

Cantor-Bernstein's Theorem in a Semiring

MARCEL CRABBÉ

Consider a commutative semiring with sum $+$, product \cdot , the identity 1 for \cdot , and the identity 0 for $+$, which is absorbing for \cdot .

Suppose moreover that the semiring contains the element \aleph_0 , and, to the usual axioms for commutative semirings, we add:

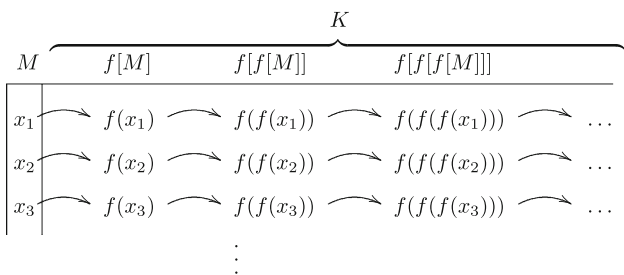
$$\aleph_0 = \aleph_0 + 1 \tag{1}$$

$$\text{if } \kappa + \mu \leq \kappa, \text{ then } \mu \cdot \aleph_0 \leq \kappa \tag{2}$$

Here the relation \leq is defined by: $\kappa \leq \mu$ if and only if $\kappa + v = \mu$, for some v . This relation is clearly transitive and reflexive.

Without being committed to implementation of cardinal numbers in a specific set theory, it is easily seen that all these axioms are satisfied in the usual interpretation, even without assuming the axiom of choice: the sum and the product of cardinal numbers are explained through union of disjoint sets and cartesian product, respectively; 0 is the cardinal of the empty set, 1 is the cardinal of a singleton, and \aleph_0 the cardinal of the set of natural numbers. I observe that according to this interpretation, $\kappa \leq \mu$ means that there is an injective function of a set of cardinal κ into a set of cardinal μ .

Finally, to see that axiom (2) is verified in this interpretation, let K, M be two disjoint sets of cardinal κ and μ , respectively. Let also f be an injective mapping from $K \cup M$ into K . Then to each x in M and natural number n , we associate the result of applying the $(n + 1)$ -fold iterate of f to x :



That is, the function g defined by the recursive equations

$$\begin{aligned} g(x, 0) &= f(x) \\ g(x, n + 1) &= f(g(x, n)) \end{aligned}$$

is an injective mapping of $M \times \mathbb{N}$ into K .

PROPOSITION 1 If $\kappa + \mu \leq \kappa$, then $\kappa + \mu = \kappa$.

PROOF. If $\kappa + \mu \leq \kappa$, then $\mu \cdot \aleph_0 \leq \kappa$, by axiom (2). This means that, for some λ , $\mu \cdot \aleph_0 + \lambda = \kappa$. We then have

$$\begin{aligned} \kappa + \mu &= \mu \cdot \aleph_0 + \mu + \lambda \\ &= \mu \cdot (\aleph_0 + 1) + \lambda \\ &= \mu \cdot \aleph_0 + \lambda \\ &= \kappa \end{aligned} \tag{1}$$

Cantor-Bernstein's Theorem states that \leq is a partial order:

$$\text{if } \kappa \leq \mu \text{ and } \mu \leq \kappa, \text{ then } \kappa = \mu.$$

PROOF. If $\mu \leq \kappa$, there exists λ such that $\mu + \lambda = \kappa$. Then, if $\kappa \leq \mu$, we have $\mu + \lambda \leq \mu$ and, by Proposition 1, $\mu + \lambda = \mu$, i.e., $\kappa = \mu$.

REMARKS

1. Axioms (1) and (2) are equivalent to the statement:

$$\kappa + \mu = \kappa \quad \text{if and only if} \quad \mu \cdot \aleph_0 \leq \kappa \tag{3}$$

which generalizes the well-known feature of Dedekind-infinite cardinals, namely,

$$\kappa + 1 = \kappa \quad \text{if and only if} \quad \aleph_0 \leq \kappa$$

2. \aleph_0 is the unique element satisfying (3).
3. The axioms allow one to derive directly the properties $\aleph_0 + \aleph_0 = \aleph_0$ and $\aleph_0 \cdot \aleph_0 = \aleph_0$.

Indeed, as $\aleph_0 + 1 + 1 = \aleph_0$, we have $(1 + 1) \cdot \aleph_0 \leq \aleph_0$ and $\aleph_0 + \aleph_0 \leq \aleph_0$. Hence, by Proposition 1, $\aleph_0 + \aleph_0 = \aleph_0$.

It follows that $\aleph_0 \cdot \aleph_0 \leq \aleph_0$. But $\aleph_0 \cdot \aleph_0 = (\aleph_0 + 1) \cdot \aleph_0 = \aleph_0 \cdot \aleph_0 + \aleph_0$. Therefore, $\aleph_0 \leq \aleph_0 \cdot \aleph_0$ and, by Cantor-Bernstein's Theorem, $\aleph_0 \cdot \aleph_0 = \aleph_0$.

4. Remark 1 and the fact that $\aleph_0 \cdot \aleph_0 = \aleph_0$ immediately entail

$$\kappa + \mu = \kappa \quad \text{if and only if} \quad \kappa + \mu \cdot \aleph_0 = \kappa$$

From this we get

$$\kappa + \kappa = \kappa \quad \text{if and only if} \quad \aleph_0 \cdot \kappa = \kappa$$

Institut Supérieur de Philosophie
Université Catholique de Louvain
1348 Louvain-La-Neuve
Belgium
e-mail: marcel.crabbe@uclouvain.be