The AM-GM Inequality and CBS Inequality Are Equivalent

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This note is partly motivated by a recent one having the same nature [1], giving a short proof of the equivalence of the AM-GM (i.e., arithmetic-mean/geometric-mean) inequality and the Bernoulli inequality. The present note provides a very short proof of equivalence of the AM-GM inequality and the CBS (i.e., Cauchy-Bunyakovsky-Schwarz) inequality.

AM-GM Inequality If $c_1, \dots, c_n$ are nonnegative real numbers, then

$$\frac{1}{n} \sum_{k=1}^n c_k \geq \prod_{k=1}^n c_k^{\frac{1}{n}}. \quad (0.1)$$

CBS Inequality If $a_1, \dots, a_n, b_1, \dots, b_n$ are nonnegative real numbers, then

$$\sum_{k=1}^n a_k \sum_{k=1}^n b_k \geq \left(\sum_{k=1}^n \sqrt{a_k b_k} \right)^2. \quad (0.2)$$

(In both statements we clearly can assume without loss of generality that the scalars considered are positive.)

It may be quibbled that any two true statements are equivalent, so knowing these two inequalities suffices to prove their equivalence. Still it is striking that they can be proved one from the other so directly, as I will show.

To show (0.1) implies (0.2), we only need a special case of (0.1),

$$\frac{c_1 + c_2}{2} \geq \sqrt{c_1 c_2}. \quad (0.3)$$

Substituting into (0.3), we have

$$\frac{a_i \sum_{k=1}^n b_k + b_i \sum_{k=1}^n a_k}{2} \geq \sqrt{a_i b_i} \sqrt{\sum_{k=1}^n a_k \sum_{k=1}^n b_k}$$

for $i = 1, \dots, n$. Summing over i , we obtain

$$\sum_{k=1}^n a_k \sum_{k=1}^n b_k \geq \left(\sum_{i=1}^n \sqrt{a_i b_i} \right) \sqrt{\sum_{k=1}^n a_k \sum_{k=1}^n b_k}.$$

Thus (0.2) follows.

To show the converse, letting $b_k = \frac{1}{n^2}$ in (0.2) for all k , we have

$$\frac{1}{n} \sum_{k=1}^n a_k \geq \left(\frac{1}{n} \sum_{k=1}^n \sqrt{a_k} \right)^2. \quad (0.4)$$

Repeatedly using (0.4), we obtain

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n a_k &\geq \left(\frac{1}{n} \sum_{k=1}^n \sqrt{a_k} \right)^2 \\ &\geq \left(\frac{1}{n} \sum_{k=1}^n a_k^{\frac{1}{4}} \right)^4 \geq \dots \geq \left(\frac{1}{n} \sum_{k=1}^n a_k^{\frac{1}{2^m}} \right)^{2^m} \geq \dots \end{aligned} \quad (0.5)$$

By L'Hospital's rule, it is easy to see that

$$\lim_{x \rightarrow 0+} \ln \left(\frac{1}{n} \sum_{k=1}^n a_k^x \right) / x = \frac{1}{n} \sum_{k=1}^n \ln a_k,$$

so

$$\lim_{x \rightarrow 0+} \left(\frac{1}{n} \sum_{k=1}^n a_k^x \right)^{\frac{1}{x}} = \prod_{k=1}^n a_k^{\frac{1}{n}}.$$

Thus in (0.5), we can pass to the limit $m \rightarrow \infty$, giving

$$\frac{1}{n} \sum_{k=1}^n a_k \geq \prod_{k=1}^n a_k^{\frac{1}{n}}; \text{ hence (0.2) implies (0.1).}$$

REFERENCE

1. L. Maligranda, The AM-GM inequality is equivalent to the Bernoulli inequality, *Math. Intelligencer*, 34, no.1 (2012), 1–2.

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