Erratum to: Risk-Sensitive Control with Near Monotone Cost

Anup Biswas · V.S. Borkar · K. Suresh Kumar

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In this note we give further corrections to [1] in addition to [2]. Awkwardness of such an exercise notwithstanding, the results come out much stronger than what was originally proposed: a key assumption (A2) can be significantly weakened.

In the proof of Theorem 2.1, the argument after (2.14) is not correct. So we give an alternate proof for [1, Theorem 2.1].

Note that $u^\alpha(\theta, x)$ given by (2.5) of [1] satisfies the p.d.e.

$$\alpha\theta \frac{\partial u^\alpha}{\partial \theta} = \inf_{v_1 \in V_1} \left[\nabla u^\alpha \cdot b(x, v_1) + \theta r(x, v_1) u^\alpha \right] + \frac{1}{2} \text{tr}(a(x) \nabla^2 u^\alpha), u^\alpha(0, x) = 1.$$

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A. Biswas

Centre for Applicable Mathematics, Tata Institute of Fundamental Research, Bangalore 560065, India

e-mail: anup@math.tifrbng.res.in

V.S. Borkar (✉)

School of Technology and Computer Science, Tata Institute of Fundamental Research, Mumbai 400005, India

e-mail: borkar@tifr.res.in

K. Suresh Kumar

Department of Mathematics, Indian Institute of Technology Bombay, Mumbai 400076, India

e-mail: suresh@math.iitb.ac.in

Also,

$$\alpha\theta \frac{\partial u^\alpha}{\partial \theta} = \theta \left(\alpha\phi^\alpha + \alpha\theta \frac{\partial \phi^\alpha}{\partial \theta} \right) u^\alpha.$$

Treating $\theta(\alpha\phi^\alpha + \alpha\theta \frac{\partial \phi^\alpha}{\partial \theta}) = g^\alpha(\theta, x)$ as a prescribed function, one can see that for each $\theta > 0$, $u^\alpha(\theta, x)$ is a solution to the p.d.e.

$$\inf_{v_1 \in V_1} \left[\nabla u^\alpha \cdot b(x, v_1) + (\theta r(x, v_1) - g^\alpha(\theta, x))u^\alpha \right] + \frac{1}{2} \operatorname{tr}(a(x) \nabla^2 u^\alpha) = 0.$$

From [1, Lemma 2.1], we have

$$\|g^\alpha\|_\infty \leq 3\theta \|r\|_\infty.$$

Set

$$\bar{u}^\alpha(\theta, x) = \frac{u^\alpha(\theta, x)}{u^\alpha(\theta, 0)}.$$

Then for each fixed $\theta > 0$, $\bar{u}^\alpha$ is a solution to the p.d.e.

$$\inf_{v_1 \in V_1} \left[\nabla \bar{u}^\alpha \cdot b(x, v_1) + (\theta r(x, v_1) - g^\alpha(\theta, x))\bar{u}^\alpha \right] + \frac{1}{2} \operatorname{tr}(a(x) \nabla^2 \bar{u}^\alpha) = 0. \quad (0.1)$$

Using the Harnack's inequality [3, p. 199], we have for each $R > 0$,

$$\sup_{x \in B_R} \bar{u}^\alpha(\theta, x) \leq C \inf_{x \in B_R} \bar{u}^\alpha(\theta, x) \leq C$$

where C denotes a constant independent of k and α . Now using [3, Theorem 9.11, p. 235] we have for each $R' < R$

$$\|\bar{u}^\alpha\|_{W^{2,p}(B_{R'})} \leq K \|\bar{u}^\alpha\|_{L^p(B_R)} \leq KC \operatorname{meas}(B_R)^{\frac{1}{p}}.$$

By standard approximation arguments using compact and continuous embedding theorems, there exists $\hat{u} \in W_{loc}^{2,p}(\mathbb{R}^n)$ such that

$$\bar{u}^\alpha \rightarrow \hat{u} \in W_{loc}^{1,p}(\mathbb{R}^n), \quad \bar{u}^\alpha \rightarrow \hat{u} \in W_{loc}^{2,p}(\mathbb{R}^n) \quad (\text{weakly})$$

along a subsequence as $\alpha \rightarrow 0$. By [1, Lemmas 2.1 and 2.2], it follows that along a further subsequence,

$$\alpha\phi^\alpha(\cdot, \cdot) \rightarrow \rho_1(\cdot) \quad (0.2)$$

in weak* topology of L^∞ on $(0, 1) \times \mathbb{R}^n$, for some $\rho_1(\cdot)$ which is a function of θ alone. That $\rho_1(\cdot)$ is a function of θ alone is proved as follows: by [1, Lemma 2.2],

$$\lim_{\alpha \downarrow 0} \int_{\mathbb{R}^n} \alpha \nabla \phi^\alpha(\theta, x) f(x) dx = 0$$

for each $f \in C_0^\infty(\mathbb{R}^n)$. Thus the distributional derivative of ρ_1 in x is identically zero, implying the claim. Also by [1, Lemma 2.1], for each fixed $\theta = \theta_0 > 0$, $\{\alpha \frac{\partial \phi^\alpha}{\partial \theta} | \alpha > 0\}$ is bounded in $L^\infty([\theta_0, 1] \times \mathbb{R}^n)$. Hence along a further subsequence

$$\alpha \frac{\partial \phi^\alpha}{\partial \theta} \rightarrow \rho_2(\cdot, \cdot) \quad (0.3)$$

weakly in $L_{loc}^2([\theta_0, 1] \times \mathbb{R}^n)$ for some $\rho_2 \in L_{loc}^2([\theta_0, 1] \times \mathbb{R}^n)$. It follows from (0.2) and (0.3) that $\rho_2(\cdot, \cdot) = \rho_1'$ in the sense of distribution, where ρ_1' is the distributional derivative (in θ) of ρ_1 . Hence $\rho_2(\cdot, \cdot)$ is also a function of θ alone. Thus we have: for each $\theta > 0$ there exists a constant ρ such that

$$\alpha \phi^\alpha + \alpha \theta \frac{\partial \phi^\alpha}{\partial \theta} \rightarrow \rho.$$

Hence for each $\theta > 0$,

$$g^\alpha(\theta, x) \rightarrow \theta \rho, \quad x \in \mathbb{R}^n$$

as $\alpha \rightarrow 0$ along the subsequence. Now by letting $\alpha \rightarrow 0$ in (0.1) along the subsequence, we can show that $(\rho, \hat{u})$ satisfies the p.d.e.

$$\theta \rho \hat{u} = \inf_{v_1 \in V_1} \left[L \hat{u}(x, v_1) + \theta r(x, v_1) \hat{u} \right]. \quad (0.4)$$

Then $(\rho, \hat{\phi})$ where $\hat{u} = e^{\theta \hat{\phi}}$ satisfies the p.d.e. (2.12) of [1]. This completes the proof. Some minor points:

1. Lemma 2.2 follows from (3.9) and using the fact that $\frac{1}{1+\|x\|^{n+1}}$ is bounded from below on each compact subset of $\mathbb{R}^n$. In particular we do not require the rest of the argument that follows (3.9), nor does this result require (A2).
2. Since we are not using (A2) of [1] in the proof of Lemma 2.2, Theorem 2.1 of [1] holds without (A2). We may replace (A2) with the following weaker stability assumption:

There exists a $w \in C^2(\mathbb{R}^n)$ satisfying:

$$(a) \quad \lim_{\|x\| \rightarrow \infty} w(x) = \infty, \quad w \geq 0.$$

(b) There exists $R > 0$ and $\epsilon > 0$ such that

$$Lw(x, u) < -\epsilon \quad \text{for } u \in V_1, \|x\| > R$$

where

$$Lf(x, v_1) = (\nabla f)b(x, v_1) + \frac{1}{2} \text{trace}(a(x) \nabla^2 f).$$

Then the proof of Theorem 2.2 given in [2] follows. Note that we need the above stability assumption for the application of Fatou's lemma in [2, lines 5–6 of last page], where it ensures the a.s. finiteness of τ_C .

3. The function $b : \mathbb{R}^n \times V_1 \rightarrow \mathbb{R}^n$ in Example 2.1 and the functions $b : \mathbb{R}^n \times V_1 \rightarrow \mathbb{R}^n$, $r : \mathbb{R}^n \times V_1 \rightarrow \mathbb{R}$ in Example 2.2 need to be affine in the second argument.

References

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