  $C(z)$  is smooth and hence that the equation holds pointwise on  $\mathbb{C}$ . □

### 4 Universality Results

In this section, we prove Theorems 1.3 and 1.4. The proof of Theorem 1.4 relies on certain apriori estimates, whose proofs are postponed to Sect. 6.

#### 4.1 Homogeneous Singularities

Assume that  $Q$  has a homogeneous singularity of type  $2k - 2$  at the origin, i.e., that the canonical decomposition is of the form  $Q = Q_0 + Re H$ ,  $H = c \zeta^{2k}$ , where  $Q_0$  is positively homogeneous of degree  $2k$ . As always, we write  $\mu_0$  for the measure  $d\mu_0 = e^{-Q_0} dA$ .

We now recall the kernel  $L_n$  (defined in (2.3)):

$$L_n(z, w) = k_n(z, w)e^{-H_n(z)/2 - \bar{H}_n(w)/2}, \quad (H_n(z) = nH(r_n z), \quad r_n = n^{-1/2k}).$$

In the present case,  $L_n(z, w) = k_n(z, w)e^{-c z^{2k}/2 - \bar{c} \bar{w}^{2k}/2}$ . By Lemma 2.2,  $L_n$  is the reproducing kernel for the space

$$\mathcal{H}_n = \{f(z) = q(z) \cdot e^{-c z^{2k}/2}; q \in \text{Pol}(n)\}$$

regarded as a subspace of  $L^2(\mu_0)$ . (This is because  $\tilde{\mu}_n = \mu_0$  for the measure  $\tilde{\mu}_n$  in (2.4).)

Since the spaces  $\mathcal{H}_n$  are increasing,  $\mathcal{H}_n \subset \mathcal{H}_{n+1}$ , where the inclusions are isometric, it follows that a unique limiting holomorphic kernel  $L = \lim L_n$  exists. By Theorem 1.7, the kernel  $L$  is the reproducing kernel for a contractively embedded subspace  $\mathcal{H}_*$  of  $L^2_a(\mu_0)$ , which must contain the dense subset  $U = \bigcup \mathcal{H}_n$ . Furthermore, by the reproducing property of  $L_n$ , we have for each element  $f(z) = q(z) \cdot e^{-c z^{2k}/2} \in U$

that  $\langle f, L_{n,z} \rangle_{L^2(\mu_0)} = f(z)$ , whenever  $n > \text{degree } q$ . It follows that

$$f(z) = \lim_{n \rightarrow \infty} \langle f, L_{n,z} \rangle_{L^2(\mu_0)} = \langle f, L_z \rangle_{L^2(\mu_0)}, \quad f \in U.$$

Since  $U$  is dense in  $L^2_a(\mu_0)$ ,  $L$  must equal to the reproducing kernel  $L_0$  of  $L^2_a(\mu_0)$ . The proof of Theorem 1.3 is complete.  $\square$

### 4.2 Rotational Symmetry

Referring to the canonical decomposition  $Q = Q_0 + Re H + Q_1$ , we now suppose that  $Q_0(z) = Q_0(|z|)$ , and we fix a rotationally symmetric limiting holomorphic kernel

$$L(z, w) = E(z\bar{w}).$$

Writing  $E(z) = \sum_0^\infty a_j z^j$ , the mass-one inequality

$$\int e^{-Q_0(w)} |L(z, w)|^2 dA(w) \leq L(z, z)$$

is seen to be equivalent to that

$$\sum |a_j|^2 |z|^{2j} \|w^j\|_{L^2(\mu_0)}^2 \leq \sum a_j |z|^{2j}. \tag{4.1}$$

To use Ward’s equation, we first compute the Cauchy transform  $C(z)$  as follows:

$$\begin{aligned} C(z) &= \frac{1}{L(z, z)} \int_{\mathbb{C}} \frac{e^{-Q_0(w)}}{z - w} |L(z, w)|^2 dA(w) \\ &= \frac{1}{E(|z|^2)} \sum_{j,k} a_j \bar{a}_k z^j \bar{z}^k \int_{\mathbb{C}} \frac{e^{-Q_0(w)}}{z - w} \bar{w}^j w^k dA(w) \\ &= \frac{1}{E(|z|^2)} \sum_{j,k} a_j \bar{a}_k z^j \bar{z}^k \frac{1}{\pi} \int_0^\infty e^{-Q_0(r)} r^{j+k} dr \int_0^{2\pi} \frac{e^{i(k-j)\theta}}{z/r - e^{i\theta}} d\theta. \end{aligned}$$

However, as is shown in [3], we have that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i(k-j)\theta}}{z/r - e^{i\theta}} d\theta = \begin{cases} -(z/r)^{k-j-1} & \text{if } |z| < r, \quad k - j \geq 1, \\ (z/r)^{k-j-1} & \text{if } |z| > r, \quad k - j \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\begin{aligned}
 C(z) &= \frac{2}{E(|z|^2)} \sum_{j,k} a_j \bar{a}_k z^j \bar{z}^k \\
 &\quad \left( \int_0^{|z|} e^{-Q_0(r)} r^{j+k} \left(\frac{z}{r}\right)^{k-j-1} \chi(k \leq j) \, dr \right. \\
 &\quad \left. - \int_{|z|}^\infty e^{-Q_0(r)} r^{j+k} \left(\frac{z}{r}\right)^{k-j-1} \chi(k \geq j+1) \, dr \right) \\
 &= A(z) - B(z),
 \end{aligned}$$

where

$$\begin{aligned}
 A(z) &= \frac{2}{E(|z|^2)} \sum_{j,k} a_j \bar{a}_k z^j \bar{z}^k \int_0^{|z|} e^{-Q_0(r)} r^{j+k} \left(\frac{z}{r}\right)^{k-j-1} \, dr, \\
 B(z) &= \frac{2}{E(|z|^2)} \sum_{j,k} a_j \bar{a}_k z^j \bar{z}^k \int_0^\infty e^{-Q_0(r)} r^{j+k} \left(\frac{z}{r}\right)^{k-j-1} \chi(k \geq j+1) \, dr.
 \end{aligned}$$

The term  $A(z)$  can be written as

$$\begin{aligned}
 A(z) &= \frac{1}{z E(|z|^2)} \sum_{j,k} a_j \bar{a}_k |z|^{2k} \int_0^{|z|^2} e^{-Q_0(\sqrt{r})} r^j \, dr \\
 &= \frac{1}{z} \int_0^{|z|^2} e^{-Q_0(\sqrt{r})} E(r) \, dr,
 \end{aligned}$$

which gives

$$\bar{\partial} A(z) = e^{-Q_0(z)} E(|z|^2) = R(z).$$

The term  $B(z)$  is computed as follows:

$$\begin{aligned}
 B(z) &= \frac{1}{E(|z|^2)} \sum_{k=1}^\infty \bar{a}_k z^{k-1} \bar{z}^k \sum_{j=0}^{k-1} a_j \int_0^\infty e^{-Q_0(\sqrt{r})} r^j \, dr \\
 &= \frac{1}{E(|z|^2)} \sum_{k=1}^\infty \bar{a}_k z^{k-1} \bar{z}^k \sum_{j=0}^{k-1} a_j \|z^j\|_{Q_0}^2.
 \end{aligned}$$

Noting that

$$\partial_z \log L(z, z) = \frac{\partial_z E(|z|^2)}{E(|z|^2)} = \frac{1}{E(|z|^2)} \sum_{k=1}^\infty k \bar{a}_k z^{k-1} \bar{z}^k,$$



we infer that Ward’s equation

$$\bar{\partial}A - \bar{\partial}B = R - \Delta_z \log L(z, z)$$

is equivalent to that  $\bar{\partial}(B - \partial_z \log L(z, z)) = 0$ . This in turn, is equivalent to that the function

$$\frac{1}{E(|z|^2)} \sum_{k=1}^{\infty} \bar{a}_k z^{k-1} \bar{z}^k \left( k - \sum_{j=0}^{k-1} a_j \|z^j\|_{L^2(\mu_0)}^2 \right)$$

be entire. It is easy to check that this is the case if and only if all coefficients in the sum vanish, that is, if and only if for each  $k \geq 1$ , we have that

$$a_k = 0 \quad \text{or} \quad \sum_{j=0}^{k-1} a_j \|z^j\|_{L^2(\mu_0)}^2 = k. \tag{4.2}$$

We now apply the growth estimate in Theorem 1.6, part (ii), which says that

$$E(|z|^2) = \Delta Q_0(z) e^{Q_0(z)} (1 + o(1)) \quad \text{as } z \rightarrow \infty. \tag{4.3}$$

We claim that this implies the second alternative in (4.2).

Indeed, (4.3) is clearly not satisfied if  $E$  is constant. Next note that the mass-one inequality (4.1) and the zero-one law (Theorem 1.5) imply that  $0 < a_0 \leq 1/\|1\|_{L^2(\mu_0)}^2$ . Since  $E(z)$  is not a polynomial by (4.3), for any  $k$  there exists  $N \geq k$  such that  $a_N \neq 0$ . By (4.2), we obtain that if  $a_N \neq 0$  but  $a_j = 0$  for all  $j$  with  $1 \leq j \leq N - 1$ , then  $N = 1$  and  $a_0 = 1/\|1\|_{L^2(\mu_0)}^2$ . By a simple induction, we then have  $a_k = 1/\|z^k\|_{L^2(\mu_0)}^2$  for all  $k \geq 0$ . Thus, we have

$$E(z) = \sum_{j=0}^{\infty} \frac{1}{\|z^j\|_{L^2(\mu_0)}^2} z^j.$$

Since the polynomial  $\phi_j(z) = z^j/\|z^j\|_{L^2(\mu_0)}$  is the  $j$ :th orthonormal polynomial with respect to the measure  $\mu_0$ , we have

$$L(z, w) = E(z\bar{w}) = \sum_{j=0}^{\infty} \phi_j(z)\bar{\phi}_j(w) = L_0(z, w),$$

where  $L_0$  is the Bergman kernel for the space  $L_a^2(\mu_0)$ . The proof is complete. □

### 5 Asymptotics for $L_0(z, z)$

In this section, we prove part (i) of Theorem 1.6.

To this end, let  $A_0(z, w)$  be the Hermitian polynomial such that  $A_0(z, z) = Q_0(z)$ , and write

$$L_0^\sharp(z, w) = [\partial_1 \bar{\partial}_2 A_0](z, w) \cdot e^{A_0(z, w)}.$$

We write  $L_z^\sharp(w)$  for  $L_0^\sharp(w, z)$  and, for suitable functions  $u$ ,

$$\pi^\sharp u(z) = \langle u, L_z^\sharp \rangle_{L^2(\mu_0)} = \int_{\mathbb{C}} u \bar{L}_z^\sharp e^{-Q_0} dA.$$

Below, we fix a  $z$  with  $|z|$  large enough; we must estimate  $L_0(z, z)$ . We also fix a number  $\delta_0 = \delta_0(z) > 0$  and write  $\chi_z$  for a fixed  $C^\infty$ -smooth test-function with  $\chi_z(w) = 1$  when  $|w - z| \leq \delta_0$  and  $\chi_z(w) = 0$  when  $|w - z| \geq 2\delta_0$ .

We will use the following estimate.

**Lemma 5.1** *If  $|1 - w/z|$  is sufficiently small, then  $2\operatorname{Re} A_0(z, w) \leq Q_0(z) + Q_0(w) - c|z|^{2k-2}|w - z|^2$ , where  $c$  is a positive constant.*

*Proof* We write  $h = w - z$ . By Taylor’s formula,  $A_0(w, z) = Q_0(z) + \sum_1^{2k} \frac{\partial^j Q_0(z)}{j!} h^j$ . Similarly,  $A_0(w, w) = Q_0(z) + \sum_{i+j \geq 1} \frac{\partial^i \bar{\partial}^j Q_0(z)}{i!j!} h^i \bar{h}^j$ . Hence

$$2\operatorname{Re} A_0(z, w) - Q_0(z) - Q_0(w) + \Delta Q_0(z)|h|^2 = - \sum_{i, j \geq 1, i+j \geq 3} \frac{\partial^i \bar{\partial}^j Q_0(z)}{i!j!} h^i \bar{h}^j. \tag{5.1}$$

However, since  $Q_0$  is homogeneous of degree  $2k$ , the derivative  $\partial^i \bar{\partial}^j Q_0$  is homogeneous of degree  $2k - i - j$ . Hence

$$\left| \partial^i \bar{\partial}^j Q_0(z) \right| |w - z|^{i+j} \leq C|z|^{2k-2}|w - z|^2 |1 - w/z|^{i+j-2}.$$

Thus, if  $i + j \geq 3$  and  $|1 - w/z|$  is sufficiently small, then the left-hand side in (5.1) is dominated by an arbitrarily small multiple of  $|z|^{2k-2}|z - w|^2$ . On the other hand, by homogeneity and positive definiteness of  $\Delta Q_0$ , we have that  $\Delta Q_0(z)|z - w|^2 \geq c'|z|^{2k-2}|z - w|^2$ , where  $c'$  is a positive constant. The lemma thus follows with any positive constant  $c < c'$  □

As always, we write  $d\mu_0 = e^{-Q_0} dA$ ;  $L_a^2(\mu_0)$  denotes the associated Bergman space of entire functions, and  $L_0$  is the Bergman kernel of that space.

**Lemma 5.2** *Let  $|z| \geq 1$  and  $\delta_0$  be a positive number with  $\delta_0/|z|$  sufficiently small. Then there is a constant  $C = C(\delta_0)$  such that, for all functions  $u \in L_a^2(\mu_0)$ ,*

$$\left| u(z) - \pi^\sharp[\chi_z u](z) \right| \leq C \|u\|_{L^2(\mu_0)} (\delta_0^{-1} + 1) e^{Q_0(z)/2}.$$

*Proof* Note that

$$\begin{aligned} \pi^\sharp[\chi_z u](z) &= \int_{\mathbb{C}} \chi_z(w) u(w) [\partial_1 \bar{\partial}_2 A_0](z, w) \cdot e^{A_0(z,w) - A_0(w,w)} dA(w) \\ &= - \int_{\mathbb{C}} \frac{u(w) \chi_z(w) F(z, w)}{w - z} \bar{\partial}_w \left[ e^{A_0(z,w) - A_0(w,w)} \right] dA(w), \end{aligned} \tag{5.2}$$

where

$$F(z, w) = \frac{(w - z) [\partial_1 \bar{\partial}_2 A_0](z, w)}{\bar{\partial}_2 A_0(w, w) - \bar{\partial}_2 A_0(z, w)}. \tag{5.3}$$

Now fix  $w$ . The denominator  $P(z) = \bar{\partial}_2 A_0(w, w) - \bar{\partial}_2 A_0(z, w)$  is by Taylor’s formula equal to the polynomial

$$-\Delta Q_0(w) \cdot (z - w) - \frac{\partial \Delta Q_0(w)}{2} \cdot (z - w)^2 - \dots - \frac{\partial^{k-1} \Delta Q_0(w)}{k!} \cdot (z - w)^k.$$

Here the derivative  $\partial^j \Delta Q_0(w) = |w|^{2k-2-j} \partial^j \Delta Q_0(w/|w|)$  is positively homogeneous of degree  $2k - 2 - j$ . Set  $c(w) = \Delta Q_0(w/|w|)$ . We then have that

$$P(z) = c(w) |w|^{2k-2} \cdot (w - z) + O((w - z)^2), \quad (z \rightarrow w).$$

Since also  $\partial_1 \bar{\partial}_2 A_0(z, w) = c(z) |z|^{2k-2} (1 + O(w - z))$ , we have by (5.3),

$$F(z, w) = 1 + O(w - z), \quad (w \rightarrow z). \tag{5.4}$$

By the form of  $F$  it is also clear that

$$\bar{\partial}_2 F(z, w) = O(w - z), \quad (w \rightarrow z). \tag{5.5}$$

An integration by parts in (5.2) gives  $\pi^\sharp[\chi_z u](z) = u(z) + \epsilon_1 + \epsilon_2$ , where

$$\begin{aligned} \epsilon_1 &= \int \frac{u(w) \bar{\partial} \chi_z(w) F(z, w)}{w - z} e^{A_0(z,w) - A_0(w,w)} dA(w), \\ \epsilon_2 &= \int \frac{u(w) \chi_z(w) \bar{\partial}_2 F(z, w)}{w - z} e^{A_0(z,w) - A_0(w,w)} dA(w). \end{aligned}$$

Inserting the estimates (5.4) and (5.5), using also that  $\bar{\partial} \chi_z(w) = 0$  when  $|w - z| \leq \delta_0$ , we find that

$$\begin{aligned} |\epsilon_1| &\leq C \delta_0^{-1} \int |u(w)| |\bar{\partial} \chi_z(w)| e^{Re A_0(z,w) - Q_0(w)} dA(w), \\ |\epsilon_2| &\leq C \int \chi_z(w) |u(w)| e^{Re A_0(z,w) - Q_0(w)} dA(w). \end{aligned}$$

To estimate  $\epsilon_1$ , we use Lemma 5.1 to get

$$e^{Re A_0(z,w) - Q_0(w)/2} \leq C e^{Q_0(z)/2 - c|z-w|^2}. \tag{5.6}$$

This gives

$$\begin{aligned} |\epsilon_1| e^{-Q_0(z)/2} &\leq C \delta_0^{-1} \int |u(w)| |\bar{\partial} \chi_z(w)| e^{-Q_0(w)/2} dA(w) \\ &\leq C \delta_0^{-1} \|u\|_{L^2(\mu_0)} \|\bar{\partial} \chi_z\|_{L^2} \leq C' \|u\|_{L^2(\mu_0)}. \end{aligned}$$

To estimate  $\epsilon_2$ , we note that (again by (5.6))

$$|\epsilon_2| e^{-Q_0(z)/2} \leq C \|u\|_{L^2(\mu_0)} \left( \int_{|w-z| \leq 2\delta_0} e^{-c|z-w|^2} dA(w) \right)^{1/2} \leq C \|u\|_{L^2(\mu_0)}.$$

The proof is complete. □

Let  $\pi_0 : L^2(\mu_0) \rightarrow L^2_a(\mu_0)$  be the Bergman projection,  $\pi_0[f](z) = \langle f, L_z \rangle_{L^2(\mu_0)}$ , where we write  $L_z(w)$  for  $L_0(w, z)$ . Noting that

$$(\pi^\sharp[\chi_z L_z](z))^* = \langle \chi_z L_z, L_z^\sharp \rangle^* = \langle \chi_z L_z^\sharp, L_z \rangle = \pi_0[\chi_z L_z^\sharp](z),$$

we see that

$$|L_z(z) - \pi_0[\chi_z L_z^\sharp](z)| = |L_z(z) - \pi^\sharp[\chi_z L_z](z)|.$$

If we now choose  $u = L_z$  in Lemma 5.2 and recall that  $\|L_z\|_{L^2(\mu_0)}^2 = L_0(z, z)$ , we obtain the estimate

$$|L_0(z, z) - \pi_0[\chi_z L_z^\sharp](z)| \leq C \sqrt{L_0(z, z)} \cdot e^{Q_0(z)/2}, \quad |z| \geq 1. \tag{5.7}$$

**Lemma 5.3** *There is a constant  $C$  such that for all  $|z| \geq 1$  and all  $\delta_0 = \delta_0(z) > 0$  with  $\delta_0/|z|$  small enough,*

$$\left| \Delta Q_0(z) e^{Q_0(z)} - \pi_0[\chi_z L_z^\sharp](z) \right| \leq C |z|^{k-1} e^{Q_0(z)}.$$

*Proof* Consider the function  $u_0 = \chi_z L_z^\sharp - \pi_0[\chi_z L_z^\sharp]$ . This is the norm-minimal solution in  $L^2(\mu_0)$  to the problem  $\bar{\partial} u = (\bar{\partial} \chi_z) \cdot L_z^\sharp$ .

Since  $Q_0$  is strictly subharmonic on the support of  $\chi_z$ , we can apply the standard Hörmander estimate (e.g., [19, p. 250]) to obtain

$$\begin{aligned} \|u\|_{L^2(\mu_0)}^2 &\leq \int_{\mathbb{C}} |\bar{\partial}\chi_z|^2 |L_z^\sharp|^2 \frac{e^{-Q_0}}{\Delta Q_0} dA \\ &\leq C |z|^{-(2k-2)} \|\bar{\partial}\chi_z\|_{L^2}^2 \\ &\quad \sup_{\delta_0 \leq |w-z| \leq 2\delta_0} |[\partial_1 \bar{\partial}_2 A_0](z, w)|^2 e^{2\operatorname{Re} A_0(z, w) - A_0(w, w)}, \end{aligned}$$

where we used homogeneity of  $\Delta Q_0$ .

By Taylor’s formula and the estimate (5.6), we have when  $\delta_0 \leq |w - z| \leq 2\delta_0$ ,

$$|[\partial_1 \bar{\partial}_2 A_0](z, w)|^2 e^{2\operatorname{Re} A_0(z, w) - A_0(w, w)} \leq C \Delta Q_0(z)^2 e^{Q_0(z) - 2c|z|^{2k-2}|z-w|^2}.$$

By the homogeneity of  $\Delta Q_0$ , we thus obtain the estimate

$$\|u\|_{L^2(\mu_0)} \leq C |z|^{k-1} e^{Q_0(z)/2 - c'\delta_0^2|z|^{2k-2}}. \tag{5.8}$$

We now pick another (small) number  $\delta > 0$  and invoke the following pointwise- $L^2$  estimate (see, e.g., [4, Lemma 3.1] or the proof of the inequality (2.2)):

$$|u(z)|^2 e^{-Q_0(z)} \leq C e^{c''\delta \Delta Q_0(z)|z|} \delta^{-2} \int_{D(z, \delta)} |u(w)|^2 e^{-Q_0(w)} dA(w). \tag{5.9}$$

Combining with (5.8), this gives

$$|u(z)|^2 e^{-Q_0(z)} \leq C \delta^{-2} e^{-c'\delta_0^2|z|^{2k-2} + c''\delta|z|^{2k-1}} |z|^{2k-2} e^{Q_0(z)}.$$

Choosing  $\delta_0$  a small multiple of  $|z|^{1/2}$  and then  $\delta$  small enough, we insure that the right-hand side is dominated by  $C|z|^{2k-2}e^{Q_0(z)}$ , as desired. □

*Proof of Part (i) of Theorem 1.6* By the estimate (5.7) and Lemma 5.3, we have

$$\left| \Delta Q_0(z) e^{Q_0(z)} - L_0(z, z) \right| \leq C_1 \sqrt{L_0(z, z)} e^{Q_0(z)/2} + C_2 |z|^{k-1} e^{Q_0(z)}.$$

Writing  $R_0(z) = L_0(z, z)e^{-Q_0(z)}$ , this becomes

$$\left| |z|^{2k-2}c(z) - R_0(z) \right| \leq C_1 \sqrt{R_0(z)} + C_2 |z|^{k-1}, \quad c(z) = \Delta Q_0(z/|z|). \tag{5.10}$$

We must prove that the left-hand side in (5.10) is dominated by  $M|z|^{1-k} \Delta Q_0(z)$  for all large  $|z|$ , where  $M$  is a suitable constant. If this is false, there are two possibilities. If  $R_0(z) \leq (1 - M|z|^{1-k})\Delta Q_0(z)$  for arbitrarily large  $|z|$ , then (5.10) implies

$$M|z|^{k-1}c(z) \leq C_1 \sqrt{R_0(z)} + C_2 |z|^{k-1} \leq (C'_1 + C_2)|z|^{k-1},$$

and we reach a contradiction for large enough  $M$ .

In the remaining case, we have  $R_0(z) \geq (1 + M|z|^{1-k})\Delta Q_0(z)$ . Then (5.10) gives the estimate  $R_0(z) \geq cM^2|z|^{2k-2}$  for some  $c > 0$ . Since  $\Delta Q_0(z) \leq c'|z|^{2k-2}$  for some  $c' > 0$ , we obtain

$$R_0(z) - \Delta Q_0(z) \geq (cM^2 - c')|z|^{2k-2}.$$

Choosing  $M$  large enough, we obtain  $R_0(z) \geq C_3M|z|^{4k-4}$  by (5.10) again. Repeating the above argument gives  $R_0(z) \geq C_pM|z|^{2p}$  for all sufficiently large  $|z|$  for some constant  $C_p > 0$ . On the other hand, we will show that

$$R_0(z) \leq C(1 + |z|^{4k-2}) \tag{5.11}$$

for all  $z$ , which will give the desired contradiction. To see this, note that for functions  $u \in L_a^2(\mu_0)$ , the estimate (5.9) gives

$$|u(z)|^2 e^{-Q_0(z)} \leq C\delta^{-2} e^{C|z|^{2k-1}\delta} \|u\|_{L^2(\mu_0)}^2, \quad (|z| \geq 1, 0 < \delta < 1).$$

Taking  $\delta = |z|^{1-2k}$ , we obtain  $|u(z)|^2 \leq C|z|^{4k-2} e^{Q_0(z)} \|u\|_{L^2(\mu_0)}^2$ . Since

$$L_0(z, z) = \sup\{|u(z)|^2 ; u \in L_a^2(\mu_0), \|u\|_{L^2(\mu_0)} \leq 1\},$$

we now obtain the estimate (5.11). □

### 6 Apriori Estimates for the One-Point Function

In this section, we prove part (ii) of Theorem 1.6.

As before, we write  $Q = Q_0 + Re H + Q_1$  for the canonical decomposition of  $Q$  at 0, and we write  $\mu_0$  for the measure  $d\mu_0 = e^{-Q_0} dA$ . In this section, the assumption that 0 is in the bulk of the droplet will become important.

Our arguments below essentially follow by adaptation of the previous section.

Fix a point  $\zeta$  in a small neighborhood of 0 with  $|\zeta| \geq r_n$ . We also fix a number  $\delta_0 = \delta_0(\zeta) \geq \text{const.} > 0$  with  $\delta_0(\zeta) \cdot r_n/|\zeta|$  uniformly small, and a smooth function  $\psi$  with  $\psi = 1$  in  $D(0, \delta_0)$  and  $\psi = 0$  outside  $D(0, 2\delta_0)$ . We define a function  $\chi_\zeta = \chi_{\zeta,n}$  by

$$\chi_\zeta(\omega) = \psi((\omega - \zeta)/r_n).$$

Let  $A(\eta, \omega)$  be a Hermitian-analytic function in a neighborhood of  $(0, 0)$ , satisfying  $A(\eta, \eta) = Q(\eta)$ . We shall essentially apply the definition of the approximating kernel (denoted by  $L_0^\sharp$  in the preceding section) with “ $A_0$ ” replaced by “ $nA$ .” We denote this kernel by  $L_n^\sharp$ , viz.

$$L_n^\sharp(\zeta, \eta) = n \partial_1 \bar{\partial}_2 A(\zeta, \eta) \cdot e^{nA(\zeta, \eta)}.$$

The corresponding “approximate projection” is defined on suitable functions  $u$  by

$$\pi_n^\sharp u(\zeta) = \langle u, \mathbf{L}_\zeta^\sharp \rangle_{L^2(\mu_n)}, \quad d\mu_n = e^{-nQ} dA,$$

where, for convenience, we write  $\mathbf{L}_\zeta^\sharp$  instead of  $\mathbf{L}_{n,\zeta}^\sharp$ .

**Lemma 6.1** *Suppose that  $u$  is holomorphic in a neighborhood of  $\zeta$  and  $\delta_0(\zeta) \cdot r_n / |\zeta| \leq \varepsilon_0$  (small enough). Then there is a constant  $C = C(\varepsilon_0)$  such that, when  $r_n \leq |\zeta| \leq r_n \log n$ ,*

$$|u(\zeta) - \pi_n^\sharp[\chi_\zeta u](\zeta)| \leq C(1 + (\delta_0 r_n)^{-1}) \|u\|_{L^2(\mu_n)} e^{nQ(\zeta)/2}.$$

*Proof* It will be sufficient to indicate how the proof of Lemma 5.2 is modified in the present setting. We start as earlier, by writing

$$\pi_n^\sharp[\chi_\zeta f](\zeta) = - \int \frac{u(\omega) \chi_\zeta(\omega) F(\zeta, \omega)}{\omega - \zeta} \bar{\partial}_\omega \left[ e^{-n(A(\omega, \omega) - A(\zeta, \omega))} \right] dA(\omega),$$

where

$$F(\zeta, \omega) = \frac{(\omega - \zeta) \partial_1 \bar{\partial}_2 A(\zeta, \omega)}{\bar{\partial}_2 A(\omega, \omega) - \bar{\partial}_2 A(\zeta, \omega)}.$$

Here, we may replace “ $A$ ” by “ $A_0$ ” to within negligible terms, for the relevant  $\zeta$  and  $\omega$ . More precisely, Taylor’s formula gives that

$$\bar{\partial}_2 A(\omega, \omega) - \bar{\partial}_2 A(\zeta, \omega) = \Delta Q_0(\omega)(1 + O(r_n \log n)) \cdot (\omega - \zeta) + O((\omega - \zeta)^2), \tag{6.1}$$

$$\partial_1 \bar{\partial}_2 A(\zeta, \omega) = \partial_1 \bar{\partial}_2 A_0(\zeta, \omega)(1 + O(r_n \log n)), \tag{6.2}$$

when  $r_n \leq |\zeta| \leq r_n \log n$  and  $|\omega - \zeta| \leq 2\delta_0 r_n$ .

From (6.1) and the form of  $F$ , we see (as in the proof of Lemma 5.2) that

$$F(\zeta, \omega) = 1 + O(\zeta - \omega), \quad \bar{\partial}_2 F(\zeta, \omega) = O(\omega - \zeta). \tag{6.3}$$

We continue to write  $\pi_n^\sharp u(\zeta) = u(\zeta) + \epsilon_1 + \epsilon_2$ , where

$$\begin{aligned} \epsilon_1 &= \int \frac{u(\omega) \cdot \bar{\partial} \chi_\zeta(\omega) \cdot F(\zeta, \omega)}{\omega - \zeta} e^{-n[A(\omega, \omega) - A(\zeta, \omega)]} dA(\omega), \\ \epsilon_2 &= \int \frac{u(\omega) \cdot \chi_\zeta(\omega) \cdot \bar{\partial}_2 F(\zeta, \omega)}{\omega - \zeta} e^{-n(A(\omega, \omega) - A(\zeta, \omega))} dA(\omega). \end{aligned}$$

To estimate  $\epsilon_1$  and  $\epsilon_2$ , we note that there is a positive constant  $c$  such that

$$e^{-n(Q_0(\omega)/2 - \text{Re } A_0(\zeta, \omega))} \leq C e^{nQ_0(\zeta)/2 - cn|\zeta|^{2k-2}|\zeta - \omega|^2}, \quad |\omega - \zeta| \leq 2\delta_0 r_n. \tag{6.4}$$

See Lemma 5.1.

Inserting the estimates in (6.3) and (6.4), using also that  $\bar{\partial}\chi_\zeta(w) = 0$  when  $|\zeta - w| \leq \delta_0 r_n$ , we find that if  $|\zeta| \geq r_n$ ,

$$\begin{aligned} |\epsilon_1| e^{-nQ(\zeta)/2} &\leq C\delta_0^{-1}r_n^{-1} \int |u(\omega)| |\bar{\partial}\chi_\zeta(\omega)| e^{-nQ(\omega)/2} dA(\omega), \\ |\epsilon_2| e^{-nQ(\zeta)/2} &\leq C \int \chi_\zeta(\omega) |u(\omega)| e^{-nQ(\omega)/2} e^{-cnr_n^{2k-2}|\zeta-\omega|^2} dA(\omega). \end{aligned}$$

Using the Cauchy–Schwarz inequality, we find now that

$$(|\epsilon_1| + |\epsilon_2|) e^{-nQ(\zeta)/2} \leq C(1 + \delta_0^{-1}r_n^{-1}) \|u\|_{L^2(\mu_n)}.$$

The proof is complete. □

Choosing  $u(\eta) = \mathbf{k}_n(\eta, \zeta)$ , where  $\mathbf{k}_n$  is the Bergman kernel for the subspace  $\mathcal{P}_n$  of  $L^2(\mu_n)$ , we obtain the following estimate, valid when  $r_n \leq |\zeta| \leq r_n \log n$ :

$$\left| \mathbf{k}_n(\zeta, \zeta) - \pi_n \left[ \chi_\zeta \mathbf{L}_\zeta^\sharp \right] (\zeta) \right| \leq Cr_n^{-1} \sqrt{\mathbf{k}_n(\zeta, \zeta)} \cdot e^{nQ(\zeta)/2}. \tag{6.5}$$

Here  $\pi_n : L^2(\mu_n) \rightarrow \mathcal{P}_n$  is the orthogonal projection,  $\pi_n u(\zeta) = \langle u, \mathbf{k}_{n,\zeta} \rangle_{L^2(\mu_n)}$ . (Cf. (5.7) for details on the derivation of equation (6.5) from Lemma 6.1.)

**Lemma 6.2** *For all  $\zeta$  in the annulus  $r_n \leq |\zeta| \leq \log n \cdot r_n$ , and for  $\delta_0(\zeta) \cdot r_n$  a small enough multiple of  $|\zeta|$ , we have the estimate*

$$\left| \pi_n \left[ \chi_\zeta \mathbf{L}_\zeta^\sharp \right] (\zeta) - n\Delta Q(\zeta) e^{nQ(\zeta)} \right| \leq C\sqrt{nr_n}^{-1} |\zeta|^{k-1} e^{nQ(\zeta)}.$$

*Proof* Let  $u_0 = \chi_\zeta \mathbf{L}_\zeta^\sharp - \pi_n \left[ \chi_\zeta \mathbf{L}_\zeta^\sharp \right]$  be the norm-minimal solution in  $L^2(\mu_n)$  to the problem  $\bar{\partial}u_0 = \bar{\partial}f$ , where  $f = \chi_\zeta \mathbf{L}_\zeta^\sharp$ . We will prove that the problem  $\bar{\partial}u = \bar{\partial}f$  has a solution  $u$  with  $u - f \in \text{Pol}(n)$  and

$$\|u\|_{L^2(\mu_n)} \leq Cn^{-1/2} |\zeta|^{-(k-1)} \left\| \bar{\partial} \left[ \chi_\zeta \mathbf{L}_\zeta^\sharp \right] \right\|_{L^2(\mu_n)}. \tag{6.6}$$

This is done by a standard device, which now we briefly recall.

Let  $\check{Q}$  be the ‘‘obstacle function’’ pertaining to  $Q$ . The main facts about this function to be used here are the following (cf. [28] for details). The obstacle function can be defined as  $\check{Q} = \gamma - 2U^\sigma$ , where  $U^\sigma$  is the logarithmic potential of the equilibrium measure and  $\gamma$  is a constant chosen so that  $\check{Q} = Q$  on  $S$ . One has that  $\check{Q}$  is harmonic outside  $S$ , and that its gradient is Lipschitz continuous on  $\mathbb{C}$ . Furthermore,  $\check{Q}(\omega)$  grows like  $2 \log |\omega| + O(1)$  as  $\omega \rightarrow \infty$ .

We use the obstacle function to form the strictly subharmonic function  $\phi(\omega) = \check{Q}(\omega) + n^{-1} \log(1 + |\omega|^2)$ , and we go on to define a measure  $\mu'_n$  by  $d\mu'_n(\omega) = e^{-n\phi(\omega)} dA(\omega)$ . Write  $\mathcal{P}'_n$  for the subspace of  $L^2(\mu'_n)$  of holomorphic polynomials of



degree at most  $n - 1$ , and let  $\pi'_n$  be the corresponding orthogonal projection. Finally, we write

$$v_0 = f - \pi'_n f.$$

Since  $\phi$  is now strictly subharmonic, the standard Hörmander estimate can be applied. It gives

$$\|v_0\|_{L^2(\mu'_n)}^2 \leq \int_{\mathbb{C}} |\bar{\partial} f|^2 \frac{e^{-n\phi}}{n\Delta\phi} dA.$$

Since  $\chi_\zeta$  is supported in the disk  $D(\zeta, 2\delta_0 r_n)$ , and since  $\Delta\check{Q} = \Delta Q = \Delta Q_0 \cdot (1 + o(1))$  there, we see that

$$\|v_0\|_{L^2(\mu'_n)} \leq Cn^{-1/2} |\zeta|^{-(k-1)} \|\bar{\partial} f\|_{L^2(\mu_n)}.$$

Next we use the estimate  $n\phi \leq nQ + \text{const.}$  which holds by the growth assumption on  $Q$  near infinity. This gives  $\|v_0\|_{L^2(\mu_n)} \leq C\|v_0\|_{L^2(\mu'_n)}$ , and so we have shown (6.6) with  $u = v_0$ .

Since  $n\phi(\omega) = (n+1) \log |\omega|^2 + O(1)$  as  $\omega \rightarrow \infty$ , we have that  $L_a^2(\mu'_n) = \text{Pol}(n)$ . Hence  $u = v_0$  solves, in addition to (6.6), the problem

$$\bar{\partial} u = \bar{\partial} f \quad \text{and} \quad u - f \in \text{Pol}(n).$$

Using the form of  $\bar{\partial} f = \bar{\partial} \chi_\zeta \cdot \mathbf{L}_\zeta^\sharp$  and the estimate (6.4), we find that for  $|\omega - \zeta| \leq \delta_0 r_n$ ,

$$|\bar{\partial} u(\omega)|^2 e^{-nQ(\omega)} \leq C(n\Delta Q_0(\zeta))^2 |\bar{\partial} \chi_\zeta(\omega)|^2 e^{nQ(\zeta) - 2nc|\zeta|^{2k-2} |\omega - \zeta|^2}.$$

By the homogeneity of  $\Delta Q_0$  and the fact that  $\bar{\partial} \chi_\zeta = 0$  when  $|\omega - \zeta| \leq \delta_0 r_n$ , this gives the estimate

$$\|\bar{\partial} f\|_{L^2(\mu_n)} \leq Cn |\zeta|^{2k-2} e^{nQ(\zeta)/2} e^{-cn|\zeta|^{2k-2} (\delta_0 r_n)^2}.$$

Applying (6.6), we now get

$$\|u\|_{L^2(\mu_n)} \leq C\sqrt{n} |\zeta|^{k-1} e^{nQ(\zeta)/2} e^{-cn(\delta_0 r_n)^2 |\zeta|^{2k-2}}. \tag{6.7}$$

We now pick a small constant  $\delta$  (independent of  $n$ ) and use the pointwise- $L^2$  estimate

$$|u(\zeta)|^2 e^{-nQ(\zeta)} \leq C(r_n \delta)^{-2} e^{c'n r_n \delta |\zeta|^{2k-1}} \|u\|_{L^2(\mu_n)}^2.$$

Choosing  $\delta_0 r_n$  as a small multiple of  $|\zeta|$  and then  $\delta$  small enough, we can now use (6.7) to deduce that

$$|u(\zeta)| e^{-nQ(\zeta)/2} \leq Cr_n^{-1} \sqrt{n} |\zeta|^{k-1} e^{nQ(\zeta)/2},$$

finishing the proof.  $\square$

*Proof of Theorem 1.6, part (ii)* Fix  $\varepsilon > 0$  and take  $\zeta$  with  $r_n \leq |\zeta| \leq \log n \cdot r_n$ . By the estimate (6.5) and Lemma 6.2, we have for all large  $n$  that

$$|\mathbf{R}_n(\zeta) - n\Delta Q_0(\zeta)| \leq C_1 r_n^{-1} \sqrt{\mathbf{R}_n(\zeta)} + C_2 r_n^{-k-1} |\zeta|^{k-1}$$

for some constants  $C_1, C_2$ . Multiplying through by  $r_n^2$  and writing  $R_n(z) = r_n^2 \mathbf{R}_n(\zeta)$ ,  $z = r_n^{-1} \zeta$ , we get

$$|R_n(z) - \Delta Q_0(z)| \leq C_1 \sqrt{R_n(z)} + C_2 |z|^{k-1}.$$

It follows that each limiting 1-point function  $R$  must satisfy

$$\left| R(z) - c(z) |z|^{2k-2} \right| \leq C_1 \sqrt{R(z)} + C_2 |z|^{k-1}, \quad |z| \geq 1,$$

where  $c(z) = \Delta Q_0(z/|z|) > 0$ . The proof of part (i) of Theorem 1.6 shows that this is only possible if  $R(z) = \Delta Q_0(z)(1 + O(z^{1-k}))$  as  $z \rightarrow \infty$ .  $\square$

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