Asymptotic Growth of Solutions of Neutral Type Systems

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Abstract We consider a differential system of neutral type with distributed delay. We obtain a precise norm estimation of semigroup generated by the operator corresponding to the system in question. Our result is based on a spectral analysis of the operator and some uniform estimation of norms of the exponentials of matrices. We also discuss the stability properties of corresponding solutions and the existence of the fastest growing solution.

Keywords Delay systems · Neutral type systems · Asymptotic behaviour of solutions · Maximal asymptotics

1 Introduction

One of the important problems in the theory of functional differential equations is the estimation of the asymptotic behaviour of their solutions. It is also related to stability analysis of those equations. Even in the case when stability is studied, there remains the question about the rate of growth or decay of individual solutions and their dependence on initial states. All those questions pertain directly to the equations with delay. Among the works devoted to this problem, we can single out the works of D.A. Medvedev and V.V. Vlasov [9], W.E. Brumley [3], J.K. Hale, S.M. Verduyn Lunel [4, 5], D.A. O'Connor, T.J. Tarn [11], R. Rabah, G.M. Sklyar [12, 13], R. Rabah et al. [16], S.M. Verduyn Lunel, D.V. Yakubovich [22]. The fundamental approach for estimation of asymptotic growth of solutions is interpretation of

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the equation as a system in an abstract Banach space and a spectral analysis of the corresponding semigroup. Within this approach traditionally the encountered problem studying is to estimate the norm of the semigroup $\|T(t)\|$, $t \geq 0$. In the context of asymptotic stability this problem leads to study of so-called exponential stability, because the relation $\|T(t)\| \rightarrow 0$, $t \rightarrow \infty$, is possible only if $\|T(t)\| \leq Me^{-\lambda t}$ for some positive M and λ , so the rate of decay of the norm is exponential. This kind of stability is mostly studied for particular classes of equations, for example, for retarded systems by J.K. Hale, S.M. Verduyn Lunel [4], V.B. Kolmanovskii and V.R. Nosov [7]. The strong stability takes an important place in the asymptotic theory of semigroups of last decades, where by strong stability we mean the situation when all the trajectories of the equation $T(t)x$, $x \in X$, tend to 0 as $t \rightarrow +\infty$ while the norm $\|T(t)\|$ does not decay. Analysis of this kind of stability began in the book of B. Sz. Nagy, C. Foias, [20] for contractions in Hilbert space and was extended by G.M. Sklyar, V.Ya. Shirman [19] to the case of equations in Banach space. Later it was developed by W. Arendt and C.J.K. Batty [1], Yu.I. Lyubich, V.Q. Phong [8], and then in many other works (see [21] and references therein). Recently some further progress in the analysis of strong stability was achieved, namely we mean some new results concerning the estimation of individual trajectories on nonclosed sets obtained by A. Borichev, Y. Tomilov [2] and some estimations of so-called fast solutions by A. Haraux [6]. Recall that in the case of the strong stability the rate of decay of particular trajectories in general can be arbitrarily slow. Some estimations of this rate are obtained recently by Medvedev, Vlasov, Wu [9, 23]. For delayed systems the effect of the strong stability appears in the case of systems of neutral type. This question was considered in [3], where nonexponential stability for delayed systems was studied. An extensive analysis of this problem was given in [15]. In the recent work [17] a development of the idea of strong stability is proposed. The concept of maximal asymptotics (solution of fastest growth) is introduced there and some conditions under which no such solution exists are obtained. The latter means that there is no $x \in X$ such that $\|T(t)x\|/\|T(t)\| \rightarrow 1$, $t \rightarrow +\infty$. This concept encouraged us to consider the question of existence of the maximal asymptotics for neutral type equations. Namely it concerns the case when the growth of solutions is lower than the exponential one.

In the present work we consider delay systems of neutral type of the form

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + \int_{-1}^0 A_2(\theta)\dot{z}(t+\theta)d\theta + \int_{-1}^0 A_3(\theta)z(t+\theta)d\theta, \quad (1)$$

where A_{-1} is a $n \times n$ invertible complex matrix, A_2 and A_3 are $n \times n$ matrices of functions from $L_2(-1, 0)$. We develop methods of analysis of stability of neutral type systems proposed in [15]. Now we consider a more general problem—describing the asymptotic behaviour of solutions. Our goal is to estimate solutions' growth speed. Note that recently in [9] an extensive research of a more general class of neutral type equations was given. In particular an upper estimation of solutions' growth of nonhomogenous equation was obtained. In our paper we specify results of [9] for the case of Eq. (1) and we get a more detailed estimation in terms of spectral properties of matrix A_{-1} . Our approach allows us to obtain not only the upper estimation of solutions' growth but also the lower estimation of the norm of the corresponding

semigroup, and the condition under which the upper bound can be reached. Based on those results we consider a problem of null-reachability of a control system connected with (1). Besides, we also give the answer to the question on existence of the fastest growing solution for Eq. (1) (so-called maximal asymptotics, cf. [18]). Following [15] we rewrite Eq. (1) in the operator form

$$\dot{x} = \mathcal{A}x, \quad x \in M_2, \quad (2)$$

where $M_2 = \mathbb{C}^n \times L_2(-1, 0; \mathbb{C}^n)$, the operator $\mathcal{A}$ is then given by

$$\mathcal{A} \begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} = \begin{pmatrix} \int_{-1}^0 A_2(\theta) \dot{z}_t(\theta) d\theta + \int_{-1}^0 A_3(\theta) z_t(\theta) d\theta \\ dz_t(\theta)/d\theta \end{pmatrix}, \quad (3)$$

and the domain of $\mathcal{A}$ is as follows:

$$\mathcal{D}(\mathcal{A}) = \{(y, z(\cdot)) : z \in H^1(-1, 0; \mathbb{C}^n), y = z(0) - A_{-1}z(-1)\} \subset M_2. \quad (4)$$

Detailed spectral analysis of rewritten system (2)–(4) was presented in [15] and for a more general class of systems in [9]. The crucial role in our further considerations plays the fact that operator $\mathcal{A}$ generates C_0 -semigroup and has finite-dimensional, invariant subspaces which constitute a Riesz basis in M_2 (see Theorem 16 in [15]).

The work is organized as follows. In Sect. 2 we describe the asymptotic behaviour of the norm of the semigroup generated by the operator $\mathcal{A}$ defined by (3)–(4). We show that the upper and lower bounds of this norm are both of the form $Ce^{\omega t}(t^m + 1)$, where ω, m depends only on matrix A_{-1} (Theorem 2.1). We give two examples of Eq. (1), with the same matrix A_{-1} showing that the norm of semigroup can achieve either upper or lower bound. In general, for fixed A_{-1} constant C can not be chosen uniformly with respect to A_2, A_3 , unless one includes additional assumptions. However, we prove (Theorem 2.6) that the constant C can be found uniformly with respect to A_2, A_3 , if the validity of estimations is required only on some subspace of a finite codimension. The choice of this subspace depends on A_2 and A_3 .

In Sect. 3 we consider the problem of existence of the fastest growing solution. We study the case when the asymptotic behaviour of the norm of semigroup $e^{\mathcal{A}t}$ is precisely estimated by Theorem 2.1, namely when matrix A_{-1} has only one Jordan block corresponding to the eigenvalue of maximal modulus. In this case we give the conditions of nonexistence of maximal asymptotics of that semigroup (Theorem 3.1). The result is based on general theorems on the maximal asymptotics [17, 18]. We also give the explicit form of the maximal asymptotics of Eq. (2) (Theorem 3.2) and discuss its existence in the general case (Theorem 3.3).

In Sect. 4 we present some applications of obtained results for controlled systems of neutral type. First we consider the feedback control of a special type and give the uniform estimation of semigroup's growth rate (Statement 4.1). Next we estimate the null-reachability sets of a controlled system under some restrictions on a norm of applied control. Namely we describe the asymptotic behaviour of diameters of those sets (Statement 4.2).

In Appendix we recall some properties of divided differences and prove some statements about special determinants which we use in the proof of auxiliary results (Lemma 2.3).

2 Upper and Lower Estimation of the Norm of the Semigroup

We consider the differential equation given by (2)–(4), which was derived from the neutral type Eq. (1). If A_2 and A_3 in (3) equal zero, we denote the operator $\mathcal{A}$ by $\tilde{\mathcal{A}}$ and we add tilde to all the symbols related to this special case (e.g. $\tilde{\lambda}$, $\tilde{\omega}$, $\tilde{V}$, etc.). We recall (see Theorems 7, 15, 16 [15]) some properties of operators $\mathcal{A}$ and $\tilde{\mathcal{A}}$, which we will need in order to describe the corresponding semigroups. Denote the eigenvalues of the matrix A_{-1} by $\mu_m, m = 1, \dots, \ell$ ($|\mu_1| \geq |\mu_2| \geq \dots \geq |\mu_\ell|$), and their multiplicities by p_m ($\sum p_m = n$). The eigenvalues of $\tilde{\mathcal{A}}$ are complex logarithms of μ_m and zero i.e.

$$\sigma(\tilde{\mathcal{A}}) = \{\tilde{\lambda}_m^{(k)} = \ln |\mu_m| + i(\arg \mu_m + 2k\pi), \mu_m \in \sigma(A_{-1}), m = 1, \dots, \ell; k \in \mathbb{Z}\} \cup \{0\}.$$

Almost all eigenvalues of $\mathcal{A}$ lie close to $\tilde{\lambda}_m^{(k)}$. More precisely, for k large enough they are contained in the discs $L_m^{(k)}$ centered at $\tilde{\lambda}_m^{(k)}$ of radii $r_k \rightarrow 0$ (see Theorem 4 [16]). The sum of multiplicities of eigenvalues of $\mathcal{A}$ lying in each disc centered at $\tilde{\lambda}_m^{(k)}$ equals the multiplicity of $\tilde{\lambda}_m^{(k)}$ and μ_m , that is p_m . Let us denote $\mathcal{A}$ -invariant subspaces $V_m^{(k)} = P_m^{(k)} M_2$, where $P_m^{(k)} x = \frac{1}{2\pi i} \int_{L_m^{(k)}} R(\mathcal{A}, \lambda) x d\lambda$ are Riesz projectors, $m = 1, \dots, \ell, k \in \mathbb{Z}$. The sequence of p_m -dimensional subspaces $V_m^{(k)}, m = 1, \dots, \ell, |k| \geq N$, and some $2(N+1)n$ -dimensional subspace W_N constitute $\mathcal{A}$ -invariant Riesz basis of space M_2 . Notice that the subspaces $\tilde{V}_m^{(k)}, m = 1, \dots, \ell, |k| \geq N$, and some $2(N+1)n$ -dimensional subspace $\tilde{W}_N$ constitute $\tilde{\mathcal{A}}$ -invariant Riesz basis of the space M_2 , which is quadratically close to the previous one. Theorem 7 states in particular that to each $\tilde{\lambda}_m^{(k)} \in \sigma(\tilde{\mathcal{A}}) \setminus \{0\}$ and each Jordan chain of eigen- and rootvectors of the matrix A_{-1} corresponds the Jordan chain of $\tilde{\mathcal{A}}: v_{m,j}^{(k),0}, v_{m,j}^{(k),1}, \dots, v_{m,j}^{(k),p_{m,j}-1}$, i.e. the vectors v^s (the indices k, m, j are omitted) satisfy the relation $(\tilde{\mathcal{A}} - \lambda I)v^s = v^{s-1}$. They are given by

$$v_{m,j}^{(k),s} = \begin{pmatrix} 0 \\ e^{\lambda_m^{(k)} \theta} P_{m,j}^s(\theta) \end{pmatrix}, \quad (5)$$

where $P_{m,j}^s(\theta)$ is some polynomial independent of $k, s = 0, \dots, p_{m,j} - 1; m = 1, \dots, \ell; k \in \mathbb{Z}; j = 1, \dots, v_m$.

To estimate the norm of semigroup generated by the operator $\mathcal{A}$ defined by (3)–(4) we have to know how exactly the spectrum of $\mathcal{A}$ looks like. Let us denote $\sup\{\operatorname{Re} \lambda : \lambda \in \sigma(\mathcal{A})\}$ by ω and $\sup\{\operatorname{Re} \lambda : \lambda \in \sigma(\tilde{\mathcal{A}})\}$ by $\tilde{\omega}$. Notice that $\omega \geq \tilde{\omega}$. The case when $\omega > \tilde{\omega}$ is not interesting because in this case there is only a finite number of eigenvalues of $\mathcal{A}$, whose real parts are close to ω and thus there is an eigenvalue λ_0 of $\mathcal{A}$ with real part ω . It is easy to see that this eigenvalue will determine the behaviour of semigroup $e^{\mathcal{A}t}$, that is for some positive constants m, M and $t \geq 0$ the inequalities hold

$$me^{\omega t} (t^{q_0-1} + 1) \leq \|e^{\mathcal{A}t}\| \leq Me^{\omega t} (t^{q_0-1} + 1),$$

where $1 \leq q_0 \leq n$ is the maximal size of Jordan block corresponding to λ_0 . In the case when $\omega = \tilde{\omega}$ the following theorem describes the behaviour of $\|e^{\mathcal{A}t}\|$.

Theorem 2.1 *Let us consider a semigroup $e^{\mathcal{A}t}$, $t \geq 0$, where operator $\mathcal{A}$ is defined by (3)–(4). If $\operatorname{Re} \lambda < \sup\{\operatorname{Re} \tau : \tau \in \sigma(\mathcal{A})\}$ for any $\lambda \in \sigma(\mathcal{A})$ then there exist the constants M_p, M_q , such that the following inequalities hold*

- (i) $\|e^{\mathcal{A}t}\| \leq f_p(t) = M_p e^{\omega t} (t^{p-1} + 1),$
- (ii) $\|e^{\mathcal{A}t}\| \geq f_q(t) = M_q e^{\omega t} (t^{q-1} + 1),$

where q is the maximal size of Jordan block of matrix A_{-1} corresponding to the eigenvalue with maximal modulus and p is the sum of sizes of all Jordan blocks corresponding to the maximal eigenvalue of A_{-1} .

Proof (i) In general, there can be more than one eigenvalue of maximal modulus, that is $|\mu_1| = |\mu_2| = \dots = |\mu_{m_0}| > |\mu_{m_0+1}| \geq \dots \geq |\mu_\ell|$ for some integer $m_0 \leq \ell$. Let us denote the sum of sizes of Jordan blocks of matrix A_{-1} corresponding to the eigenvalues $\mu_1, \dots, \mu_{m_0}$ by $p_1, \dots, p_{m_0}$ respectively, then $p = \max\{p_1, \dots, p_{m_0}\}$. Theorem 4 from [12] implies that there exists N large enough, such that we can split the spectrum of operator $\mathcal{A}$ into three parts. The first part, σ_N , consists of eigenvalues contained in the separated circles $\{L_m^{(k)}\}_{m=1, \dots, m_0}$ with centers in $\tilde{\lambda}_m^{(k)}$ and radii $r_k \rightarrow 0$, such that the number of eigenvalues of $\mathcal{A}$ in a given circle $L_m^{(k)}$ equals p_m (counting with multiplicities). The second part is called σ_ε . Therefore we have $\sigma_N = \sigma(\mathcal{A}) \cap \bigcup_{m=1, \dots, m_0} \bigcup_{|k| > N} L_m^{(k)}$ and $\sigma_\varepsilon = \sigma(\mathcal{A}) \setminus \sigma_N$. Notice that there exists $\varepsilon > 0$ small enough such that every eigenvalue $\lambda \in \sigma_\varepsilon$ satisfies $\operatorname{Re} \lambda \leq \omega - \varepsilon$. Denote the images of spectral projectors corresponding to $\sigma_N, \sigma_\varepsilon$ by $\mathcal{Q}_N$ and $\mathcal{Q}_\varepsilon$. Those subspaces are $\mathcal{A}$ -invariant and their direct sum equals M_2 . Thus $\mathcal{A}|_{\mathcal{Q}_\varepsilon}$ generates C_0 -semigroup and from Hille-Yosida theorem we have

$$\|e^{\mathcal{A}|_{\mathcal{Q}_\varepsilon} t}\| \leq M_\varepsilon e^{\omega t}, \quad t \geq 0. \quad (6)$$

The operator $\mathcal{A}|_{\mathcal{Q}_N}$ satisfies the assumptions of Theorem 1 in [10] thus it easily follows that

$$\|e^{\mathcal{A}|_{\mathcal{Q}_N} t}\| \leq M_N e^{\omega t} (t^{p-1} + 1), \quad t \geq 0. \quad (7)$$

Subspaces $\mathcal{Q}_N, \mathcal{Q}_\varepsilon$ are direct sums of basis subspaces. Thus they also constitute the Riesz basis from subspaces and it is easy to see that there exists a constant M_p such that

$$\|e^{\mathcal{A}t}\| \leq M_p e^{\omega t} (t^{p-1} + 1), \quad t \geq 0. \quad (8)$$

(ii) It suffices to show that for any positive ε and T there exists $x \in M_2$ such that for all $t \in [0, T]$ the inequality $\|e^{\mathcal{A}t} x\| \geq (f_q(t) - \varepsilon) \|x\|$ holds.

Let us denote the maximal size of Jordan block of matrix A_{-1} corresponding to μ_m by q_m for $m = 1, 2, \dots, m_0$, and $\max\{q_1, \dots, q_{m_0}\}$ by q . Without loss of generality we assume that $q = q_1$. Let us consider $x \in V_1^{(k)}$. Later on for simplicity we will omit index 1 and write shortly V_k instead of $V_1^{(k)}$, λ_k instead of $\lambda_1^{(k)}$, A_k instead of $A_1^{(k)}$ etc. Then $\|e^{\mathcal{A}t} x\|_{M_2} = \|e^{A_k t} x\|_{V_k}$, where operator $A_k : V_k \rightarrow V_k$ has eigenvalues close to $\tilde{\lambda}_k$. Representing A_k as $A_k = A_k^0 + \tilde{\lambda}_k I$, where I is identity operator, we get for exponents $\|e^{A_k t} x\|_{V_k} = |e^{\tilde{\lambda}_k t}| \|e^{A_k^0 t} x\|_{V_k}$. On the other hand $\tilde{\lambda}_k$ are the complex

logarithms of μ_1 , so $\operatorname{Re} \tilde{\lambda}_k = \ln |\mu_1| = \tilde{\omega} = \omega$ and our inequality can be rewritten in the form

$$\|e^{\mathcal{A}t}x\|_{M_2} = e^{\omega t} \|e^{A_k^0 t}x\|_{V_k}. \quad (9)$$

Because the families of subspaces $\{V_m^{(k)}\}_{k \in \mathbb{Z}; m=1, \dots, \ell}$ and $\{\tilde{V}_m^{(k)}\}_{k \in \mathbb{Z}; m=1, \dots, \ell}$ constitute quadratically close Riesz basis, there exists a bounded operator T_N , with bounded inverse, which is close to identity and transforms almost all subspaces $V_m^{(k)}$ to $\tilde{V}_m^{(k)}$. Such operator T_N can be defined on every basis subspace $V_m^{(k)}$ by the formula

$$T_N|_{V_m^{(k)}}x = \tilde{P}_m^{(k)}x, \quad \text{for } |k| > N, m = 1, 2, \dots, \ell, \quad (10)$$

where $\tilde{P}_m^{(k)} : V_m^{(k)} \rightarrow \tilde{V}_m^{(k)}$ is a Riesz projector given by

$$\tilde{P}_m^{(k)}x = \frac{1}{2\pi i} \int_{L_m^{(k)}} R(\tilde{\mathcal{A}}, \lambda)x d\lambda$$

and

$$T_N|_{V_m^{(k)}}x = x, \quad \text{for } |k| \leq N, m = 1, 2, \dots, \ell. \quad (11)$$

It is easy to see that the operator T_N is bounded on M_2 and close to identity. Therefore T_N is invertible, its inverse T_N^{-1} is bounded and transforms all but finitely many subspaces $V_m^{(k)}$ onto $\tilde{V}_m^{(k)}$. Using T_N we can define the operator $\mathcal{B}_k : \tilde{V}_k \rightarrow \tilde{V}_k$ close to $A_k^0 : V_k \rightarrow V_k$ for each $|k| > N$ by the formula $\mathcal{B}_k = T_N A_k^0 T_N^{-1}$. Let $\tilde{x} = T_N x$. Then we get

$$\|e^{\mathcal{B}_k t} \tilde{x}\| = \|T_N e^{A_k^0 t} x\| \leq \|T_N\| \|e^{A_k^0 t} x\|. \quad (12)$$

From here and (9) we obtain

$$\|e^{\mathcal{A}t}x\|_{M_2} \geq \|T_N\|^{-1} e^{\omega t} \|e^{\mathcal{B}_k t} \tilde{x}\|_{\tilde{V}_k}. \quad (13)$$

The eigen- and rootvectors of operator $\tilde{\mathcal{A}}$ are given by (5), we can see that the number and lengths of all Jordan chains of operators $\tilde{A}_k = \tilde{\mathcal{A}}|_{\tilde{V}_k}$ are independent of k . Thus all operators $\tilde{A}_k^0 := \tilde{A}_k - \tilde{\lambda}_k I$ have the same matrix, say A_0 in the basis $\{v^{(k),s}\}_{s=1}^{p_1}$. We will now compare $\tilde{A}_k^0$ to each $\mathcal{B}_k$. Using the variation of a constant to a system

$$\begin{cases} \dot{y} = \tilde{A}_k^0 y + (\mathcal{B}_k - \tilde{A}_k^0)y, \\ y(0) = \tilde{x}, \end{cases}$$

we obtain

$$e^{\mathcal{B}_k t} \tilde{x} = e^{\tilde{A}_k^0 t} \tilde{x} + \int_0^t e^{\tilde{A}_k^0(t-\tau)} (\mathcal{B}_k - \tilde{A}_k^0) e^{\mathcal{B}_k \tau} \tilde{x} d\tau.$$

Hence

$$\|e^{\mathcal{B}_k t} \tilde{x} - e^{\tilde{A}_k^0 t} \tilde{x}\| \leq \|\mathcal{B}_k - \tilde{A}_k^0\| \int_0^t \|e^{\tilde{A}_k^0(t-\tau)}\| \|e^{\mathcal{B}_k \tau} \tilde{x}\| d\tau, \quad (14)$$

for any $|k| > N$. Let us define a family of operators $S_k : \tilde{V}_k \rightarrow \mathbb{C}^{p_1}$ by the formula $S_k \tilde{x}_k = \alpha^{(k)}$, where $\alpha^{(k)} = [\alpha_1^{(k)}, \dots, \alpha_{p_1}^{(k)}]$ is the vector of coefficients of $\tilde{x}_k$ in the basis $\{v_i^{(k)}\}_{i=1}^{p_1}$. It is easy to see that each S_k is invertible. Using the fact that Jordan chains of operator $\tilde{A}$ constitute the Riesz basis in the closure of their linear span it is also easy to show that the norms of S_k, S_k^{-1} are uniformly bounded. With this notation (14) reads

$$\|e^{\mathcal{B}_k t} \tilde{x} - e^{\tilde{A}_k^0 t} \tilde{x}\| \leq \|\mathcal{B}_k - \tilde{A}_k^0\| \int_0^t \|S_k e^{A_0(t-\tau)} S_k^{-1}\| \|S_k e^{B_k \tau} S_k^{-1} \tilde{x}\| d\tau, \quad (15)$$

where $B_k = S_k^{-1} \mathcal{B}_k S_k$. To better elaborate the estimation (15) we need the following lemmas.

Lemma 2.2 *For every $\delta > 0$ there exists k_0 large enough, that for every $|k| > k_0$ we have $\|\mathcal{B}_k - \tilde{A}_k^0\| < \delta$.*

Proof Let us take any $\tilde{x} \in \tilde{V}_k, |k| > N$ and estimate $\|\mathcal{B}_k \tilde{x} - \tilde{A}_k^0 \tilde{x}\|$. From the definition of $\mathcal{B}_k$ we have $\mathcal{B}_k \tilde{x} = T_N|_{V_k} A_k^0 T_N^{-1}|_{\tilde{V}_k} \tilde{x}$, where T_N is defined by (10)–(11). We get $\|\mathcal{B}_k \tilde{x} - \tilde{A}_k^0 \tilde{x}\| = \|\tilde{P}_k A_k^0 x - \tilde{A}_k^0 \tilde{P}_k x\|$, where $x = T_N^{-1}|_{\tilde{V}_k} \tilde{x}$. One can now extend operators A_k^0 and $\tilde{A}_k^0$ to the whole space M_2 by formulas $\bar{A}_k^0 x = \frac{1}{2\pi i} \int_{L_k} (\lambda - \lambda_k) R(\mathcal{A}, \lambda) x d\lambda$ and $\hat{A}_k^0 x = \frac{1}{2\pi i} \int_{L_k} (\lambda - \lambda_k) R(\tilde{\mathcal{A}}, \lambda) x d\lambda$. It is easy to see that $\bar{A}_k^0|_{V_k} = A_k^0$ and $\hat{A}_k^0|_{\tilde{V}_k} = \tilde{A}_k^0$. Using this and the fact that $\tilde{A}_k^0 \tilde{P}_k = \hat{A}_k^0 \tilde{P}_k = \tilde{P}_k \hat{A}_k^0$ we have $\|\mathcal{B}_k \tilde{x} - \tilde{A}_k^0 \tilde{x}\| = \|\tilde{P}_k (\bar{A}_k^0 - \hat{A}_k^0) x\|$. Notice that the Riesz projectors $\tilde{P}_k$ are uniformly bounded by some constant C and $\|\bar{A}_k^0 - \hat{A}_k^0\|$ can be estimated by $\|\bar{A}_k^0 - \hat{A}_k^0\| \leq r \sup_{\lambda \in L_k} \|R(\mathcal{A}, \lambda) - R(\tilde{\mathcal{A}}, \lambda)\|$. It is shown in [15] that such a difference of resolvents satisfies

$$\sup_{\lambda \in L_k} \|R(\mathcal{A}, \lambda) - R(\tilde{\mathcal{A}}, \lambda)\| \leq \gamma_k,$$

where $\gamma_k \rightarrow 0$ and $r := \sup\{r_k\}$ is a fixed radius of all $L_m^{(k)}$, thus we obtain $\|\mathcal{B}_k \tilde{x} - \tilde{A}_k^0 \tilde{x}\| \leq Cr \gamma_k \|\tilde{x}\| \leq \delta \|\tilde{x}\|$. This completes the proof of Lemma 2.2. $\square$

Lemma 2.3 *Let $A_0 \in M_n(\mathbb{C})$ be a matrix of Jordan form of eigenvalue 0, and let $\mathcal{F}$ be any family of matrices from a neighbourhood of A_0 t.i. $\mathcal{F} \subset U_\delta(A_0) = \{A \in M_n(\mathbb{C}) : \|A - A_0\| < \delta\}$. Then there exists such a constant M_δ , that for all $A \in \mathcal{F}$*

$$\|e^{At}\| \leq M_\delta e^{\omega t} (t^{n-1} + 1), \quad (16)$$

where $\omega = \sup_{A \in \mathcal{F}} \max_{\lambda \in \sigma(A)} \operatorname{Re} \lambda$.

Proof Without loss of generality we will assume that every $A \in \mathcal{F}$ has n different eigenvalues or we can replace such matrix by the sequence of matrices with different eigenvalues tending to matrix A . For any matrix $A \in \mathcal{F}$ we can write e^{At} as a finite

sum of powers of A with polynomial coefficients $\alpha_k^A(t)$ and estimate its norm by triangle inequality,

$$\|e^{At}\| \leq \sum_{k=0}^{n-1} |\alpha_k^A(t)| \cdot \|A^k\|.$$

There exists a constant $m(n, \delta)$ such for every Jordan form matrix A_0 of eigenvalue 0 and for every $A \in \mathcal{F}$ satisfies $\|A^k\| \leq m(n, \delta)$ for each $k = 0, \dots, n-1$. Using this we get the estimation

$$\|e^{At}\| \leq m \sum_{k=0}^{n-1} |\alpha_k^A(t)|. \quad (17)$$

Now we apply Statement 1 and Statement 2 of [Appendix](#) to determinant $\alpha_k^A(t)$ and expand it along $(k+1)$ -th row,

$$|\alpha_k^A(t)| = \left| \sum_{i=1}^n (-1)^{k+1+i} t^{i-1} f[\lambda_i t, \dots, \lambda_1 t] P_i^{(k)}(\lambda) \right|,$$

where $P_i^{(k)}(\lambda)$, $i, k \in \{1, \dots, n\}$ are polynomials (independent of A) of variables $\lambda_1, \dots, \lambda_n$ and $f[\lambda_i t, \dots, \lambda_1 t]$ is a divided difference of i -th order of function $f(z) = e^z$. We estimate $|\alpha_k^A(t)|$ using triangle inequality,

$$|\alpha_k^A(t)| \leq \sum_{i=1}^n t^{i-1} |f[\lambda_i t, \dots, \lambda_1 t]| \cdot |P_i^{(k)}(\lambda)|.$$

At the right-hand side we increase t^{i-1} to $(t^{n-1} + 1)$ and use Property 3 of divided differences (see [Appendix](#)), obtaining

$$|\alpha_k^A(t)| \leq e^{\omega t} (t^{n-1} + 1) \sum_{i=1}^n |f[(\lambda_i - \omega)t, \dots, (\lambda_1 - \omega)t]| \cdot |P_i^{(k)}(\lambda)|.$$

Terms $(\lambda_i - \omega)$ are in $\mathbb{C}^-$ for $i = 1, \dots, n$, thus we can use Property 2 of [Appendix](#) which states that there exists such a constant M , that $|f[(\lambda_i - \omega)t, \dots, (\lambda_1 - \omega)t]| < M$ for $i = 1, \dots, n$. Let us denote the closure of the set $\bigcup_{A \in \mathcal{F}} \sigma(A)$ by σ_F . Set σ_F is compact, so the family of polynomials $P_i^{(k)}(\lambda)$ is bounded on $\prod_{j=1}^n \sigma_F \subset \mathbb{C}^n$. Let $P = \max_{k,i} \max_{\lambda} |P_i^{(k)}(\lambda)|$. Then we have the estimation

$$|\alpha_k^A(t)| \leq e^{\omega t} (t^{n-1} + 1) M P \sum_{i=1}^n 1 = e^{\omega t} (t^{n-1} + 1) M P n.$$

Applying it to (17) we get

$$\|e^{At}\| \leq m \sum_{k=0}^{n-1} |\alpha_k^A(t)| \leq m \sum_{k=0}^{n-1} e^{\omega t} (t^{n-1} + 1) M P n = e^{\omega t} (t^{n-1} + 1) m M P n^2.$$

Denoting $M_\delta = mMPn^2$ we have the estimation

$$\|e^{At}\| \leq M_\delta e^{\omega t} (t^{n-1} + 1),$$

which concludes the proof of Lemma 2.3. $\square$

Proof of Theorem 2.1 (cont.) Let us now fix time $T > 0$ and $\varepsilon > 0$. It is obvious that $\|e^{A_0(t-\tau)}\|$ and $\|e^{B_k\tau}\|$ are bounded uniformly for $\tau \in [0, t]$, $t \in [0, T]$. From Lemma 2.2 the family of matrices B_k is close to matrix A_0 , hence it satisfies assumptions of Lemma 2.3. Norms $\|e^{B_k\tau}\|$ are also bounded uniformly with respect to k , thus we can rewrite inequality (15) in the form

$$\|e^{B_k t} \tilde{x} - e^{\tilde{A}_k^0 t} \tilde{x}\| \leq M_T \|B_k - \tilde{A}_k^0\| \|\tilde{x}\|, \quad t \in [0, T], \quad (18)$$

where constant M_T is independent of k . Using the above and (13) we obtain

$$\|e^{A_t} x\|_{M_2} \geq \|T_N\|^{-1} e^{\omega t} (\|e^{\tilde{A}_k^0 t} \tilde{x}\| - M_T \|B_k - \tilde{A}_k^0\| \|\tilde{x}\|). \quad (19)$$

Now, let us assume that $x \in V_k$ is of the form $x := T_N^{-1} \tilde{x}_k^q$, where $\tilde{x}_k^q \in \tilde{V}_k$ is the rootvector of $\tilde{A}_k$ of the maximal order q . That means $\|e^{\tilde{A}_k^0 t} \tilde{x}_k^q\|^2 = \|e^{0 \cdot t} (\tilde{x}_k^q + t \tilde{x}_k^{q-1} + \dots + \frac{t^{q-1}}{(q-1)!} \tilde{x}_k^1)\|^2$. Jordan chains of operator $\tilde{A}$ constitute a Riesz basis in M_2 (see Remark 9 in [15]) so there exists a constant $c > 0$ such that $\|\tilde{x}_k^q + t \tilde{x}_k^{q-1} + \dots + \frac{t^{q-1}}{(q-1)!} \tilde{x}_k^1\|^2 \geq c^2 (\|\tilde{x}_k^q\|^2 + \|t \tilde{x}_k^{q-1}\|^2 + \dots + \|\frac{t^{q-1}}{(q-1)!} \tilde{x}_k^1\|^2)$. Thus we have $\|e^{\tilde{A}_k^0 t} \tilde{x}_k^q\|^2 \geq c^2 (\|\tilde{x}_k^q\|^2 + \|\frac{t^{q-1}}{(q-1)!} \tilde{x}_k^1\|^2)$. Notice also that $\|\tilde{x}_k^1\|, \dots, \|\tilde{x}_k^q\|$ are independent of k (see Theorem 7 and Remark 9 in [15]), which gives us $\|e^{\tilde{A}_k^0 t} \tilde{x}_k^q\|^2 \geq c^2 (\|\tilde{x}_k^q\|^2 + \|\frac{t^{q-1} c_1}{(q-1)!}\|^2 \|\tilde{x}_k^q\|^2)$ for some constant c_1 , again independent of k . Reassum- ing, we have

$$\|e^{\tilde{A}_k^0 t} \tilde{x}_k^q\| \geq c(1 + t^{q-1}) \|\tilde{x}_k^q\|$$

for some new positive constant c independent of k . Inserting $x = x_k^q = T_N^{-1} \tilde{x}_k^q$ to (19) and using the above we conclude that

$$\|e^{A_t} x_k^q\|_{M_2} \geq \|T_N\|^{-1} e^{\omega t} (c(1 + t^{q-1}) - M_T \|B_k - A_0\|) \|\tilde{x}_k^q\|.$$

Denoting constant $c\|T_N\|^{-1}$ by M_q and choosing $\delta = \frac{\varepsilon\|T_N\|}{M_T} \min\{1, e^{-\omega T}\}$ we obtain by Lemma 2.2

$$\|e^{A_t} x_k^q\|_{M_2} \geq (f_q(t) - \varepsilon) \|x_k^q\|, \quad t \in [0, T], \quad |k| > k_0,$$

and the proof of Theorem 2.1 is complete. $\square$

Remark 1 Recently an upper bound of solutions' growth rate of a general nonhomogeneous equation was given in [9], see Theorem 7.4.1. In the case of Eq. (2) it takes the form $\|x(t)\| \leq M e^{\omega t} t^{N_0}$, where N_0 is a generally unknown constant, which depends on the location of the spectrum of operator $\mathcal{A}$. In this context we would

like to note that our result (Theorem 2.1(i)) concretizes those estimations. Besides, our approach allows us to obtain also a lower bound of the norm of a corresponding semigroup of matrix A_{-1} .

Remark 2 In Theorem 2.1 terms $(t^{p-1} + 1)$ and $(t^{q-1} + 1)$ can be replaced by t^{p-1} and t^{q-1} but the estimation will be valid for $t > 1$.

Corollary 2.4 *If we omit the assumption that $\operatorname{Re} \lambda < \omega$, $\lambda \in \sigma(\mathcal{A})$, then using the same methods as in the proof of Theorem 2.1 we can prove that there exist positive constants m, M such that*

- (i) $\|e^{At}\| \leq M e^{\omega t} t^{\max\{p, q_0\}-1}, t > 1,$
- (ii) $\|e^{At}\| \geq m e^{\omega t} t^{\max\{q, q_0\}-1}, t > 1,$

where q_0 is the maximal size of Jordan block corresponding to eigenvalues $\lambda \in \sigma(\mathcal{A}) : \operatorname{Re} \lambda = \omega$.

Without assumption $\operatorname{Re} \lambda < \omega$, $\lambda \in \sigma(\mathcal{A})$ we can not estimate the multiplicity of eigenvalues of $\mathcal{A}$ with real parts equal ω in terms of matrix A_{-1} . If we take $\max\{p, q_0\}$ instead of p in the proof of Theorem 2.1 we get assertion (i). Assertion (ii) follows from the fact that in the proof of Theorem 2.1 we did not use assumption $\operatorname{Re} \lambda < \omega$ and from validity of the estimation $\|e^{At}\| \geq m e^{\omega t} t^{q_0-1}, t > 1$.

Corollary 2.5 *If matrix A_{-1} has only one Jordan block corresponding to the eigenvalue of maximal modulus, that is $p = q$, then the norm of the semigroup e^{At} behaves exactly like the function f_q in the sense $f_q(t) \leq \|e^{At}\| \leq C f_q(t)$ for some constant C .*

In general, $p \neq q$ and the estimation given in Theorem 2.1 is not precise, the upper and lower estimations differ by degree of the polynomial. The following examples show that for a fixed A_{-1} the both estimations, upper and lower, can be reached for some A_2, A_3 . It means that in general Theorem 2.1 cannot be improved without further assumptions on A_2, A_3 .

Example 2.1 We consider Eq. (1) with $A_{-1} = I$, $A_2 = f(\theta)I$, $A_3 = g(\theta)I$, where $f, g \in L_2(-1, 0)$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We show that there exist functions f, g such that the lower estimation from Theorem 2.1 is reached. Equation (1) has the form

$$\dot{z}(t) = I \left(\dot{z}(t-1) + \int_{-1}^0 f(\theta) \dot{z}(t+\theta) d\theta + \int_{-1}^0 g(\theta) z(t+\theta) d\theta \right). \quad (20)$$

Matrix A_{-1} has one eigenvalue $\mu_1 = 1$ of multiplicity $p_1 = 2$, thus operator $\tilde{\mathcal{A}}$ has a sequence of eigenvalues of the form $\{\tilde{\lambda}_1^k = 2k\pi i\}_{k \in \mathbb{Z}}$. If $z = (z^{(1)}, z^{(2)})^T$, then the Eq. (20) can be splitted into two independent equations,

$$\begin{cases} \dot{z}^{(1)}(t) = \dot{z}^{(1)}(t-1) + \int_{-1}^0 f(\theta) \dot{z}^{(1)}(t+\theta) d\theta + \int_{-1}^0 g(\theta) z^{(1)}(t+\theta) d\theta, \\ \dot{z}^{(2)}(t) = \dot{z}^{(2)}(t-1) + \int_{-1}^0 f(\theta) \dot{z}^{(2)}(t+\theta) d\theta + \int_{-1}^0 g(\theta) z^{(2)}(t+\theta) d\theta, \end{cases} \quad (21)$$

where each equation is a scalar version of Eq. (1). Using Theorem 8 from [16] to both scalar equations we get that for any sequence $\tau^{(k)}$ quadratically close to sequence $\tilde{\lambda}_1^{(k)}$ there are such functions f, g that $\lambda_1^{(k)} = \tau^{(k)}$. Hence in the case of Eqs. (21) there exist functions f, g such that the spectrum $\lambda_1^{(k)}$ of corresponding operator $\mathcal{A}$ is in the left open halfplane, thus assumptions of Theorem 2.1 are satisfied. In this case $p = 2$ and $q = 1$ so the estimation

$$M_q \leq \|e^{\mathcal{A}t}\| \leq M_p(t + 1)$$

from Theorem 2.1 is not precise. On the other hand it is clear that there exists a Riesz basis consisted of eigenvectors of operator $\mathcal{A}$. Thus it is easy to see that $\|e^{\mathcal{A}t}\| \leq M_1$. Hence we proved that there exist functions f, g such that $M_q \leq \|e^{\mathcal{A}t}\| \leq M_1$, which means that the lower estimation from Theorem 2.1 can be reached.

Example 2.2 We consider (1) with the same matrix $A_{-1} = I$ and $A_2 = f(\theta)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $A_3 = g(\theta)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $f, g \in L_2(-1, 0)$. Now Eq. (1) has the form

$$\dot{z}(t) = \dot{z}(t-1)I + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \left(\int_{-1}^0 f(\theta) \dot{z}_t(\theta) d\theta + \int_{-1}^0 g(\theta) z_t(\theta) d\theta \right). \quad (22)$$

The corresponding characteristic function Δ is of the form

$$\begin{aligned} \Delta(\lambda) = \det & \left[I \left(\lambda e^{-\lambda} + \lambda \int_{-1}^0 f(s) e^{\lambda s} ds + \int_{-1}^0 g(s) e^{\lambda s} ds - \lambda \right) \right. \\ & \left. + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \left(\lambda \int_{-1}^0 f(s) e^{\lambda s} ds + \int_{-1}^0 g(s) e^{\lambda s} ds - \lambda \right) \right], \end{aligned}$$

and it equals zero only if

$$\lambda = \lambda e^{-\lambda} + \lambda \int_{-1}^0 f(s) e^{\lambda s} ds + \int_{-1}^0 g(s) e^{\lambda s} ds. \quad (23)$$

Once again there exist functions f, g such that all eigenvalues of corresponding operator $\mathcal{A}$ are in the open left halfplane, quadratically close to $\{\tilde{\lambda}_1^k = 2k\pi i\}_{k \in \mathbb{Z}}$ and each one has a two-dimensional Jordan block. Thus we can easily show that $\|e^{\mathcal{A}t}\|$ behaves like a function $M(t + 1)$ i.e. there exist positive constants C_1, C_2 such that $C_1(t + 1) \leq \|e^{\mathcal{A}t}\| \leq C_2(t + 1)$.

The examples above show that for fixed matrix A_{-1} and proper choice of A_2, A_3 the top or bottom estimation of the norm of semigroup $e^{\mathcal{A}t}$ can be reached. One can ask if the constants M_p, M_q can be chosen independently on $\mathcal{A}$. The answer is negative in general. Next theorem states that constants M_p, M_q can be chosen uniformly, but the estimations are valid for initial states from subspace of M_2 of finite codimension $2(N + 1)n$ (see [15] Theorem 15), which depends on $\mathcal{A}$, namely we have

Theorem 2.6 Let us consider a semigroup $e^{\mathcal{A}t}$, $t \geq 0$, where operator $\mathcal{A}$ is defined by (3)–(4). If we assume that $\operatorname{Re} \lambda < \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(\mathcal{A})\}$ for any $\lambda \in \sigma(\mathcal{A})$ then there exist the constants M_q, M_p , which are independent of A_2, A_3 such that the following inequalities hold

1. $\|e^{\mathcal{A}t}x\| \leq f_p(t)\|x\| = M_p e^{\omega t} (t^{p-1} + 1)\|x\|$, $x \in M_2^{(N)}$,
2. $\|e^{\mathcal{A}t}\|_{M_2^{(N)}} \geq f_q(t) = M_q e^{\omega t} (t^{q-1} + 1)$,

where $M_2^{(N)}$ is some $\mathcal{A}$ -invariant subspace of M_2 of finite codimension $2(N+1)n$, q is the maximal size of Jordan block of matrix A_{-1} corresponding to eigenvalue μ_1 and p is the sum of sizes of all Jordan blocks corresponding to μ_1 .

Proof For any $\mathcal{A}$, subspaces $\{V_m^{(k)}\}_{m=1, \dots, \ell; |k| > N}$ constitute the Riesz basis in their linear span, say $M_2^{(N)}$. Thus for any $x \in M_2^{(N)}$ we have $x = \sum_{m=1}^{\ell} \sum_{|k| > N} x_m^{(k)}$, $x_m^{(k)} \in V_m^{(k)}$, and

$$c_A \sum_{\substack{m=1, \dots, \ell \\ |k| > N}} \|x_m^{(k)}\|^2 \leq \|x\|^2 \leq C_A \sum_{\substack{m=1, \dots, \ell \\ |k| > N}} \|x_m^{(k)}\|^2, \quad x_m^{(k)} \in V_m^{(k)}. \quad (24)$$

We will show that for any $\mathcal{A}$ there exists N such that the constants C_A and $1/c_A$ are uniformly bounded with respect to A_2, A_3 . Indeed the basis subspaces $V_m^{(k)}$ are images of basis subspaces $\tilde{V}_m^{(k)}$ under operator T_N , which is bounded and has bounded inverse for N large enough (see proof of Theorem 2.1). Thus operator T_N transforms Riesz basis to another Riesz basis, but of course the choice of N depends on $\mathcal{A}$. Hence the constants can be chosen as $C_A = c_A^{-1} = \|T_N\|^2 \|T_N^{-1}\|^2 \|\tilde{T}\|^2 \|\tilde{T}^{-1}\|^2$, where the operator $\tilde{T}$ transforms Riesz basis from subspaces $\tilde{V}_k$ onto orthonormal basis.

Space $M_2^{(N)}$ is the direct sum of $Q_m^{(N)}$, $m = 1, 2, \dots, \ell$, where $Q_m^{(N)} = \bigoplus_{|k| > N} V_m^{(k)}$. For any $x \in M_2^{(N)}$, using the above and (24), we have

$$\|e^{\mathcal{A}t}x\|^2 \leq \left\| \sum_{m=1}^{\ell} \sum_{|k| > N} e^{\mathcal{A}t} x_m^{(k)} \right\|^2 \leq C_A \sum_{m=1}^{\ell} \sum_{|k| > N} \|e^{A_m^{(k)}t} x_m^{(k)}\|^2.$$

Then

$$\|e^{\mathcal{A}t}x\|^2 \leq C_A \sum_{m=1}^{\ell} e^{2\omega_m t} \sum_{|k| > N} \|e^{A_m^{(k),0}t} x_m^{(k)}\|^2,$$

where $\omega_m = \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A_m^{(k)}), |k| > N\}$ and $A_m^{(k),0} = A_m^{(k)} - \tilde{\lambda}_m^{(k)} I$. Substituting $x_m^{(k)} = T_N^{-1} \tilde{x}_m^{(k)}$ we obtain

$$\|e^{\mathcal{A}t}x\|^2 \leq C_A \|T_N^{-1}\|^2 \sum_{m=1}^{\ell} e^{2\omega_m t} \sum_{|k| > N} \|e^{B_m^{(k)}t} \tilde{x}_m^{(k)}\|^2, \quad (25)$$

where $\mathcal{B}_m^{(k)}$ is defined by $\mathcal{B}_m^{(k)} = T_N A_m^{(k),0} T_N^{-1}$. For any fixed ε and A_2, A_3 there exists N large enough (see Lemma 2.2) that $\|\mathcal{B}_m^{(k)} - \tilde{\mathcal{A}}_m^{(k),0}\| < \varepsilon$. Thus from Lemma 2.3 we obtain that there exists constant M , which is independent of $\mathcal{A}$, such that

$$\|e^{\mathcal{B}_m^{(k)} t} \tilde{x}_m^{(k)}\| \leq M(t^{p_m-1} + 1) \|\tilde{x}_m^{(k)}\|.$$

Hence (25) can be rewritten in the form

$$\|e^{\mathcal{A}t} x\|^2 \leq C_A \|T_N^{-1}\|^2 M^2 \sum_{m=1}^{\ell} e^{2\omega_m t} (t^{p_m-1} + 1)^2 \sum_{|k|>N} \|\tilde{x}_m^{(k)}\|^2.$$

For each $m > m_0$ there exists constant M_0 such that $e^{\omega_m t} (t^{p_m-1} + 1) \leq M_0 e^{\omega t}$, $t \geq 0$, $m > m_0$. Using the fact that $C_A = 1/c_A$, $M_0 \geq 1$ and $\|\tilde{x}\| \leq \|T_N\| \cdot \|x\|$ we obtain

$$\|e^{\mathcal{A}t} x\|^2 \leq (C_A M M_0 \|T_N\| \|T_N^{-1}\|)^2 e^{2\omega t} (t^{p-1} + 1)^2 \|x\|^2.$$

Hence there exists constant $M_p := C_A M M_0 \|T_N\| \|T_N^{-1}\|$, which is independent of A_2, A_3 such that for some large enough N ,

$$\|e^{\mathcal{A}t} x\| \leq M_p e^{\omega t} (t^{p-1} + 1)^2 \|x\|, \quad x \in M_2^{(N)}.$$

To prove part (ii) one needs to check that all constants used in proof of Theorem 2.1(ii) can be chosen independently on A_2, A_3 for some N large enough. In particular constant M_T is determined by the constant M_δ , which depends only on dimension n and δ (see Lemma 2.3). For any A_2, A_3 we can choose N large enough such that $1/2 < \|T_N\|$, $\|T_N^{-1}\| < 3/2$ (see proof of Theorem 2.1(i)). The rest of the proof is the same as in Theorem 2.1. $\square$

3 Maximal Asymptotics and Estimation of Individual Solutions Growth

Knowing the asymptotic behaviour of $\|e^{\mathcal{A}t}\|$ one can analyse the behaviour of individuals $\|e^{\mathcal{A}t} x\|$. The question about existence of the solution of the fastest growth is of particular interest. This problem was studied for an abstract linear differential equation in Banach space [17]. We say that equation $\dot{x} = Ax$ (or semigroup $e^{\mathcal{A}t}$, $t \geq 0$) has maximal asymptotics if there exists a real, positive function $f(t)$, $t \geq 0$, such that the function $\frac{\|e^{\mathcal{A}t}\|}{f(t)}$ is bounded on $[a, +\infty)$, for some $a \geq 0$, and there exists at least one $x_0 \in X$ such that

$$\lim_{t \rightarrow +\infty} \frac{\|e^{\mathcal{A}t} x_0\|}{f(t)} = 1.$$

It is easy to see that every finite-dimensional system has maximal asymptotics. Thus the question about maximal asymptotics is interesting only in the case of infinite-dimensional systems. In particular, it was shown (see Theorem 5, 9 [17]) that if $\rho(A) \cap \{\lambda : \operatorname{Re} \lambda = \omega_0\}$ is at most countable, and the adjoint operator A^* does not possess eigenvalues with real part ω_0 then the equation $\dot{x} = Ax$ does not have any

maximal asymptotics. If, in addition, function $\log f(t)$ concaves down then the function f increases faster than any solution of the system $\dot{x} = Ax$, i.e.

$$\lim_{t \rightarrow +\infty} \frac{\|e^{At}x\|}{f(t)} = 0,$$

for any $x \in X$. As a consequence of the above and of previous section results we obtain the following.

Theorem 3.1 *Given operator A from Eq. (2), and eigenvalues $\lambda \in \sigma(A)$ satisfying $\operatorname{Re} \lambda < \omega$, with $\omega = \sup_{\lambda \in \sigma(A)} \operatorname{Re} \lambda$, we have*

$$\lim_{t \rightarrow \infty} \frac{\|e^{At}x\|}{M_p e^{\omega t} t^{p-1}} = 0 \quad (26)$$

for any initial state $x \in M_2$. If, in addition, the matrix A_{-1} has only one Jordan block corresponding to maximal (modulus) eigenvalue, then for any positive function $g(t)$, with $\frac{g(t)}{M_p e^{\omega t} (t^{p-1})} \rightarrow 0$ as $t \rightarrow +\infty$, there exists an initial state $x \in M_2$ such that

$$\frac{\|e^{At}x\|}{g(t)} \text{ is unbounded on the interval } [0, +\infty).$$

Proof The function $M_p e^{\omega t} (t^{p-1} + 1)$ is the upper bound of $\|e^{At}\|$ and logarithm of $M_p e^{\omega t} (t^{p-1} + 1)$ concaves down. Therefore (26) follows directly from Theorem 9 [17]. Now let us assume that matrix A_{-1} has only one Jordan block corresponding to maximal (modulus) eigenvalue (i.e. $p = q$) and for any initial state $x \in M_2$ the quotient $\frac{\|e^{At}x\|}{g(t)}$ is bounded on the interval $[0, +\infty)$. From Banach-Steinhaus Theorem we obtain that the operator $e^{At}/g(t)$ is bounded by some constant M . But by Theorem 2.1 $\|e^{At}\| \geq m M_p e^{\omega t} (t^{p-1})$, which means

$$m \frac{M_p e^{\omega t} (t^{p-1})}{g(t)} \leq \frac{\|e^{At}\|}{g(t)} \leq M$$

and that contradicts the fact that $\frac{M_p e^{\omega t} (t^{p-1})}{g(t)}$ tends to infinity. $\square$

In the following theorems we describe the conditions under which the Eq. (2) does or does not have maximal asymptotics. We recall that $\omega = \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A)\}$, $\tilde{\omega} = \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(\tilde{A})\}$, and we denote by q_0 the maximal order of rootvector corresponding to eigenvalue $\lambda \in \sigma(A)$ with $\operatorname{Re} \lambda = \omega$. If $\{\lambda \in \sigma(A) : \operatorname{Re} \lambda = \omega\} = \emptyset$ then we put $q_0 := 0$. We notice that in general $\omega \geq \tilde{\omega}$ and the following theorem describes asymptotic behaviour of Eq. (2) in both cases $\omega > \tilde{\omega}$ and $\omega = \tilde{\omega}$.

Theorem 3.2 *If a real, positive function f is maximal asymptotics of Eq. (2) then $q_0 \geq 1$ and $f \approx t^{q_0-1} e^{\omega t}$ i.e. there exist positive constants c, C with*

$$0 < c = \liminf_{t \rightarrow +\infty} \frac{t^{q_0-1} e^{\omega t}}{f} \leq \limsup_{t \rightarrow +\infty} \frac{t^{q_0-1} e^{\omega t}}{f} = C < +\infty.$$

Moreover, function $t^{q_0-1}e^{\omega t}$ is also maximal asymptotics of this equation.

Proof If there is no eigenvalue of $\mathcal{A}$ on the line $\operatorname{Re} z = \omega$ (i.e. $q_0 = 0$), then Eq. (2) can not possess any maximal asymptotics (see [17] for more details). Therefore $q_0 \geq 1$. Now we prove that $C < +\infty$. We denote by x_0 the rootvector of operator $\mathcal{A}$ of order q_0 corresponding to an eigenvalue of real part ω . The solution corresponding to the state x_0 satisfies the following inequality

$$\frac{\|e^{At}x_0\|}{t^{q_0-1}e^{\omega t}} \geq m, \quad t \geq t_0, \quad (27)$$

where m, t_0 are some positive constants. Function f is maximal asymptotics, what means, in particular, that there exist constants M, t_0 such that for all $x \in M_2$

$$\frac{\|e^{At}x\|}{f(t)} \leq M\|x\|, \quad t \geq t_0. \quad (28)$$

Putting $x = x_0$ into (28) and combining with (27) we get

$$\frac{t^{q_0-1}e^{\omega t}}{f(t)} \leq \frac{M\|x_0\|}{m} \quad \text{for } t \geq t_0$$

and that proves the first part of the theorem.

To prove that $c > 0$, we proceed by contradiction. Let us assume that $c = 0$. This implies that there exists a sequence $t_j \rightarrow +\infty$ such that

$$\lim_{j \rightarrow +\infty} \frac{t_j^{q_0-1}e^{\omega t_j}}{f(t_j)} = 0.$$

Now, for any $x \in M_2$ we have

$$\lim_{j \rightarrow +\infty} \frac{\|e^{At_j}x\|}{f(t_j)} = 0. \quad (29)$$

Indeed, to see this, we observe first, that if the state $x \in V_m^{(k)}$, for $k \in \mathbb{Z}, m = 1, \dots, \ell$, then (29) follows from the fact that every subspace $V_m^{(k)}$ is finite-dimensional and for each m, k there exist constants $M_{m,k}, t_{m,k}$ such that $\|e^{At}x\| \leq M_{m,k}t^{q_0-1}e^{\omega t}\|x\|$, $x \in V_m^{(k)}, t > t_{m,k}$. Hence

$$0 \leq \frac{\|e^{At_j}x\|}{f(t_j)} \leq M_{m,k}\|x\| \cdot \frac{t_j^{q_0-1}e^{\omega t_j}}{f(t_j)} \xrightarrow{j \rightarrow +\infty} 0.$$

If x is a finite combination of states from $V_m^{(k)}$ then (29) follows for the same reason. Let now $x \in \bigoplus_{k \in \mathbb{Z}; m=1, \dots, \ell} V_m^{(k)}$. We take sequence $\{x_n\} \subset \bigoplus_{k \in \mathbb{Z}; m=1, \dots, \ell} V_m^{(k)}$, such that $x_n \rightarrow x$. Using triangle inequality and (28) we obtain

$$0 \leq \frac{\|e^{At}x\|}{f(t)} \leq M\|x - x_n\| + \frac{\|e^{At}x_n\|}{f(t)}, \quad t \geq t_0.$$

Taking $t = t_j$ and letting j tend to $+\infty$ in the above, we obtain for every $n \in \mathbb{N}$

$$0 \leq \lim_{j \rightarrow +\infty} \frac{\|e^{\mathcal{A}t_j} x\|}{f(t_j)} \leq M\|x - x_n\| + 0.$$

Then we apply the limit with $n \rightarrow +\infty$ in the above inequality to get

$$\lim_{j \rightarrow +\infty} \frac{\|e^{\mathcal{A}t_j} x\|}{f(t_j)} = 0, \quad \text{for any } x \in \overline{\bigoplus_{k \in \mathbb{Z}; m=1, \dots, \ell} V_m^{(k)}} = M_2.$$

But since the function f is maximal asymptotics, then there exists a state $x_0 \in M_2$, such that

$$\lim_{j \rightarrow +\infty} \frac{\|e^{\mathcal{A}t_j} x_0\|}{f(t_j)} = 1,$$

a contradiction. Finally, we observe that the proved inequality implies boundedness of the function

$$\varphi(t) = \frac{\|e^{\mathcal{A}t} x\|}{t^{q_0-1} e^{\omega t}}, \quad t \geq a,$$

for all $x \in M_2$. Besides, for some m_0, k_0 , there exists a rootvector $x_0 \in V_{m_0}^{(k_0)}$ such that

$$\lim_{t \rightarrow +\infty} \frac{\|e^{\mathcal{A}t} x_0\|}{t^{q_0-1} e^{\omega t}} = 1.$$

Thus the function $t^{q_0-1} e^{\omega t}$ is also maximal asymptotics. $\square$

Theorem 3.3 *The following hold.*

- (i) If $\omega > \tilde{\omega}$ then Eq. (2) possesses the maximal asymptotics, given by $t^{q_0-1} e^{\omega t}$.
- (ii) If $\omega = \tilde{\omega}$ then the existence of maximal asymptotics depends on relation between parameters q, q_0, p . (1) if $q_0 < q$ then Eq. (2) does not possess any maximal asymptotics, (2) if $q \leq q_0 < p$ then Eq. (2) may possess and may not possess the maximal asymptotics, (3) if $q_0 \geq p$ then Eq. (2) possesses the maximal asymptotics, which is $t^{q_0-1} e^{\omega t}$.

Proof (i) If there exists $\lambda \in \sigma(\mathcal{A})$, $\operatorname{Re} \lambda \geq \tilde{\omega}$, then for at least one eigenvalue $\lambda_0 \in \sigma(\mathcal{A})$ we have $\operatorname{Re} \lambda_0 = \max\{\operatorname{Re} \lambda, \lambda \in \sigma(\mathcal{A})\}$ and the size of Jordan block corresponding to λ_0 equals q_0 . There is only finite number of eigenvalues $\lambda \in \sigma(\mathcal{A})$ with $\operatorname{Re} \lambda_0 - \operatorname{Re} \lambda < \frac{1}{2}(\omega - \tilde{\omega})$ because the radii r_k of circles centered at $\tilde{\omega} + i(\arg \mu_0 + 2k\pi)$ containing the eigenvalues of operator $\mathcal{A}$ tend to 0 as $k \rightarrow \infty$. Thus $\|e^{\mathcal{A}t}\| = O(t^{q_0-1} e^{\lambda_0 t})$. On the other hand there is rootvector x_0 corresponding to λ_0 such that

$$\lim_{t \rightarrow \infty} \frac{\|e^{\mathcal{A}t} x_0\|}{t^{q_0-1} e^{\omega t}} = 1.$$

This means that the function $t^{q_0-1} e^{\omega t}$ is maximal asymptotics.

(ii) Let $q_0 < q$. We argue by contradiction. Let $f(t)$ be maximal asymptotics. According to Theorem 3.2 we have $f \approx t^{q_0-1}e^{\omega t}$. On the other hand it is shown (Remark 3 [17]) that $f \approx \|e^{At}\|$. From Theorem 2.1 we have also $\|e^{At}\| \geq Mt^{q-1}e^{\omega t}$. This implies that $q = q_0$, a contradiction. Let $q_0 \geq p$. It is easy to see that the function $f(t) = e^{\omega t}t^{q_0-1}$ is maximal asymptotics. Indeed from Corollary 2.4 for each $x \in M_2$ we have $\frac{\|e^{At}x\|}{f(t)} \leq M\|x\|$ and there exists a rootvector x_0 of order q_0 such that $\frac{\|e^{At}x_0\|}{f(t)} \rightarrow 1$.

If $q_0 \geq p$ then it is clear that there is a rootvector x_0 corresponding to λ_0 such that

$$\lim_{t \rightarrow \infty} \frac{\|e^{At}x_0\|}{t^{q_0-1}e^{\omega t}} = 1.$$

The inequality

$$\frac{\|e^{At}\|}{t^{q_0-1}e^{\omega t}} \leq M$$

follows directly from the Corollary 2.4. Thus the function $t^{q_0-1}e^{\omega t}$ is a maximal asymptotics.

If $q \leq q_0 < p$ then we do not know the exact behaviour of $\|e^{At}\|$. As we see (Example 2.1 and Example 2.2) the function $\varphi(t) = \|e^{At}\|$ may behave asymptotically like $t^{q_0-1}e^{\omega t}$ or $t^{p-1}e^{\omega t}$. According to Theorem 3.2 in the first case the function $t^{q_0-1}e^{\omega t}$ is maximal asymptotics, in the second case function $t^{q_0-1}e^{\omega t}$ is the best candidate for being maximal asymptotics but it is easy to see that

$$\frac{\|e^{At}\|}{t^{q_0-1}e^{\omega t}}$$

is not bounded and hence Eq. (2) does not have any maximal asymptotics. □

4 Selected Applications for Control Systems

In this section we give some applications of our results for control systems. Namely, we consider a controlled neutral type system of the form

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + \int_{-1}^0 A_2(\theta)\dot{z}(t+\theta)d\theta + \int_{-1}^0 A_3(\theta)z(t+\theta)d\theta + Bu, \quad (30)$$

where B is $n \times r$ matrix. Such system can be rewritten in the operator form

$$\dot{x} = \mathcal{A}x + \mathcal{B}u, \quad x \in D(\mathcal{A}) \subset M_2, \quad (31)$$

where $\mathcal{A}$ is introduced above and $\mathcal{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}$. We study two problems. First we consider the closed-loop control for system (31) with a feedback law of the form

$$u(t) = \mathcal{F} \begin{pmatrix} y \\ z_t(\cdot) \end{pmatrix} = \int_{-1}^0 F_2(\theta)\dot{z}(t+\theta)d\theta + \int_{-1}^0 F_3(\theta)z(t+\theta)d\theta, \quad (32)$$

where $F_2(\cdot), F_3(\cdot) \in L_2(-1, 0; \mathbb{C}^{r \times n})$. We want to investigate how adding of a control of this type can affect the system. In particular, we check the asymptotic behaviour of solutions under such controls. Next we consider the case of programming controls satisfying the constraint $\|u(t)\| \leq 1$. Our goal is to describe the asymptotic behaviour of the diameters of the sets of null-reachable states.

Let us discuss the first question in detail. It was shown in [16] that for any linear feedback control of the form (32) the displaced eigenvalues of operator $\mathcal{A} + \mathcal{BF}$ are still in the discs $L_m^{(k)}$ centered in $\tilde{\lambda}_m^{(k)}$, for $|k|$ large enough. On the other hand for any choice of small enough displacements there is a feedback, which dislocates eigenvalues to desired points under the condition that all eigenvalues $\tilde{\lambda}_m^{(k)}$ are simple. This means that for any choice of sequences $\tau_m^{(k)} \in L_m^{(k)}$, $k \in \mathbb{Z}$, $m = 1, \dots, \ell$ there exists a linear feedback $\mathcal{F}$ of the form (32) such that $\lambda_m^{(k)} = \tau_m^{(k)}$. The growth of the norm of the semigroup corresponding to Eq. (31) under no control is described by Corollary 2.4 as

$$me^{\omega t} t^{\max\{q, q_0\}-1} \leq \|e^{\mathcal{A}t}\| \leq Me^{\omega t} t^{\max\{p, q_0\}-1}, \quad t > 1. \quad (33)$$

The following Statement describes how the behaviour of solutions can be changed under a closed-loop control (32).

Statement 4.1 *Let us consider a control system (31). If there exists a feedback $\mathcal{F}$ of the form (32), which displaces the spectrum of operator $\mathcal{A}$ in such a way that $\operatorname{Re} \lambda < \tilde{\omega}$ for all $\lambda \in \sigma(\mathcal{A} + \mathcal{BF})$, then for some positive constants m, M we have*

$$me^{\tilde{\omega}t} t^{q-1} \leq \|e^{(\mathcal{A}+\mathcal{BF})t}\| \leq Me^{\tilde{\omega}t} t^{p-1}, \quad t > 1. \quad (34)$$

In particular this means that no choice of a control of the form (32) can decrease the exponents $\tilde{\omega}, q$ or increase p .

Note that for a simple $\tilde{\lambda} : \operatorname{Re} \tilde{\lambda} = \tilde{\omega}$ the assumption of Statement 4.1 is satisfied.

Proof Assertion follows from the fact that the system (31)–(32) can be represented as

$$\dot{x} = \mathcal{A}_1 x, \quad x \in D(\mathcal{A}_1) \subset M_2,$$

where $\mathcal{A}_1 = \mathcal{A} + \mathcal{BF}$, $D(\mathcal{A}_1) = D(\mathcal{A})$. Operator $\mathcal{A}_1$ generates a C_0 -semigroup and also is of the form (3) with the same matrix A_{-1} and disturbed A_2, A_3 (see [16] for more details). Thus assertion follows directly from Theorem 2.1. $\square$

Consider now the problem of null-controllability for the system (31). The detailed analysis of this problem for the case of unrestricted u is given in the work [14]. In particular it was proven that for sufficiently large time T if some special conditions on the matrices of system (30) are satisfied then system (31) is exact controllable (see Theorem 1.2 [14] for more details). The latter means that the null-reachability set is maximal possible, i.e.

$$\mathcal{R}_T = \left\{ \int_0^T e^{\mathcal{A}t} \mathcal{B}u(t) dt, u \in L_2([0, T], \mathbb{C}^r) \right\} = D(\mathcal{A}).$$

Let us consider the case of a bounded control: $\|u(t)\| \leq 1$. In this case null-reachability set $\tilde{\mathcal{R}}_T$ is given by

$$\tilde{\mathcal{R}}_T = \left\{ \int_0^T e^{-At} \mathcal{B}u(t) dt, u \in L_2([0, T], \mathbb{C}^r), \|u(t)\|_{\mathbb{C}^r} \leq 1 \right\} \subset D(\mathcal{A}).$$

Our goal is to find the asymptotical growth of the diameter of set $\tilde{\mathcal{R}}_T$ in time. Since $\tilde{\mathcal{R}}_T \subset D(\mathcal{A})$ the natural norm for our estimation is

$$\|x\|_{D(\mathcal{A})} = \|\mathcal{A}x\|_{M_2} + \|x\|_{M_2}.$$

First we prove the following lemma.

Lemma 4.2 *Let $T = 1$, then there exists a constant C —determined by the parameters of system (30)—such that for any $x \in \tilde{\mathcal{R}}_1$*

$$\|x\|_{D(\mathcal{A})} \leq C.$$

Proof Let $x \in \tilde{\mathcal{R}}_1$, we have

$$\|x\|_{D(\mathcal{A})}^2 = \left\| \int_0^1 e^{-At} \sum_{j=1}^r b_j u_j(t) dt \right\|_{D(\mathcal{A})}^2 \leq \sum_{j=1}^r \left\| \int_0^1 e^{-At} b_j u_j(t) dt \right\|_{D(\mathcal{A})}^2,$$

where $b_j, j = 1, \dots, r$ are the columns of operator $\mathcal{B}$. Since the family $\{V_m^{(k)} : |k| > N, k \in \mathbb{Z}\} \cup \{W_N\}$ constitutes Riesz basis in M_2 , then

$$\|x\|_{D(\mathcal{A})}^2 \leq C_1 \sum_{m=1}^{\ell} \sum_{|k| > N} k^2 \left\| P_m^{(k)} \int_0^1 e^{-At} \sum_{j=1}^r b_j u_j(t) dt \right\|_{M_2}^2,$$

where $C_1 > 0$ is a constant. Without loss of generality we assume that $r = \ell = 1$ and omit indices j, m ,

$$\begin{aligned} & \|x\|_{D(\mathcal{A})}^2 \\ & \leq C_2 \sum_{|k| \geq N} k^2 \left(\left\| \int_0^1 (P_k e^{-At} - \tilde{P}_k e^{-\tilde{A}t}) bu(t) dt \right\|_{M_2}^2 + \left\| \int_0^1 \tilde{P}_k e^{-\tilde{A}t} bu(t) dt \right\|_{M_2}^2 \right). \end{aligned}$$

We estimate both integrals using the properties of operators $\tilde{P}_k, P_k, S_k, A_0$ defined and discussed in detail in Sect. 2. Riesz projectors $\tilde{P}_k$ restricted to subspaces V_k are close to identity, hence there exists a constant m_1 such that $\|\tilde{P}_k x\| \geq m_1 \|x\|$ for any $x \in V_k$ and $|k|$ large enough. Applying this fact and Lemma 3.1 [14], which states that there exists a sequence $\{\alpha_k\} \in \ell_2$ such that $\|\tilde{P}_k (P_k e^{-At} - \tilde{P}_k e^{-\tilde{A}t}) bu(t)\|_{M_2} \leq \frac{\alpha_k}{|k|}$ we can estimate the first integral as follows

$$\|x\|_{D(\mathcal{A})}^2 \leq C_3 + C_2 \sum_{|k| \geq N} k^2 \left\| \int_0^1 \tilde{P}_k e^{-\tilde{A}t} bu(t) dt \right\|_{M_2}^2,$$

where C_3 is a new constant. Next we use $\tilde{P}_k e^{\tilde{A}t} = e^{\tilde{\lambda}_k t} S_k^{-1} e^{A_0 t} S_k$ and the fact that the norms of S_k, S_k^{-1} are uniformly bounded, thus

$$\|x\|_{D(\mathcal{A})}^2 \leq C_3 + C_4 \sum_{|k| \geq N} k^2 \left\| \int_0^1 e^{\tilde{\lambda}_k t} e^{A_0 t} S_k \tilde{P}_k b u(t) dt \right\|_{M_2}^2.$$

Since $\tilde{\lambda}_k = \operatorname{Re} \tilde{\lambda}_1 + i(2k\pi + c)$, where c is some constant, we have

$$\|x\|_{D(\mathcal{A})}^2 \leq C_3 + C_5 \sum_{|k| \geq N} k^2 \left| \int_0^1 e^{i2k\pi t} e^{\operatorname{Re} \tilde{\lambda}_1 t} e^{A_0 t} u(t) dt \right|^2 \|\tilde{P}_k b\|_{M_2}^2.$$

Applying Lemma 3.2 [14], which states that $\|\tilde{P}_k b\| \leq \frac{\alpha}{|k|}$, for some constant α and any k , we get

$$\|x\|_{D(\mathcal{A})}^2 \leq C_3 + C_5 \sum_{|k| \geq N} \left| \langle e^{\operatorname{Re} \tilde{\lambda}_1 \cdot} e^{A_0 \cdot} u(\cdot), e^{i2k\pi \cdot} \rangle_{L^2(0,1)} \right|^2.$$

The last series converges and its sum is at most $\|e^{\operatorname{Re} \tilde{\lambda}_1 \cdot} e^{A_0 \cdot} u(\cdot)\|_{L^2(0,1)}^2$. Thus under our assumption that $\|u(t)\| \leq 1$, we conclude that there exists a constant C such that $\|x\|_{D(\mathcal{A})}^2 \leq C$. $\square$

The result obtained above allows us to describe asymptotic behaviour of diameter of the set $\tilde{\mathcal{R}}_T$. To this end we take $T = n$ for $n \in \mathbb{N}$ and estimate $\operatorname{diam} \tilde{\mathcal{R}}_n$. Since set $\tilde{\mathcal{R}}_n$ is symmetric, $\operatorname{diam} \tilde{\mathcal{R}}_n = 2 \sup\{\|x\| : x \in \tilde{\mathcal{R}}_n\}$. The elements of $\tilde{\mathcal{R}}_n$ are of the form $\int_0^n e^{At} \mathcal{B}u(t) dt$, hence we can split this integral as $\sum_{k=0}^{n-1} \int_k^{(k+1)} e^{At} \mathcal{B}u(t) dt$ and substitute $s = t - k$. This way we obtain

$$\operatorname{diam} \tilde{\mathcal{R}}_n \leq \operatorname{diam} \tilde{\mathcal{R}}_1 \sum_{k=0}^{n-1} \|e^{Ak}\|_{M_2}.$$

The last norm can be estimated using Theorem 2.1, thus

$$\operatorname{diam} \tilde{\mathcal{R}}_n \leq \operatorname{diam} \tilde{\mathcal{R}}_1 \sum_{k=0}^{n-1} e^{\omega k} k^{p-1}. \quad (35)$$

Lemma 4.2 and (35) imply the following

Statement 4.3 *The asymptotics of $\tilde{\mathcal{R}}_T$ is as follows: if $\omega < 0$ then there exists a positive constant M such that*

$$\operatorname{diam} \tilde{\mathcal{R}}_T \rightarrow M, \quad T \rightarrow +\infty;$$

if $\omega = 0$ then there exists a positive constant M such that

$$\frac{\operatorname{diam} \tilde{\mathcal{R}}_T}{T^p} \rightarrow M, \quad T \rightarrow +\infty,$$

if $\omega > 0$ then there exists a positive constant M such that

$$\frac{\text{diam } \tilde{\mathcal{R}}_T}{e^{\omega T} T^{p-1}} \rightarrow M, \quad T \rightarrow +\infty.$$

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Appendix

In this section we recall some properties of divided differences and the statements on determinants used in the proof of Lemma 2.3.

Property 1 Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function, then divided difference has a representation as a series of the form

$$f[x_0, x_1, \dots, x_m] = \sum_{n=m}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{i_0+\dots+i_m=n-m} x_0^{i_0} x_1^{i_1} \dots x_m^{i_m}. \quad (36)$$

Corollary 1 Divided difference is symmetric, that is

$$f[x_0, x_1, \dots, x_n] = f[x_{\sigma(0)}, x_{\sigma(1)}, \dots, x_{\sigma(n)}] \quad \text{for any permutation } \sigma.$$

Corollary 2 Divided difference $f[x_0, x_1, \dots, x_n]$ can be extended to the whole $\mathbb{C}^{n+1}$ by the formula (36).

Property 2 Let us denote $\{x \in \mathbb{C} : \text{Re } x \leq 0\}$ by $\mathbb{C}^-$. If $f(x) = e^x$ then its divided difference is bounded on $(\mathbb{C}^-)^{n+1}$, i.e. for any n there exists constant $M_n > 0$, such that $|f[x_0, x_1, \dots, x_n]| \leq M_n$ for all $x \in (\mathbb{C}^-)^{n+1}$.

Property 3 For any $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^-)^n$, there exists constant $M(n, \delta)$ independent of λ, t , such that for all $t \geq 1$

$$|f[\lambda_1 t, \lambda_2 t, \dots, \lambda_n t]| \leq \frac{M(n, \delta)}{t^{n-p}}, \quad t \geq 1.$$

Property 4 If $f(x) = \exp(x)$, $x \in \mathbb{C}$, then for any $z \in \mathbb{C}$ and $n \in \mathbb{N}$ we have

$$f[x_0, x_1, \dots, x_n] = \exp(z) f[x_0 - z, x_1 - z, \dots, x_n - z]. \quad (37)$$

Statement 1 If matrix $A \in M_n(\mathbb{C})$ has n different eigenvalues λ_i then the coefficients $\alpha_k(t)$ of the representation $e^{At} = \sum_{k=0}^{n-1} \alpha_k(t) A^k$ can be written in the form $\alpha_k(t) =$

$\frac{W_k(t)}{W}$, where

$$W_k(t) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \dots & \lambda_n^{k-1} \\ e^{\lambda_1 t} & e^{\lambda_2 t} & \dots & e^{\lambda_n t} \\ \lambda_1^{k+1} & \lambda_2^{k+1} & \dots & \lambda_n^{k+1} \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{vmatrix}, \quad W = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{vmatrix}.$$

Proof Let x_i , $i = 1, \dots, n$ be the eigenvectors of matrix A corresponding to eigenvalues λ_i , $i = 1, \dots, n$ respectively. So we have a system of n equations

$$e^{At} x_i = \sum_{k=0}^{n-1} \alpha_k(t) A^k x_i, \quad i = 1, \dots, n.$$

Since $e^{At} x_i = e^{\lambda_i t} x_i$ and $A^k x_i = \lambda_i^k x_i$, we have

$$e^{\lambda_i t} x_i = \sum_{k=0}^{n-1} \alpha_k(t) \lambda_i^k x_i, \quad i = 1, \dots, n,$$

which is equivalent to

$$e^{\lambda_i t} = \sum_{k=0}^{n-1} \alpha_k(t) \lambda_i^k, \quad i = 1, \dots, n.$$

The above is a linear system with respect to $\alpha_k(t)$. Matrix of coefficients is a Vandermonde's matrix, with the determinant equal to W . Determinant W is non zero because $\lambda_i \neq \lambda_j$ for $i \neq j$. Thus using Cramer's rule we obtain a solution $\alpha_k(t) = \frac{W_k(t)}{W}$. This concludes the proof. $\square$

Statement 2 The quotient of determinants $\frac{W_k(t)}{W}$ can be written as

$$\begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ \lambda_1 & 1 & 0 & \dots & 0 \\ \lambda_1^2 & \lambda_2 + \lambda_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda_1^{k-1} & \sum_{i_2+i_1=k-2} \lambda_2^{i_2} \lambda_1^{i_1} & \sum_{i_3+i_2+i_1=k-3} \lambda_3^{i_3} \lambda_2^{i_2} \lambda_1^{i_1} & \dots & 0 \\ f[\lambda_1 t] & t f[\lambda_2 t, \lambda_1 t] & t^2 f[\lambda_3 t, \lambda_2 t, \lambda_1 t] & \dots & t^{n-1} f[\lambda_n t, \dots, \lambda_1 t] \\ \lambda_1^{k+1} & \sum_{i_2+i_1=k} \lambda_2^{i_2} \lambda_1^{i_1} & \sum_{i_3+i_2+i_1=k-1} \lambda_3^{i_3} \lambda_2^{i_2} \lambda_1^{i_1} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \sum_{i_2+i_1=n-2} \lambda_2^{i_2} \lambda_1^{i_1} & \sum_{i_3+i_2+i_1=n-3} \lambda_2^{i_2} \lambda_2^{i_2} \lambda_1^{i_1} & \dots & 1 \end{vmatrix},$$

where $f(x) = e^x$.

Proof Let $g_p(x) = x^p$, we can rewrite determinant $W_k(t)$ as follows

$$W_k(t) = \begin{vmatrix} g_0(\lambda_1) & g_0(\lambda_2) & \dots & g_0(\lambda_n) \\ g_1(\lambda_1) & g_1(\lambda_2) & \dots & g_1(\lambda_n) \\ \vdots & \vdots & & \vdots \\ g_{k-1}(\lambda_1) & g_{k-1}(\lambda_2) & \dots & g_{k-1}(\lambda_n) \\ f(\lambda_1 t) & f(\lambda_2 t) & \dots & f(\lambda_n t) \\ g_{k+1}(\lambda_1) & g_{k+1}(\lambda_2) & \dots & g_{k+1}(\lambda_n) \\ \vdots & \vdots & & \vdots \\ g_{n-1}(\lambda_1) & g_{n-1}(\lambda_2) & \dots & g_{n-1}(\lambda_n) \end{vmatrix}.$$

Denote j -th column of $W_k(t)$ by K_j and transform it in $n-1$ steps. First step: from the last column K_n subtract the previous one, i.e. K_{n-1} , divide it by $\lambda_n - \lambda_{n-1}$ and put the same factor in front of determinant. We do the same with columns $K_{n-1}, \dots, K_2$, i.e. we subtract the previous column and pull the corresponding difference of lambdas out of determinant. We denote it by $K_j \mapsto \frac{K_j - K_{j-1}}{\lambda_j - \lambda_{j-1}}$ for $j = n, n-1, \dots, 2$. This operation does not change the value of the determinant. Using the definition of divided differences, we can rewrite W_k as

$$W_k(t) = \prod_{j=2}^n (\lambda_j - \lambda_{j-1}) \begin{vmatrix} g_0[\lambda_1] & g_0[\lambda_2, \lambda_1] & \dots & g_0[\lambda_n, \lambda_{n-1}] \\ g_1[\lambda_1] & g_1[\lambda_2, \lambda_1] & \dots & g_1[\lambda_n, \lambda_{n-1}] \\ \vdots & \vdots & & \vdots \\ g_{k-1}[\lambda_1] & g_{k-1}[\lambda_2, \lambda_1] & \dots & g_{k-1}[\lambda_n, \lambda_{n-1}] \\ f[\lambda_1 t] & t f[\lambda_2 t, \lambda_1 t] & \dots & t^{n-1} f[\lambda_n t, \lambda_{n-1} t] \\ g_{k+1}[\lambda_1] & g_{k+1}[\lambda_2, \lambda_1] & \dots & g_{k+1}[\lambda_n, \lambda_{n-1}] \\ \vdots & \vdots & & \vdots \\ g_{n-1}[\lambda_1] & g_{n-1}[\lambda_2, \lambda_1] & \dots & g_{n-1}[\lambda_n, \lambda_{n-1}] \end{vmatrix}.$$

In m -th step we transform $K_j \mapsto \frac{K_j - K_{j-1}}{\lambda_j - \lambda_{j-m}}$ for $j = n, n-1, \dots, 1+m$. After $n-1$ steps, we obtain $W_k(t)$ in the following form

$$W_k(t) = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \begin{vmatrix} g_0[\lambda_1] & g_0[\lambda_2, \lambda_1] & \dots & g_0[\lambda_n, \lambda_{n-1}, \dots, \lambda_1] \\ g_1[\lambda_1] & g_1[\lambda_2, \lambda_1] & \dots & g_1[\lambda_n, \lambda_{n-1}, \dots, \lambda_1] \\ \vdots & \vdots & & \vdots \\ g_{k-1}[\lambda_1] & g_{k-1}[\lambda_2, \lambda_1] & \dots & g_{k-1}[\lambda_n, \lambda_{n-1}, \dots, \lambda_1] \\ f[\lambda_1 t] & tf[\lambda_2 t, \lambda_1 t] & \dots & t^{n-1} f[\lambda_n t, \lambda_{n-1} t, \dots, \lambda_1 t] \\ g_{k+1}[\lambda_1] & g_{k+1}[\lambda_2, \lambda_1] & \dots & g_{k+1}[\lambda_n, \lambda_{n-1}, \dots, \lambda_1] \\ \vdots & \vdots & & \vdots \\ g_{n-1}[\lambda_1] & g_{n-1}[\lambda_2, \lambda_1] & \dots & g_{n-1}[\lambda_n, \lambda_{n-1}, \dots, \lambda_1] \end{vmatrix}.$$

The product $\prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i)$ is a Vandermonde's determinant W . Notice, that $g_p^{(m)}(0) = p!$, for $m = p$ and zero for others m , thus from Property 1 of divided differences we have

$$g_p[\lambda_m, \dots, \lambda_1] = \begin{cases} \sum_{i_1 + \dots + i_m = p-m+1} \lambda_m^{i_m} \dots \lambda_1^{i_1}, & p > m-1, \\ 1, & p = m-1, \\ 0, & p < m-1, \end{cases}$$

what shows us the expected form of $\frac{W_k(t)}{W}$. $\square$

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