Note

Cantor-Bernstein's Theorem in a Semiring

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Consider a commutative semiring with sum $+$, product $\cdot$, the identity 1 for $\cdot$, and the identity 0 for $+$, which is absorbing for $\cdot$.

Suppose moreover that the semiring contains the element $\aleph_0$, and, to the usual axioms for commutative semirings, we add:

$$\aleph_0 = \aleph_0 + 1 \tag{1}$$

$$\text{if } \kappa + \mu \leq \kappa, \text{ then } \mu \cdot \aleph_0 \leq \kappa \tag{2}$$

Here the relation $\leq$ is defined by: $\kappa \leq \mu$ if and only if $\kappa + \nu = \mu$, for some ν. This relation is clearly transitive and reflexive.

Without being committed to implementation of cardinal numbers in a specific set theory, it is easily seen that all these axioms are satisfied in the usual interpretation, even without assuming the axiom of choice: the sum and the product of cardinal numbers are explained through union of disjoint sets and cartesian product, respectively; 0 is the cardinal of the empty set, 1 is the cardinal of a singleton, and $\aleph_0$ the cardinal of the set of natural numbers. I observe that according to this interpretation, $\kappa \leq \mu$ means that there is an injective function of a set of cardinal κ into a set of cardinal μ.

Finally, to see that axiom (2) is verified in this interpretation, let K, M be two disjoint sets of cardinal κ and μ, respectively. Let also f be an injective mapping from $K \cup M$ into K. Then to each x in M and natural number n, we associate the result of applying the $(n+1)$-fold iterate of f to x:

$$\begin{array}{c|ccccccccc}
M & \multicolumn{9}{c}{\overbrace{\quad f[M] \qquad\qquad f[f[M]] \qquad\qquad f[f[f[M]]] \qquad\qquad}^{K}} \\
\hline
x_1 & \to & f(x_1) & \to & f(f(x_1)) & \to & f(f(f(x_1))) & \to & \dots \\
x_2 & \to & f(x_2) & \to & f(f(x_2)) & \to & f(f(f(x_2))) & \to & \dots \\
x_3 & \to & f(x_3) & \to & f(f(x_3)) & \to & f(f(f(x_3))) & \to & \dots \\
 & & & & \vdots & & & &
\end{array}$$

That is, the function g defined by the recursive equations

$$g(x,0) = f(x)$$
$$g(x,n+1) = f(g(x,n))$$

is an injective mapping of $M \times \mathbb{N}$ into K.

PROPOSITION 1 *If* $\kappa + \mu \leq \kappa$, *then* $\kappa + \mu = \kappa$.

PROOF. If $\kappa + \mu \leq \kappa$, then $\mu \cdot \aleph_0 \leq \kappa$, by axiom (2). This means that, for some λ, $\mu \cdot \aleph_0 + \lambda = \kappa$. We then have

$$\begin{aligned}
\kappa + \mu &= \mu \cdot \aleph_0 + \mu + \lambda \\
&= \mu \cdot (\aleph_0 + 1) + \lambda \\
&= \mu \cdot \aleph_0 + \lambda \\
&= \kappa && \text{axiom (1).}
\end{aligned}$$

Cantor-Bernstein's Theorem states that $\leq$ is a partial order:

$$\text{if } \kappa \leq \mu \text{ and } \mu \leq \kappa, \text{ then } \kappa = \mu.$$

PROOF. If $\mu \leq \kappa$, there exists λ such that $\mu + \lambda = \kappa$. Then, if $\kappa \leq \mu$, we have $\mu + \lambda \leq \mu$ and, by Proposition 1, $\mu + \lambda = \mu$, i.e., $\kappa = \mu$.

REMARKS

1. Axioms (1) and (2) are equivalent to the statement:

$$\kappa + \mu = \kappa \quad \text{if and only if} \quad \mu \cdot \aleph_0 \leq \kappa \tag{3}$$

which generalizes the well-known feature of Dedekind-infinite cardinals, namely,

$$\kappa + 1 = \kappa \qquad \text{if and only if} \qquad \aleph_0 \leq \kappa$$

2. $\aleph_0$ is the unique element satisfying (3).
3. The axioms allow one to derive directly the properties $\aleph_0 + \aleph_0 = \aleph_0$ and $\aleph_0 \cdot \aleph_0 = \aleph_0$.
Indeed, as $\aleph_0 + 1 + 1 = \aleph_0$, we have $(1+1) \cdot \aleph_0 \leq \aleph_0$ and $\aleph_0 + \aleph_0 \leq \aleph_0$. Hence, by Proposition 1, $\aleph_0 + \aleph_0 = \aleph_0$.
It follows that $\aleph_0 \cdot \aleph_0 \leq \aleph_0$. But $\aleph_0 \cdot \aleph_0 = (\aleph_0 + 1) \cdot \aleph_0 = \aleph_0 \cdot \aleph_0 + \aleph_0$. Therefore, $\aleph_0 \leq \aleph_0 \cdot \aleph_0$ and, by Cantor-Bernstein's Theorem, $\aleph_0 \cdot \aleph_0 = \aleph_0$.
4. Remark 1 and the fact that $\aleph_0 \cdot \aleph_0 = \aleph_0$ immediately entail

$$\kappa + \mu = \kappa \qquad \text{if and only if} \qquad \kappa + \mu \cdot \aleph_0 = \kappa$$

From this we get

$$\kappa + \kappa = \kappa \qquad \text{if and only if} \qquad \aleph_0 \cdot \kappa = \kappa$$

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