Erratum to: On 2D Newest Vertex Bisection: Optimality of Mesh-Closure and H^1 -Stability of L_2 -Projection

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Published online: 28 September 2015
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Erratum to: Constr Approx DOI 10.1007/s00365-013-9192-4

Part of the proof of [3, Theorem 3] is incorrect, while the result remains true: On page 232, we define the node-to-element distance $\delta(z_j, T) := \min_{z_k \in \mathcal{N}_\ell \cap T} \delta(z_j, z_k)$, where

- $\delta(z_j, z_k) := 0$, if $z_j = z_k$,
- $\delta(z_j, z_k) := 1$, if there exist some $T \in \mathcal{T}_\ell$ with $z_j, z_k \in T$,
- $\delta(z_j, z_k) := N$, if $N \geq 2$ is the minimal number of elements $T_1, \dots, T_N \in \mathcal{T}_\ell$ with $z_j \in T_1, T_j \cap T_{j+1} \in \mathcal{E}_\ell$ for all $j = 1, \dots, N-1$, and $z_k \in T_N$.

Communicated by Wolfgang Dahmen.

The online version of the original article can be found under doi:[10.1007/s00365-013-9192-4](https://doi.org/10.1007/s00365-013-9192-4).

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Fig. 1 With the old definition of the node-to-element distance $\delta(\cdot, \cdot)$, it holds that $\delta(z_j, T') = \delta(z_j, z_i) = 1$, but $\delta(z_k, T') = \delta(z_k, z_i) = 4$, where the *dashed red line* indicates an element chain which realizes $\delta(z_k, z_i) = 4$. This violates $|\delta(z_j, T') - \delta(z_k, T')| \leq 1$

Contrary to what we claim in [3], this definition of the node-to-element distance does not imply the condition $|\delta(z_j, T') - \delta(z_k, T')| \leq 1$ for all $T, T' \in \mathcal{T}_\ell$ and $z_j, z_k \in \mathcal{N}_\ell \cap T$ which is necessary to prove [3, Equation (14)]. A counterexample can easily be constructed; see Fig. 1.

As a remedy, we proceed as follows: Let $\mathcal{T}_\star = \text{refine}(\mathcal{T}_0)$ be an arbitrary refinement of the initial mesh $\mathcal{T}_0$. For $z \in \mathcal{N}_\star$ and $T \in \mathcal{T}_\star$, we follow [1] and define the new node-to-element distance $\delta(z, T) \in \mathbb{N}_0$ by

- $\delta(z, T) = N$ with $N \in \mathbb{N}_0$ being the *minimal* number such that elements $T_0, \dots, T_N \in \mathcal{T}_\star$ exist with $z \in T_0$, $T_{n-1} \cap T_n \neq \emptyset$ for all $n = 1, \dots, N$, and $T_N = T$.

With this definition of the node-to-element distance $\delta(z_j, T)$, we show that the proof of [3, Theorem 3] remains, in fact, correct. To that end, we rely on recent results of [2]. In the latter work, the authors define for two vertices $z \neq z' \in \mathcal{N}_\ell$ a *chain of connecting edges* as a set of vertices $\text{CE}(z, z') := \{z = z_0, z_1, \dots, z_M = z'\} \subset \mathcal{N}_\ell$ such that $\overline{z_{m-1}z_m} \in \mathcal{E}_\ell$ for $m = 1, \dots, M$. Furthermore, they set $\text{CE}(z, z) := \{z\}$. Then, they define the distance between z and z' by $\text{dist}(z, z') := N$ with $N \in \mathbb{N}_0$ being the *minimal* number such that a chain of connecting edges of length $\#\text{CE}(z, z') = N + 1$ between z and z' exists. Finally, they define the node-to-element distance

- $\text{dist}(z, T) := \min_{z' \in \mathcal{N}_\ell \cap T} \text{dist}(z, z')$.

It is easy to see that $\text{dist}(z, T) = \delta(z, T)$ with the node-to-element distance from [1], which we employ now. From [2, Sect. 5.2], we therefore obtain the following auxiliary result.

Lemma 1 *Suppose that $\mathcal{T}_0$ is an initial mesh with an arbitrary distribution of reference edges, and $\mathcal{T}_\star = \text{refine}(\mathcal{T}_0)$ is an arbitrary refinement. Then, for all $T, T' \in \mathcal{T}_\star$ and $z \in \mathcal{N}_\star \cap T$, we have*

$$\text{gen}(T') - \text{gen}(T) \leq 3\delta(z, T') + C_0,$$

where $C_0 > 0$ is a constant which depends only on $\mathcal{T}_0$. □

Now we are in the position to give a corrected proof.

Corrected proof of [3, Theorem 3] It only remains to define $d_j > 0$ for all $z_j \in \mathcal{N}_\ell$ such that the assumptions (13)–(15) of [3, Proposition 9] are satisfied. With the node-to-element distance, we define

$$d_j := \min_{T \in \mathcal{T}_\ell} 2^{(3\delta(z_j, T) - \text{gen}(T))/2},$$

while we used

$$d_j := \min_{T \in \mathcal{T}_\ell} 2^{(2\delta(z_j, T) - \text{gen}(T))/2}.$$

in our former proof in [3]. However, we can now proceed as in [3]:

- We first prove (14): Let $T \in \mathcal{T}_\ell$ and $z_j, z_k \in \mathcal{N}_\ell \cap T$. Suppose that $T' \in \mathcal{T}_\ell$ satisfies $d_k = 2^{(3\delta(z_k, T') - \text{gen}(T'))/2}$. By definition, it holds that $d_j \leq 2^{(3\delta(z_j, T') - \text{gen}(T'))/2}$ and $|\delta(z_k, T') - \delta(z_j, T')| \leq 1$. This implies

$$\frac{d_j}{d_k} \leq \frac{2^{(3\delta(z_j, T') - \text{gen}(T'))/2}}{2^{(3\delta(z_k, T') - \text{gen}(T'))/2}} \leq 2^{3(\delta(z_j, T') - \delta(z_k, T'))/2} \leq 2^{3/2} = \sqrt{8} =: C_5.$$

In particular, (13) holds due to $1 + C_5^2 + C_5^{-2} = 9 + 1/8 < 11$.

- Second, we prove the upper bound $d_j \leq C_{\text{diam}}^{-1} h_\ell(T)$ in (15) for all $T \in \mathcal{T}_\ell$ and all $z_j \in \mathcal{N}_\ell \cap T$. By definition, one has $\delta(z_j, T) = 0$. With [3, Proposition 6 (i)], we thus see

$$d_j \leq 2^{-\text{gen}(T)/2} \leq C_{\text{diam}}^{-1} |T|^{1/2} \leq C_{\text{diam}}^{-1} h_\ell(T).$$

- Third, we prove the lower bound in (15) for all $T \in \mathcal{T}_\ell$ and $z_j \in \mathcal{N}_\ell \cap T$. As above, we choose $T' \in \mathcal{T}_\ell$ with $d_j = 2^{(3\delta(z_j, T') - \text{gen}(T'))/2}$. Note that $\text{gen}(T') < \text{gen}(T)$ would imply

$$d_j \leq 2^{(3\delta(z_j, T) - \text{gen}(T))/2} = 2^{-\text{gen}(T)/2} < 2^{-\text{gen}(T')/2} \leq 2^{(3\delta(z_j, T') - \text{gen}(T'))/2} = d_j$$

and hence a contradiction to the choice of T' . We thus see $\text{gen}(T) \leq \text{gen}(T')$. This gives

$$\begin{aligned} d_j^2 &= 2^{3\delta(z_j, T') - \text{gen}(T')} = 2^{-\text{gen}(T)} 2^{3\delta(z_j, T') - |\text{gen}(T') - \text{gen}(T)|} \\ &\geq \left(C^{\text{diam}}\right)^{-2} h_\ell(T)^2 2^{3\delta(z_j, T') - |\text{gen}(T') - \text{gen}(T)|} \end{aligned}$$

according to [3, Proposition 6 (i)]. According to Lemma 1,

$$|\text{gen}(T') - \text{gen}(T)| = \text{gen}(T') - \text{gen}(T) \leq 3\delta(z_j, T') + C_0,$$

and we conclude

$$d_j^2 \geq \left(C^{\text{diam}}\right)^{-2} h_\ell(T)^2 2^{-C_0}.$$

• Altogether, the definition $C_6 := \max \left\{ C_{\text{diam}}^{-1}, (C^{\text{diam}})^{-1} 2^{-C_0/2} \right\}$ shows (15). We stress that the constant C_6 depends only on $\mathcal{T}_0$ and the uniform γ -shape regularity of meshes $\mathcal{T}_\ell$ generated by [3, Algorithm 1]. Now that we have checked the assumptions (13)–(15), [3, Proposition 9] applies and proves H^1 -stability (5) of the L_2 -projection Π_ℓ onto $\mathcal{S}^1(\mathcal{T}_\ell)$. The constant $C_2 > 0$ depends only on the initial mesh $\mathcal{T}_0$ and on the γ -shape regularity of the mesh $\mathcal{T}_\ell$, which is uniformly bounded in terms of $\mathcal{T}_0$ via [3, Proposition 6(i)]. $\square$

Acknowledgments The authors thank Kunibert Siebert for drawing their attention to the flaw in the proof of [3, Theorem 3].

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