Emergence of Freidlin–Wentzell’s transmission conditions as a result of a singular perturbation of a semigroup

Adam Bobrowski¹

Received: 30 October 2014 / Accepted: 10 May 2015 / Published online: 23 May 2015
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Abstract We present a semigroup-theoretic approach to an example being one of the main motivations of the famous Freidlin–Wentzell averaging principle. In this example, as a result of a singular perturbation of the semigroups involved, the generator of the limit semigroup is related to a transmission condition emerging in a place where no transmission conditions were needed before.

Keywords Method of images · Cosine families · Strongly-continuous semigroup · Differential operators with transmission conditions · Convergence of semigroups · Freidlin–Wentzell’s transmission conditions

1 Introduction

The theory of one-parameter semigroups of operators, of beauty and importance of its own, is deeply rooted in the theories of partial differential equations and stochastic processes [13–15, 23, 24, 32]. Although there are good textbooks on semigroups which successfully avoid the notion of a stochastic process (see e.g. [14]), it is quite impossible to imagine the early stages of the theory without the impetus it has gained from the giants like Feller [16–18]. In fact, this is not surprising at all, since the very core of both theories lies in understanding dynamics, be it stochastic or deterministic. For example, it has been realized quite early that the main theorems of probability (e.g. the Central Limit Theorem), may be proved in an elegant way using approximation

Communicated by Jerome A. Goldstein.

✉ Adam Bobrowski
a.bobrowski@pollub.pl

¹ Lublin University of Technology, Nadbystrzycka 38A, 20-618 Lublin, Poland

theorems for semigroups [4, 15, 22]. On the other hand, it transpires that if seen from a proper perspective, exponential formulae of the semigroup theory become special instances of limit theorems of probability theory [11].

The theory of semigroups of operators and that of stochastic processes, continue to influence each other with mutual benefit. Modern books on Markov processes (like the old ones, see e.g. [13, 18]) treat semigroups, or at least Feller semigroups, extensively and take advantage of the core semigroup-theoretical results [15, 25]; in particular, since the publication of the treatise [15] it has become evident that convergence of stochastic processes may be efficiently treated with the help of Trotter–Kato-type theorems. On the other hand, phenomena encountered in stochastic processes do not cease to provide new and interesting challenges for semigroup-theorists.

This article is devoted to a semigroup-theoretical approach to a motivating example of the famous averaging principle of Freidlin and Wentzell [19, 21] (see also [20])—although the principle is quite well-known in probabilistic circles, its semigroup-theoretical treatment seems to be still missing. Such a treatment is needed for a clear picture of singular perturbations involving perturbation of boundary and transmission conditions. To explain, the approximation procedure considered here is a close relative of those of [2, 5, 9, 10]. In [5, 9, 10], as a result of stochastic averaging the limit semigroup is generated by a convex combination of generators of original semigroups. In [2] it is the domain of the generator of the limit semigroup that is perturbed: this domain is the kernel of a convex combination of operators describing domains of generators of original semigroups. In other words, the boundary or transmission conditions are changed in the limit, as a result of averaging. Here, in this paper, the averaging process leads to another, related, phenomenon: it forces *emergence* of transmission conditions at a point where they were not needed before.

2 Semigroup-theoretical tools

The main semigroup-theoretic devices we need are (a) the semigroups generated by operators related to quadratic forms in Hilbert spaces [12, 26, 27] (a theory that has been intensively developing in recent years, see [1, 3, 31] and the references cited there), and (b) Kurtz’s singular perturbation theorem [15, 28, 29].

To recall, suppose a sesquilinear form $\mathfrak{a}$ in a Hilbert space $\mathbb{X}$ is sectorial and closed, and that its domain $V = V_{\mathfrak{a}}$ is dense in $\mathbb{X}$. Then, there exists a linear, closed operator A with domain $D(A)$ satisfying the following conditions:

- (a) $D(A) \subset V_{\mathfrak{a}}$ and for $x \in D(A)$, $y \in V_{\mathfrak{a}}$,

$$\mathfrak{a}(x, y) = -(Ax, y),$$

- (b) if $x \in V_{\mathfrak{a}}$ and there is a $z \in X$ such that $\mathfrak{a}(x, y) = -(z, y)$ for all $y \in V_{\mathfrak{a}}$, then $x \in D(A)$ and $Ax = z$.

Using the Lax–Milgram lemma it may be shown that the range of $\lambda - A$ is the whole of $\mathbb{X}$ provided $\Re \lambda > \gamma$ where γ is the vertex of the sector

$$\Sigma_{\theta, \gamma} = \{z \in \mathbb{C}; |\arg(z - \gamma)| \leq \theta\}.$$

Also, A is uniquely determined by the conditions given above. Moreover, A generates a holomorphic semigroup in $\mathbb{X}$.

Turning to Kurtz’s singular perturbation theorem [15, 28, 29], let $(\epsilon_n)_{n \geq 1}$ be a sequence of positive numbers converging to 0. Suppose $A_n, n \geq 1$ are generators of equi-bounded semigroups $\{e^{tA_n}, t \geq 0\}$ in a Banach space $\mathbb{X}$, and Q generates a strongly continuous semigroup $(e^{tQ})_{t \geq 0}$ such that

$$\lim_{t \rightarrow \infty} e^{tQ}x =: Px, \quad x \in \mathbb{X} \quad (2.1)$$

exists. Then P is a bounded idempotent operator and

$$\text{Ker } Q = \text{Range } P, \quad \overline{\text{Range } Q} = \text{Ker } P. \quad (2.2)$$

Let

$$\mathbb{X}' = \text{Range } P.$$

Theorem 2.1 (Kurtz’s theorem) *Let A be an operator in $\mathbb{X}$, D be a subset of its domain, and assume that*

- (a) *for $x \in D$, there exist $x_n \in D(A_n)$ such that $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} A_n x_n = Ax$,*
- (b) *for y in a core D' of Q , there exist $y_n \in D(A_n)$ such that $\lim_{n \rightarrow \infty} y_n = y$ and $\lim_{n \rightarrow \infty} \epsilon_n A_n y_n = Qy$,*
- (c) *the operator PA with domain $D \cap \mathbb{X}'$ is closable and its closure $\overline{PA}$ generates a strongly continuous semigroup in $\mathbb{X}'$.*

Then

$$\lim_{n \rightarrow \infty} e^{A_n t}x = e^{t\overline{PA}}Px, \quad x \in \mathbb{X}, t > 0. \quad (2.3)$$

For $x \in \mathbb{X}'$ the same is true for $t = 0$ as well, and the limit is almost uniform in $t \in [0, \infty)$; for other x the limit is almost uniform in $t \in (0, \infty)$.

3 The example

Let us present the details of the example this paper is devoted to. To repeat: this example was one of the main motivations for the averaging principle of Freidlin and Wentzell [19, 21] (a semigroup theoretical treatment of another motivation for this principle is discussed in the companion article [8]). Imagine N diffusions (with different diffusion and drift coefficients) on N copies of an interval with Neumann boundary conditions at the ends, see Fig. 1. Suppose also that, as in e.g. [5, 9, 10], these diffusions are coupled by Markov chains: while at the i th copy, the process behaves according to the rules governed by operator

$$A_i = a_i \frac{d^2}{dx^2} + b_i \frac{d}{dx},$$

Fig. 1 Three diffusions on three copies of the unit interval coupled by Markov chains in the right part of the interval copies

Fig. 2 The limit process on a graph is generated by an averaged operator $\bar{A}$ (a convex combination of the involved operators A_i) with Neumann boundary conditions at the graph's ends, and balance, transmission conditions at point x^* where the segments meet, i.e. at the new vertex of the graph

but after a random time depending on its position, may jump to another copy of the interval to behave according to the rules described by the operator defined there. In distinction to the situation of [5, 9, 10] we assume, however, that on the left part of the interval no communication is possible: the intensities of jumps are zero here.

Freidlin and Wentzell's result says that [19, Theorem 5.1] as the intensities of jumps (in the right part of the interval) tend to infinity, the processes involved converge weakly to a diffusion on a graph formed by identifying corresponding points of all the right-parts of the intervals, see Fig. 2. The generator of the limit process is a convex combination of the generators of the involved diffusions—a phenomenon thoroughly studied in [5, 9, 10]. However, a new phenomenon is observed here as well: at the junction x^* where the intervals meet, transmission conditions need to be introduced. They are of the form:

$$\sum_{i=1}^N a_i(x^*) \pi_i(x^*) f'_{i,-}(x^*) = \left(\sum_{i=1}^N a_i(x^*) \pi_i(x^*) \right) f'_+(x^*) \quad (3.1)$$

where $\pi_i(x^*)$, $i = 1, \dots, N$ are probabilities of the equilibrium state of the Markov chain at x^* , $f'_{i,-}(x^*)$ is the left-hand derivative of f at x^* calculated on the i th interval, and $f'_+(x^*)$ is the right-hand derivative of f at x^* calculated on the edge formed by amalgamating right parts of the original intervals.

4 Analysis in L^2

In what follows we drop secondary features of the example to focus on the reason for emergence of transmission condition (3.1) and its unique form. More specifically, we consider the case of two intervals (i.e. we take $N = 2$), and assume that diffusion coefficients are constant throughout these intervals and there is no drift at all, so that $a_i(x) = a_i > 0$ and $b_i(x) = 0$ for $x \in [0, 1]$, and some constants $a_i > 0$. (The effect of convex combination has been studied in [5, 9] for processes much more general

than diffusion processes, so we would gain no generality by introducing variable coefficients here.) Moreover, we assume that for some $x^* \in (0, 1)$, the Kolmogorov matrix does not depend on $x \in [x^*, 1]$ and equals

$$\begin{pmatrix} -\beta & \beta \\ \alpha & -\alpha \end{pmatrix} \quad (4.1)$$

where $\alpha, \beta > 0$ are given constants (so that the stationary distribution is $(\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta})$); for $x \in [0, x^*)$, the Kolmogorov matrix is 0.

Since the operator related to such a choice of Kolmogorov matrices (see below, Eq. (4.2)) does not leave the space of continuous functions invariant (the original Friedlin–Wentzell theorem [19] concerns a weak convergence, with test functions being continuous), we will work in the Hilbert space $\mathbb{H}$ of pairs $f = (f_1, f_2)$ of square integrable functions on the unit interval, equipped with the scalar product

$$f \circ g = \int_0^1 (\alpha f_1 \overline{g_1} + \beta f_2 \overline{g_2}).$$

For $\kappa > 0$, we define a sesquilinear form

$$\mathfrak{a}_\kappa = \mathfrak{a} + \kappa \mathfrak{q}$$

where for $f = (f_1, f_2)$ and $g = (g_1, g_2)$,

$$\mathfrak{a}(f, g) = \int_0^1 (\alpha a_1 f_1' \overline{g_1'} + \beta a_2 f_2' \overline{g_2'})$$

while

$$\mathfrak{q}(f, g) = \alpha \beta \int_{x^*}^1 (f_1 - f_2) \overline{(g_1 - g_2)}.$$

Here, the domain V of the form $\mathfrak{a}$ is the space of pairs $f = (f_1, f_2)$ where $f_i \in H^1[0, 1]$, i.e. f_i are absolutely continuous with square integrable derivatives $f_i' \in L^2[0, 1]$, while the domain of $\mathfrak{q}$ is the whole of $\mathbb{H}$.

It is clear that both $\mathfrak{a}$ and $\mathfrak{q}$ are symmetric, and

$$\mathfrak{a}[f] \geq 0, \text{ for } f \in V, \quad \text{and} \quad \mathfrak{q}[f] \geq 0 \text{ for } f \in \mathbb{H}.$$

A direct calculation shows that the operator related to $\mathfrak{a}$ (see Sect. 2) is

$$A(f_1, f_2) = (a_1 f_1'', a_2 f_2'')$$

with domain composed of pairs (f_1, f_2) where $f_i \in H^2[0, 1]$ (i.e., f_i are continuously differentiable functions $f_i \in L^2[0, 1]$ with $f_i' \in H^1[0, 1]$) satisfying $f_i'(0) = f_i'(1) = 0$. Moreover, the operator related to $\mathfrak{q}$ is given by

$$Q(f_1, f_2) = (\beta(f_2 - f_1)1_{[x^*, 1]}, \alpha(f_1 - f_2)1_{[x^*, 1]}), \quad (4.2)$$

where $1_{[x^*, 1]}$ is the indicator function of the interval $[x^*, 1]$. It follows that both A and Q generate semigroups of self-adjoint contraction operators, and the same is true for the operators $A_\kappa = A + \kappa Q$ related to forms $\mathfrak{a}_\kappa$.

The problem is that of finding the limit

$$\lim_{\kappa \rightarrow \infty} e^{tA_\kappa}.$$

To this end, we introduce $\mathbb{H}_0$ as the subspace of $\mathbb{H}$ composed of (f_1, f_2) such that $(f_1 - f_2)1_{[x^*, 1]} = 0$; such pairs may be identified with square integrable functions on a Y -shaped graph obtained by removing the middle segment in the left-hand part in Fig. 2.

Let $V_0 = \mathbb{H}_0 \cap V$ and let

$$\mathfrak{b}(f, g) = \int_0^{x^*} (\alpha a_1 f_1' \overline{g_1'} + \beta a_2 f_2' \overline{g_2'}) + \int_{x^*}^1 (\alpha a_1 + \beta a_2) f_1' \overline{g_1'};$$

by definition of $\mathbb{H}_0$, f_1 and g_1 in the second integral may be replaced by f_2 and g_2 , respectively, without altering $\mathfrak{b}$. Again,

$$\mathfrak{b}[f] = \int_0^{x^*} (\alpha a_1 |f_1'|^2 + \beta a_2 |f_2'|^2) + \int_{x^*}^1 (\alpha a_1 + \beta a_2) |f_1'|^2 \geq 0,$$

and $\mathfrak{b}$ is symmetric. Integration by parts shows that the related operator B in $\mathbb{H}_0$ is given by

$$B(f_1, f_2) = (\chi_1 f_1'', \chi_2 f_2''),$$

where $\chi_i = a_i 1_{[0, x^*)} + (\frac{\alpha}{\alpha + \beta} a_1 + \frac{\beta}{\alpha + \beta} a_2) 1_{[x^*, 1]}$. Its domain is composed of pairs (f_1, f_2) such that

- (A) f_i are continuous on $[0, 1]$, and continuously differentiable on $[0, x^*]$ and $[x^*, 1]$ separately (one sided derivatives at x^* may differ),
- (B) $(f_i)'_{|[0, x^*]} \in H^1[0, x^*]$ and $(f_i)'_{|[x^*, 1]} \in H^1[x^*, 1]$,
- (C) we have $f_i'(1) = f_i'(0) = 0$ and

$$(\alpha a_1 + \beta a_2) f_{1,+}'(x^*) = \alpha a_1 f_{1,-}'(x^*) + \beta a_2 f_{2,-}'(x^*), \quad (4.3)$$

where $+$ and $-$ denote the right-sided and left-sided derivatives, respectively. (Again, $f_{1,+}'(x^*)$ on the left-hand side may be replaced by $f_{2,+}'(x^*)$.)

Certainly, (4.3) is a counterpart of (3.1): these conditions inform of a flux balance; we stress that along with this transmission condition, the continuity condition at x^* is tacitly assumed (as implied by (A) above).

Theorem 4.1 *We have*

$$\lim_{\kappa \rightarrow \infty} e^{A_\kappa t} f = e^{tB} P f, \quad t > 0, f \in \mathbb{H} \quad (4.4)$$

strongly and almost uniformly in $t > 0$, where projection $P \in \mathcal{L}(\mathbb{H})$ is given by $P(f_1, f_2) = (g_1, g_2)$,

$$g_i = f_i 1_{[0, x^*)} + \left(\frac{\alpha}{\alpha + \beta} f_1 + \frac{\beta}{\alpha + \beta} f_2 \right) 1_{[x^*, 1]}.$$

To prove this result, we need Simon’s theorem [33] saying roughly that convergence of positive symmetric forms implies convergence of resolvents of the related operators. A version of this theorem, due to Kato, presented in the classic treatise [26] is slightly weaker as it assumes that the limit ‘upper bound’ form is densely defined (which is not the case in the example we study). We note that Simon’s theorem has recently been generalized by Batty and ter Elst to the case of series of sectorial forms [3].

Theorem 4.2 (Simon’s theorem) *Let $0 \leq c_1 \leq c_2 \leq \dots$ be positive symmetric closed sesquilinear forms in a Hilbert space $\mathbb{H}$. Define the form c by*

$$D(c) = \left\{ x \in \bigcap_{n \geq 1} D(c_n) : \sup_{n \geq 1} c_n[x] < \infty \right\},$$

and $c(x, y) = \lim_{n \rightarrow \infty} c_n(x, y)$. Then, c is closed, positive and symmetric. Moreover, denoting by C the closed operator related to c , defined in $\mathbb{H}_0 = \overline{D(c)}$, and by C_n the closed operators related to c_n defined in $\mathbb{H}$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} (\lambda - C_n)^{-1} y &= (\lambda - C)^{-1} y, & y \in \mathbb{H}_0, \\ \lim_{n \rightarrow \infty} (\lambda - C_n)^{-1} y &= 0, & y \in \mathbb{H}_0^\perp. \end{aligned}$$

Proof of Theorem 4.1 Since $q \geq 0$, the forms a_κ increase with κ , so that we are in the set-up of Simon’s theorem. It is clear that $\sup_{\kappa > 0} a_\kappa[f] < \infty$ iff $q[f] = 0$, i.e. iff $f \in \mathbb{H}_0$. Therefore, the limit form coincides with a restricted to V_0 and this equals b . By Simon’s theorem it follows that

$$\lim_{\kappa \rightarrow \infty} (\lambda - A_\kappa)^{-1} f = (\lambda - B)^{-1} f, \quad f \in \mathbb{H}_0.$$

Given $f \in D(B)$ and $\lambda > 0$, we may find $g \in \mathbb{H}_0$ such that $f = (\lambda - B)^{-1} g$. Taking $f_\kappa = (\lambda - A_\kappa)^{-1} g$ we see that $\lim_{\kappa \rightarrow \infty} f_\kappa = f$ and, using the resolvent equation, $\lim_{\kappa \rightarrow \infty} A_\kappa f_\kappa = Bf$. Hence, condition (a) of the Kurtz singular perturbation theorem is satisfied with $D = D(B)$.

We check the remaining conditions. For all $f \in D(A)$, $\lim_{\kappa \rightarrow \infty} \kappa^{-1} A_\kappa f = Qf$ while $D(A)$ is dense in $\mathbb{H}$ and Q is bounded, proving condition (b). Next, Q and P are related by the following formula:

$$Q = (\alpha + \beta)(P - I_{\mathbb{H}}).$$

It follows that $\lim_{t \rightarrow \infty} e^{tQ} = P$. Finally, for $f \in D(B)$ we have

$$PAf = Bf.$$

In particular, PA is closed, proving assumption (c) in Kurtz's theorem and completing the proof. $\square$

5 Analysis in L^1

The approach of the previous chapter is quite elegant: the quadratic forms contain all the information needed for the limit theorem. The arguable elegance, however, comes perhaps at the cost of blurring the mechanism of emergence of transmission conditions in the limit. To explain: the information about these conditions is compressed, or, so to say, 'zipped' in the quadratic form $\mathfrak{b}$ corresponding to the operator B . While it seems without a shadow of a doubt that the limit form cannot be anything other than $\mathfrak{b}$, from the perspective of forms it is still somewhat difficult to grasp the way transmission conditions come into existence. In other words, with forms, the picture is quite clear, but some part of the mystery is still there—unless you master the connection between the operator and the form. Therefore, in this chapter, we present another approach, where calculations are much more explicit, if a bit complex. As we shall see, convergence of semigroups involved may be deduced from convergence of solutions of a linear system of equations with four unknowns (see (5.9)).

To this end, we work with the semigroups of Markov operators (contraction operators that preserve the positive cone, see e.g. [30]) in the space of absolutely integrable functions. These semigroups are responsible for evolution of densities of the Markov processes involved (and are perhaps a bit more natural than those of the previous chapter). More specifically, we work with the space $\mathbb{X} = L^1(\mathbb{R}) \times L^1(\mathbb{R})$ identified with the space of integrable functions $f : \mathbb{R} \times \{1, 2\} \rightarrow \mathbb{R}$. In other words, each pair $(f_1, f_2) \in \mathbb{X}$ is identified with such a function, defined by $f(x, i) = f_i(x)$, $i = 1, 2$, $x \in \mathbb{R}$. The norm in $\mathbb{X}$ is given by

$$\|f\| = \|f_1\|_{L^1(\mathbb{R})} + \|f_2\|_{L^1(\mathbb{R})}.$$

Given two diffusion coefficients $a_1, a_2 > 0$ we define

$$\mathcal{A} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} a_1 f_1'' \\ a_2 f_2'' \end{pmatrix},$$

for $f_1, f_2 \in W^{2,1}(\mathbb{R})$. Then, $\mathcal{A}$ generates a semigroup of Markov operators in $\mathbb{X}$: the reader has noticed probably that we got rid of reflecting (Neumann) boundary conditions (which have no bearing on the phenomenon under study) and have allowed Brownian particles to diffuse freely on two copies of the real axis. Next, for given intensities $\alpha, \beta > 0$ we define bounded linear operators in $\mathbb{X}$ by

$$\mathcal{Q} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} (\alpha f_2 - \beta f_1) 1_{[0, \infty)} \\ (\beta f_1 - \alpha f_2) 1_{[0, \infty)} \end{pmatrix}$$

and $\mathcal{P} = \frac{1}{\alpha + \beta} \mathcal{Q} + I_{\mathbb{X}}$, so that

$$\mathcal{P} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_1 1_{(-\infty, 0)} + \alpha' (f_1 + f_2) 1_{[0, \infty)} \\ f_2 1_{(-\infty, 0)} + \beta' (f_1 + f_2) 1_{[0, \infty)} \end{pmatrix}, \quad (5.1)$$

where

$$\alpha' = \frac{\alpha}{\alpha + \beta} \quad \text{and} \quad \beta' = \frac{\beta}{\alpha + \beta}.$$

It is easy to see that $\mathcal{P}$ is a Markov operator; it follows that $\mathcal{Q}$ generates a semigroup $(e^{t\mathcal{Q}})_{t \geq 0}$ of such operators. This semigroup describes the process in which states (x, i) where $x < 0$ are absorbing, while $(x, 1)$ and $(x, 2)$ communicate as states of a Markov chain with intensity matrix (4.1). (In particular, it is the point 0 that plays now the role of x^* of the previous chapter.) Therefore, the Phillips perturbation theorem combined with the Trotter product formula, implies that for each $\kappa > 0$ the operator

$$\mathcal{A}_\kappa = \mathcal{A} + \kappa \mathcal{Q}$$

generates a semigroup of Markov operators in $\mathbb{X}$. (This semigroup is in a sense dual to that generated by A_κ of the previous chapter.)

Since we want to find the limit

$$\lim_{\kappa \rightarrow \infty} e^{t\mathcal{A}_\kappa},$$

we turn to studying the resolvent equation $\lambda f - A_\kappa f = g$, for $\lambda > 0$ and $g \in \mathbb{X}$. As we shall see, the solution $f \in D(\mathcal{A})$ (which exists and is unique, A_κ generating the contraction semigroup) may be found in a quite explicit way.

To begin with, we note that on the left half-axis, the resolvent equation takes the form

$$\lambda f_i(x) - a_i f_i''(x) = g_i(x), \quad x < 0, i = 1, 2.$$

Basic principles of ordinary differential equations tell us that there are solutions to this equation that are integrable on the left half-axis, and they are given by

$$f_i(x) = l_i e^{\sqrt{\frac{\lambda}{a_i}} x} + \frac{1}{2\sqrt{\lambda a_i}} \int_{-\infty}^0 e^{-\sqrt{\frac{\lambda}{a_i}} |x-y|} g_i(y) dy, \quad x < 0, \quad (5.2)$$

where l_1, l_2 are (yet) unknown constants.

On the right half-axis, the resolvent equation takes the form

$$\begin{aligned} (\lambda + \beta) f_1 - a_1 f_1'' - \alpha f_2 &= g_1, \\ (\lambda + \alpha) f_2 - a_2 f_2'' - \beta f_1 &= g_2. \end{aligned} \quad (5.3)$$

More precisely, this is (a part of) the resolvent equation for $\mathcal{A}_1$; the general case will be recovered later by replacing each instance of α and β by $\kappa\alpha$ and $\kappa\beta$, respectively. Moreover, g_1, g_2 are now treated as members of $L^1(\mathbb{R}^+)$ and solutions are also sought in this space. The question of existence of these solutions is answered in Lemma 5.1, below, but we need to make some preparatory remarks for this result.

The quadratic equation:

$$(\lambda + \alpha - a_2 t)(\lambda + \beta - a_1 t) = \alpha\beta$$

has precisely two real solutions:

$$t_i = \frac{(\lambda + \alpha)a_1 + (\lambda + \beta)a_2 - (-1)^i \sqrt{\Delta}}{2a_1 a_2}, \quad i = 1, 2,$$

where

$$\begin{aligned} \Delta &= [(\lambda + \alpha)a_1 + (\lambda + \beta)a_2]^2 - 4a_1 a_2 [(\lambda + \alpha)(\lambda + \beta) - \alpha\beta] \\ &> [(\lambda + \alpha)a_1 + (\lambda + \beta)a_2]^2 - 4a_1 a_2 (\lambda + \alpha)(\lambda + \beta) \\ &= [(\lambda + \alpha)a_1 - (\lambda + \beta)a_2]^2 \geq 0. \end{aligned}$$

Moreover, since $\sqrt{\Delta} < (\lambda + \alpha)a_1 + (\lambda + \beta)a_2$, these solutions are positive; we note that

$$t_1 - t_2 = \frac{\sqrt{\Delta}}{a_1 a_2}. \quad (5.4)$$

We look for (the first coordinate of) solutions of (5.3) of the form¹

$$f_1 = \sum_{i=1}^2 \left(k_i e_{t_i} + (-1)^i G_{t_i} h_i \right), \quad (5.5)$$

where k_1, k_2 are constants, $e_t(x) = e^{-\sqrt{t}x}$, $x \geq 0$ and, for $t > 0$ and $h \in L^1(\mathbb{R}^+)$,

$$G_t h(x) = \int_0^\infty e^{-\sqrt{t}|x-y|} h(y) dy, \quad x \geq 0, \quad (5.6)$$

and

$$h_i = \frac{1}{2\sqrt{t_i}\Delta} (\alpha g_2 + (\lambda + \alpha - a_2 t_i) g_1).$$

We note that G_t is a bounded linear operator in $L^1(\mathbb{R}^+)$.

¹ Such a form is suggested by straightforward, but lengthy analysis involving standard methods of ODEs with constant coefficients, accompanied with the care to obtain solutions in $L^1(\mathbb{R}^+)$.

Lemma 5.1 *The pair (f_1, f_2) , where f_1 is given by (5.5), and*

$$f_2 = \alpha^{-1}((\lambda + \beta)f_1 - a_1 f_1'' - g_1)$$

solves (5.3) (for any constants k_1, k_2).

Proof By the very definition of f_2 , the first equality in (5.3) is satisfied, and we are left with proving the second one. To this end, we note that $G_t h \in W^{2,1}(\mathbb{R}^+)$, and

$$(G_t h)'' = t G_t h - 2\sqrt{t}h, \quad h \in L^1(\mathbb{R}^+). \quad (5.7)$$

It follows that

$$a_1 f_1'' = a_1 \sum_{i=1}^2 t_i (k_i e_{t_i} + (-1)^i G_{t_i} h_i) + 2\sqrt{t_1} a_1 h_1 - 2\sqrt{t_2} a_1 h_2.$$

Since

$$2\sqrt{t_1} a_1 h_1 - 2\sqrt{t_2} a_1 h_2 = \frac{a_1 a_2}{\sqrt{\Delta}} (t_2 - t_1) g_1 = -g_1, \quad (\text{see (5.4)})$$

we obtain

$$\alpha f_2 = \sum_{i=1}^2 (\lambda + \beta - a_1 t_i) (k_i e_{t_i} + (-1)^i G_{t_i} h_i). \quad (5.8)$$

Therefore, invoking (5.7) (again) and the definition of h_i (for the first time),

$$\begin{aligned} & \alpha((\lambda + \alpha) f_2 - a_2 f_2'') \\ &= \sum_{i=1}^2 [(\lambda + \alpha)(\lambda + \beta) - a_1 t_i (\lambda + \alpha) - a_2 t_i (\lambda + \beta) + a_1 a_2 t_i^2] \\ & \quad \times (k_i e_{t_i} + (-1)^i G_{t_i} h_i) \\ & \quad + 2a_2 \sqrt{t_2} (\lambda + \beta - a_1 t_2) h_2 - 2a_2 \sqrt{t_1} (\lambda + \beta - a_1 t_1) h_1 \\ &= \alpha \beta \sum_{i=1}^2 (k_i e_{t_i} + (-1)^i G_{t_i} h_i) \\ & \quad + \frac{\alpha a_2}{\sqrt{\Delta}} (\lambda + \beta - a_1 t_2) g_2 + \frac{(\lambda + \alpha - a_2 t_2)(\lambda + \beta - a_1 t_2)}{\sqrt{\Delta}} g_1 \\ & \quad - \frac{\alpha a_2}{\sqrt{\Delta}} (\lambda + \beta - a_1 t_1) g_2 - \frac{(\lambda + \alpha - a_2 t_1)(\lambda + \beta - a_1 t_1)}{\sqrt{\Delta}} g_1, \end{aligned}$$

since t_i s are roots of the quadratic. For the same reason, the terms involving g_1 cancel out, both being equal to $\alpha \beta g_1$. To summarize, using (5.4) one more time,

$$\alpha((\lambda + \alpha)f_2 - a_2 f_2'') = \alpha\beta f_1 + \frac{\alpha a_1 a_2}{\sqrt{\Delta}}(t_1 - t_2)g_2 = \alpha\beta f_1 + \alpha g_2.$$

This proves the second equality in (5.3). $\square$

Now, the pair $(f_1, f_2) \in \mathbb{X}$ defined by (5.2), (5.5) and (5.8) belongs to $D(\mathcal{A})$ iff

$$f_i(0-) = f_i(0) \quad \text{and} \quad f_i'(0-) = f_i(0), \quad i = 1, 2.$$

The first of these conditions (compatibility of values) may be written as

$$\begin{aligned} l_1 + D_1 &= \sum_{i=1}^2 (k_i + (-1)^i C_i), \\ l_2 + D_2 &= \sum_{i=1}^2 E_i (k_i + (-1)^i C_i), \end{aligned}$$

where

$$C_i := (G_{t_i} h_i)(0), \quad D_i := \frac{1}{2\sqrt{\lambda a_i}} \int_{-\infty}^0 e^{\sqrt{\frac{\lambda}{a_i}} y} g_i(y) dy \quad \text{and} \quad E_i := \frac{\lambda + \beta - a_1 t_i}{\alpha}.$$

Since

$$(G_t h)'(0) = \sqrt{t}(G_t h)(0),$$

the other condition (compatibility of derivatives) reads

$$\begin{aligned} \sqrt{\frac{\lambda}{a_1}} l_1 - \sqrt{\frac{\lambda}{a_1}} D_1 &= \sum_{i=1}^2 \sqrt{t_i} ((-1)^i C_i - k_i), \\ \sqrt{\frac{\lambda}{a_2}} l_2 - \sqrt{\frac{\lambda}{a_2}} D_2 &= \sum_{i=1}^2 \sqrt{t_i} E_i ((-1)^i C_i - k_i). \end{aligned}$$

Hence, we have the following linear system of equations for l_1, l_2, k_1, k_2 :

$$\begin{aligned} l_1 - k_1 - k_2 &= C_2 - C_1 - D_1, \\ l_2 - E_1 k_1 - E_2 k_2 &= E_2 C_2 - E_1 C_1 - D_2, \\ \sqrt{s_1} l_1 + \sqrt{t_1} k_1 + \sqrt{t_2} k_2 &= \sqrt{t_2} C_2 - \sqrt{t_1} C_1 + \sqrt{s_1} D_1, \\ \sqrt{s_2} l_2 + \sqrt{t_1} E_1 k_1 + \sqrt{t_2} E_2 k_2 &= \sqrt{t_2} E_2 C_2 - \sqrt{t_1} E_1 C_1 + \sqrt{s_2} D_2, \end{aligned}$$

where for further simplicity of notation $s_i = \frac{\lambda}{a_i}$. The Jordan–Gauss elimination method (stopped one step before completion) yields now

$$\begin{aligned} l_1 - k_1 - k_2 &= C_2 - C_1 - D_1, \\ l_2 - E_1 k_1 - E_2 k_2 &= E_2 C_2 - E_1 C_1 - D_2, \end{aligned}$$

$$\begin{aligned}
(\sqrt{s_1} + \sqrt{t_1})k_1 + (\sqrt{s_1} + \sqrt{t_2})k_2 &= (\sqrt{t_2} - \sqrt{s_1})C_2 \\
&\quad + (\sqrt{s_1} - \sqrt{t_1})C_1 + 2\sqrt{s_1}D_1, \\
E_1(\sqrt{s_2} + \sqrt{t_1})k_1 + E_2(\sqrt{s_2} + \sqrt{t_2})k_2 &= E_2(\sqrt{t_2} - \sqrt{s_2})C_2 \\
&\quad + E_1(\sqrt{s_2} - \sqrt{t_1})C_1 + 2\sqrt{s_2}D_2. \quad (5.9)
\end{aligned}$$

Before continuing, we present a lemma summarizing asymptotic behavior of constants and functions appearing in the definition of f_1 and f_2 .

Lemma 5.2 *Let*

$$a_a = \alpha' a_1 + \beta' a_2. \quad (5.10)$$

If each occurrence of α and β is replaced by $\kappa\alpha$ and $\kappa\beta$, respectively, then

$$\begin{aligned}
\text{(i)} \quad \lim_{\kappa \rightarrow \infty} \frac{\sqrt{\Delta}}{\kappa} &= (\alpha + \beta)a_a, \quad \lim_{\kappa \rightarrow \infty} \frac{t_1}{\kappa} = \frac{\alpha + \beta}{a_1 a_2} a_a, \quad \lim_{\kappa \rightarrow \infty} t_2 = \frac{\lambda}{a_a}, \\
\text{(ii)} \quad \lim_{\kappa \rightarrow \infty} E_1 &= -\frac{a_1}{a_2}, \quad \lim_{\kappa \rightarrow \infty} E_2 = \frac{\beta}{\alpha}, \\
\text{(iii)} \quad \lim_{\kappa \rightarrow \infty} h_2 &= \frac{1}{2} \frac{\alpha'}{\sqrt{\lambda a_a}} (g_1 + g_2) := h_{2,\infty}, \\
\text{(iv)} \quad \lim_{\kappa \rightarrow \infty} C_2 &= \frac{\alpha'}{2\sqrt{\lambda a_a}} \int_0^\infty e^{-\sqrt{\frac{\lambda}{a_a}} y} (g_1(y) + g_2(y)) \, dy =: C_{2,\infty}, \\
\text{(v)} \quad \lim_{\kappa \rightarrow \infty} \sqrt{t_1} h_1 &= \frac{\alpha'}{2a_a} \left(g_2 - \frac{\beta a_2}{\alpha a_1} g_1 \right), \\
\text{(vi)} \quad \lim_{\kappa \rightarrow \infty} \sqrt{t_1} C_1 &= 0.
\end{aligned}$$

Proof Except for (vi), all claims are immediate by standard calculus, if proven consecutively. To show (vi), we note that for each $t > 0$,

$$L^1(\mathbb{R}^+) \ni h \mapsto F_t h := \int_0^\infty e^{-\sqrt{t}y} h(y) \, dy$$

is a bounded linear functional of norm 1. Moreover, for $h \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$,

$$|F_t h| \leq \frac{1}{\sqrt{t}} \|h\|_{L^\infty(\mathbb{R}^+)},$$

implying $\lim_{t \rightarrow \infty} F_t h = 0$. Therefore,

$$\lim_{t \rightarrow \infty} F_t h = 0, \quad h \in L^1(\mathbb{R}^+),$$

because $L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$ is dense in $L^1(\mathbb{R}^+)$ and the functionals are equibounded. Since $\sqrt{t_1} C_1 = F_{t_1} \sqrt{t_1} h_1$ and $\lim_{\kappa \rightarrow \infty} t_1 = \infty$, this combined with (v) completes the proof. $\square$

We are finally able to state the first of the two main results of this section: it provides information on convergence of resolvents.

Proposition 5.3 *If each occurrence of α and β is replaced by $\kappa\alpha$ and $\kappa\beta$, respectively, then:*

- (a) *For κ large enough, (5.9) has a unique solution. Moreover, equations (5.2), (5.5) and (5.8) with k_1, k_2, l_1, l_2 calculated from (5.9), give the solution to the resolvent equation for A_κ .*
- (b) *As $\kappa \rightarrow \infty$, the solutions to the resolvent equations for A_κ converge to $(f_1, f_2) = R_\lambda(g_1, g_2)$ defined by*

$$f_i(x) = \begin{cases} l_i e^{\sqrt{\frac{\lambda}{a_i}} x} + \frac{1}{2\sqrt{\lambda a_i}} \int_{-\infty}^0 e^{-\sqrt{\frac{\lambda}{a_i}} |y-x|} g_i(y) dy, & x < 0, \\ k e^{-\sqrt{\frac{\lambda}{a_a}} x} + \frac{\alpha'}{2\sqrt{\lambda a_a}} \int_0^\infty e^{-\sqrt{\frac{\lambda}{a_a}} |x-y|} (g_1 + g_2)(y) dy, & x \geq 0, i = 1, \\ \frac{\beta}{\alpha} f_1(x), & x \geq 0, i = 2, \end{cases} \quad (5.11)$$

where

$$k = \frac{\left[\frac{\beta}{\alpha} \left(\sqrt{\frac{\lambda}{a_a}} - \sqrt{\frac{\lambda}{a_2}} \right) + \frac{a_1}{a_2} \left(\sqrt{\frac{\lambda}{a_a}} - \sqrt{\frac{\lambda}{a_1}} \right) \right] C_{2,\infty} + 2\sqrt{\frac{\lambda}{a_2}} D_2 + 2\frac{a_1}{a_2} \sqrt{\frac{\lambda}{a_1}} D_1}{\frac{\beta}{\alpha} \left(\sqrt{\frac{\lambda}{a_a}} + \sqrt{\frac{\lambda}{a_2}} \right) + \frac{a_1}{a_2} \left(\sqrt{\frac{\lambda}{a_a}} + \sqrt{\frac{\lambda}{a_1}} \right)},$$

$$l_1 = k + C_{2,\infty} - D_1 \quad \text{and} \quad l_2 = \frac{\beta}{\alpha} k + \frac{\beta}{\alpha} C_{2,\infty} - D_2. \quad (5.12)$$

Proof (a) For the main determinant, say W , of the last two equations in (5.9), we have

$$\lim_{\kappa \rightarrow \infty} \frac{W}{\sqrt{t_1}} = \frac{\beta}{\alpha} \left(\sqrt{\frac{\lambda}{a_2}} + \sqrt{\frac{\lambda}{a_a}} \right) + \frac{a_1}{a_2} \left(\sqrt{\frac{\lambda}{a_1}} + \sqrt{\frac{\lambda}{a_a}} \right) \neq 0.$$

Since $\lim_{\kappa \rightarrow \infty} \sqrt{t_1} = \infty$, this shows uniqueness of k_1 and k_2 in (5.9), which in turn implies uniqueness of l_1 and l_2 (by the first two equations). The rest is clear.

(b) Using Cramer's rule for the last two equations in (5.9) and Lemma 5.2, we see that $\lim_{\kappa \rightarrow \infty} k_1 = 0$ and $\lim_{\kappa \rightarrow \infty} k_2 = k$ for k defined in (5.12). Passing to the limit in the first two equations (and using Lemma 5.2) we obtain $\lim_{\kappa \rightarrow \infty} l_1 = k + C_{2,\infty} - D_1$ and $\lim_{\kappa \rightarrow \infty} l_2 = \frac{\beta}{\alpha} k + \frac{\beta}{\alpha} C_{2,\infty} - D_2$. By (5.5) and (5.8), this shows that the solutions of the resolvent equations converge to (f_1, f_2) defined by formula (5.11) because the map $t \mapsto e_t \in L^1(\mathbb{R}^+)$ is norm-continuous (with $\lim_{t \rightarrow \infty} e_t = 0$), and the map $t \mapsto G_t \in \mathcal{L}(L^1(\mathbb{R}^+))$ is strongly continuous. $\square$

All that is left to do now is to interpret, or decipher, this result and provide the link with the Freidlin–Wentzell transmission conditions. To this end, let

$$\mathbb{X}_0 = \{(f_1, f_2) \in \mathbb{X}; \alpha f_2 1_{\mathbb{R}^+} = \beta f_1 1_{\mathbb{R}^+}\}.$$

Fig. 3 The ‘infinite y’-shaped graph; its three edges, when identified with three half-axes are domains of three functions

This subspace of $\mathbb{X}$ is isometrically isomorphic to $L^1(Y)$, the space of integrable functions on an ‘infinite y’-shaped graph depicted at Fig. 3. The latter space in turn may be identified with

$$\mathbb{Y} = L^1(\mathbb{R}^-) \times L^1(\mathbb{R}^-) \times L^1(\mathbb{R}^+),$$

the isometric isomorphism $I : \mathbb{X}_0 \rightarrow \mathbb{Y}$ being given by

$$I(f_1, f_2) = (f_1|_{\mathbb{R}^-}, f_2|_{\mathbb{R}^-}, (f_1 + f_2)|_{\mathbb{R}^+}).$$

We have $I^{-1}(f_1, f_2, f_3) = (f_1 \cup \alpha' f_3, f_2 \cup \beta' f_3)$, where for $f \in L^1(\mathbb{R}^-)$ and $g \in L^1(\mathbb{R}^+)$, $f \cup g$ is a ‘union of graphs’-function:

$$(f \cup g)(x) = \begin{cases} f(x), & x < 0 \\ g(x), & x \geq 0. \end{cases}$$

Next, let B_0 be the operator in $\mathbb{Y}$ given by

$$B_0(f_1, f_2, f_3) = (a_1 f_1'', a_2 f_2'', a_a f_3'') \quad (5.13)$$

on the domain composed of (f_1, f_2, f_3) such that $f_1, f_2 \in W^{2,1}(\mathbb{R}^-)$, $f_3 \in W^{2,1}(\mathbb{R}^+)$ satisfy the transmission conditions

$$f_1(0-) = \alpha' f_3(0+), f_2(0-) = \beta' f_3(0+) \text{ and } a_a f_3'(0+) = a_1 f_1'(0-) + a_2 f_2'(0-). \quad (5.14)$$

These transmission conditions are dual to those of the previous chapter: to be more precise, they describe the same physical/biological phenomenon, yet in a different, ‘dual’ space.

Now, the isomorphic copy of B_0 in $\mathbb{X}_0$ is given by

$$B(f_1, f_2) = (\chi_1 f_1'', \chi_2 f_2''),$$

where $\chi_i = a_i 1_{(-\infty, 0)} + a_a 1_{[0, \infty)}$, on the domain composed of (f_1, f_2) such that f_1 and f_2 are continuous on $\mathbb{R}$ (so that in particular

$$f_i(0-) = f_i(0+); \quad (5.15)$$

this corresponds to the first two conditions in (5.14)),

$$f_1|_{\mathbb{R}^-}, f_2|_{\mathbb{R}^-} \in W^{2,1}(\mathbb{R}^-), \quad f_1|_{\mathbb{R}^+} \in W^{2,1}(\mathbb{R}^+)$$

and

$$a_a f_1'(0+) = \alpha' (a_1 f_1'(0-) + a_2 f_2'(0-)). \quad (5.16)$$

(It goes without saying that $(f_1, f_2) \in \mathbb{X}_0$.)

It is quite easy to solve the resolvent equation for B . Given $\lambda > 0$ and $(g_1, g_2) \in \mathbb{X}$, we have

$$f_i(x) = \begin{cases} m_i e^{\sqrt{\frac{\lambda}{a_i}} x} + \frac{1}{2\sqrt{\lambda a_i}} \int_{-\infty}^0 e^{-\sqrt{\frac{\lambda}{a_i}} |y-x|} g_i(y) dy, & x < 0, \\ n e^{-\sqrt{\frac{\lambda}{a_a}} x} + \frac{1}{2\sqrt{\lambda a_a}} \int_0^{\infty} e^{-\sqrt{\frac{\lambda}{a_a}} |x-y|} g_1(y) dy, & x \geq 0, i = 1, \\ \frac{\beta}{\alpha} f_1(x), & x \geq 0, i = 2. \end{cases} \quad (5.17)$$

Here, the constants m_1, m_2 and n are chosen so that transmission conditions (5.15) and (5.16) are satisfied, i.e.,

$$\begin{aligned} m_1 + D_1 &= n + C, \quad m_2 + D_2 = \frac{\beta}{\alpha} (n + C) \\ a_a \sqrt{\frac{\lambda}{a_a}} (C - n) &= \alpha' \left[a_1 \sqrt{\frac{\lambda}{a_1}} (l_1 - D_1) + a_2 \sqrt{\frac{\lambda}{a_2}} (l_2 - D_2) \right], \end{aligned} \quad (5.18)$$

where $C = \frac{1}{2\sqrt{\lambda a_a}} \int_0^{\infty} e^{-\sqrt{\frac{\lambda}{a_a}} y} g_1(y) dy$ and, as previously,

$$D_i := \frac{1}{2\sqrt{\lambda a_i}} \int_{-\infty}^0 e^{\sqrt{\frac{\lambda}{a_i}} y} g_i(y) dy.$$

Finally, for $(g_1, g_2) \in \mathbb{X}_0$, $\alpha'(g_1 + g_2)|_{\mathbb{R}^+} = g_1|_{\mathbb{R}^+}$ so that $C_{2,\infty} = C$, and (5.17) is the same as (5.11), except perhaps for the constants. Moreover, a bit of algebra shows that for k of (5.12), we have then

$$\begin{aligned} k &= \frac{((\alpha a_1 + \beta a_2) \sqrt{\frac{\lambda}{a_a}} - \alpha a_1 \sqrt{\frac{\lambda}{a_1}} - \beta a_2 \sqrt{\frac{\lambda}{a_2}}) C + 2\alpha \sqrt{\lambda a_1} D_1 + 2\alpha \sqrt{\lambda a_2} D_2}{(\alpha a_1 + \beta a_2) \sqrt{\frac{\lambda}{a_a}} + \alpha a_1 \sqrt{\frac{\lambda}{a_1}} + \beta a_2 \sqrt{\frac{\lambda}{a_2}}} \\ &= \frac{(\sqrt{a_a} - \alpha' \sqrt{a_1} - \beta' \sqrt{a_2}) C + 2\alpha' \sqrt{a_1} D_1 + 2\alpha' \sqrt{a_2} D_2}{\sqrt{a_a} + \alpha' \sqrt{a_1} + \beta' \sqrt{a_2}}. \end{aligned}$$

This, however, is just the assertion that k is the same as n of (5.18). It follows that $m_i = l_i, i = 1, 2$ and so

$$R_\lambda(g_1, g_2) = (\lambda - B)^{-1} (g_1, g_2), \quad (g_1, g_2) \in \mathbb{X}_0. \quad (5.19)$$

More generally,

$$R_\lambda = (\lambda - B)^{-1} \mathcal{P}, \quad (5.20)$$

where $\mathcal{P}$ is defined in (5.1).

Theorem 5.4 *We have*

$$\lim_{\kappa \rightarrow \infty} e^{A_\kappa t} f = e^{tB} \mathcal{P} f, \quad t > 0, f \in \mathbb{X} \quad (5.21)$$

strongly and almost uniformly in $t > 0$.

Proof Relation (5.20) (in fact, (5.19) suffices) shows that condition (a) of Kurtz’s singular perturbation theorem is satisfied. The rest of the argument is precisely the same as in the proof of Theorem 4.1. $\square$

To summarize, we have proved a ‘dual’ version of the main result of the previous section. In contrast to the arguments presented there, here the question of convergence of resolvents is reduced to that of a (singular) convergence of solutions of a system of linear equations in $\mathbb{R}^4$. Probabilistically, the main theorem of this section speaks of convergence of densities of the related stochastic processes in L^1 norm.

6 A cosine family

In this (last) section, we want to show that the operator B_0 introduced in (5.13) and (5.14) generates a bounded cosine family of operators. (Of course, this implies that so does its isomorphic image B in $\mathbb{X}_0$.) To begin with, we note that in the definition of B_0 , the constants α and β do not appear directly, but merely via α' and β' . Therefore, from now on and without loss of generality, we assume that $\alpha + \beta = 1$. This will simplify notations: in (5.10) and (5.14) the primes may be dropped.

We use Lord Kelvin’s method of images [6, 7]. To this end, let

$$\mathbb{Z} = L^1(\mathbb{R}) \times L^1(\mathbb{R}) \times L^1(\mathbb{R})$$

(equipped with the norm $\|(f_1, f_2, f_3)\|_{\mathbb{Z}} = \sum_{i=1}^3 \|f_i\|_{\mathbb{R}}$), and let $C = \{C(t), t \in \mathbb{R}\}$ be the cosine family in $\mathbb{Z}$ defined as follows. For $f = (f_1, f_2, f_3) \in \mathbb{Z}$, $C(t)f$ is the triple $(k_1, k_2, k_3) \in \mathbb{Z}$ given by

$$k_i(x) = \frac{1}{2}(f_i(x + \sigma_i t) + f_i(x - \sigma_i t)), \quad x, t \in \mathbb{R}$$

where $\sigma_i = \sqrt{a_i}$, $i = 1, 2$ and $\sigma_3 = \sqrt{a_a}$. Our aim is to show that $\mathbb{Y}$ (equivalently: $L^1(Y)$) is isomorphic to a certain subspace, say $\mathbb{Z}_0$, of $\mathbb{Z}$ which is invariant for C (see Lemma 6.1). This will allow describing B_0 as the generator of the isomorphic image of C restricted to $\mathbb{Z}_0$ (see Theorem 6.2).

An $f = (f_1, f_2, f_3)$ is a member of $\mathbb{Z}_0$ iff

$$\begin{aligned} f_1(x) &= \frac{2\alpha}{1+c} \left(\frac{\sigma_1}{\sigma_3} f_1(-x) + \frac{\sigma_2}{\sigma_3} f_2 \left(-\frac{\sigma_2}{\sigma_1} x \right) + f_3 \left(\frac{\sigma_3}{\sigma_1} x \right) \right) - f_1(-x), \\ f_2(x) &= \frac{2\beta}{1+c} \left(\frac{\sigma_1}{\sigma_3} f_1 \left(-\frac{\sigma_1}{\sigma_2} x \right) + \frac{\sigma_2}{\sigma_3} f_2(-x) + f_3 \left(\frac{\sigma_3}{\sigma_2} x \right) \right) - f_2(-x), \\ f_3(-x) &= \frac{1}{1+c} \left(\frac{2\sigma_1}{\sigma_3} f_1 \left(-\frac{\sigma_1}{\sigma_3} x \right) + \frac{2\sigma_2}{\sigma_3} f_2 \left(-\frac{\sigma_2}{\sigma_3} x \right) + (1-c) f_3(x) \right), \end{aligned} \quad (6.1)$$

for almost all $x \geq 0$, where

$$c = \frac{\alpha\sqrt{a_1} + \beta\sqrt{a_2}}{\sqrt{\alpha a_1 + \beta a_2}} \in (0, 1).$$

In particular, the values of f_1 and f_2 on the left half-axis and those of f_3 on the right half-axis determine the whole triple $f \in \mathbb{Z}_0$. Therefore, there is a one-to-one correspondence between members of $\mathbb{Z}_0$ and those of $\mathbb{Y}$. Let $J : \mathbb{Z}_0 \rightarrow \mathbb{Y}$ be the appropriate mapping. Since coordinates of Jf are restrictions of those of f , it is clear that $\|Jf\|_{\mathbb{Y}} \leq \|f\|_{\mathbb{Z}_0}$. By the Open Mapping theorem

$$\|f\|_{\mathbb{Z}_0} \leq \text{const.} \|Jf\|_{\mathbb{Y}},$$

and the constant appearing here may be found using (6.1).

More interestingly, (6.1) implies

$$\begin{aligned} \int_{\mathbb{R}^+} (f_1 + f_2) + \int_{\mathbb{R}^-} f_3 &= \frac{2}{1+c} \left(\int_{\mathbb{R}^-} (f_1 + f_2) + \int_{\mathbb{R}^+} f_3 \right) - \int_{\mathbb{R}^-} (f_1 + f_2) \\ &\quad + \frac{1-c}{1+c} \int_{\mathbb{R}^+} f_3 + \frac{2}{1+c} \int_{\mathbb{R}^-} (f_1 + f_2) \quad (\text{use (5.10)}) \\ &= \frac{3-c}{1+c} \left(\int_{\mathbb{R}^-} (f_1 + f_2) + \int_{\mathbb{R}^+} f_3 \right). \end{aligned}$$

It follows that

$$\sum_{i=1}^3 \int_{\mathbb{R}} f_i = \frac{4}{1+c} \left(\int_{\mathbb{R}^-} (f_1 + f_2) + \int_{\mathbb{R}^+} f_3 \right). \quad (6.2)$$

This means that J^{-1} preserves integral modulo the constant $\frac{4}{1+c}$.

Instead of showing directly that $\mathbb{Z}_0$ is invariant for C (such a calculation is possible but unrewarding, because it does not explain why $\mathbb{Z}_0$ should be defined as above), we claim first that coordinates of each triple $f = (f_1, f_2, f_3) \in D(B_0)$ may be uniquely extended to $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) \in \mathbb{Z}_0$, where the last triple fulfills the following requirement: Fix $t \in \mathbb{R}$ and let $(k_1, k_2, k_3) = C(t)\tilde{f}$. Then

$$(k_1|_{\mathbb{R}^-}, k_2|_{\mathbb{R}^-}, k_3|_{\mathbb{R}^+}) \in D(B_0). \quad (6.3)$$

To prove the claim we let

$$g_i(x) = \tilde{f}_i(x), i = 1, 2, \quad g_3(x) = \tilde{f}_3(-x), x \geq 0,$$

to be the unknown parts of $\tilde{f}_i$ ’s. Then condition (6.3) (see transmission conditions (5.14)), is equivalent to the following system of equations for g_i ’s, where f_i ’s are known:

$$\begin{aligned} g_1(\sigma_1 t) + f_1(-\sigma_1 t) &= \alpha (f_3(\sigma_3 t) + g_3(\sigma_3 t)), \\ g_2(\sigma_2 t) + f_2(-\sigma_2 t) &= \beta (f_3(\sigma_3 t) + g_3(\sigma_3 t)), \\ f'_3(\sigma_3 t) - g'_3(\sigma_3 t) &= \frac{a_1}{a_3} (g'_1(\sigma_1 t) + f'_1(-\sigma_1 t)) + \frac{a_2}{a_3} (g'_2(\sigma_2 t) + f'_2(-\sigma_2 t)), \end{aligned}$$

where $t \geq 0$. Substituting g_1 and g_2 calculated from the first two equations, after some algebra, we obtain

$$g'_3(x) = \frac{1-c}{1+c} f'_3(x) - \frac{2a_1}{a_3(1+c)} f'_1\left(-\frac{\sigma_1}{\sigma_3}x\right) - \frac{2a_2}{a_3(1+c)} f'_2\left(-\frac{\sigma_2}{\sigma_3}x\right),$$

for almost all $x \geq 0$. Taking into account that g_3 is supposed to be integrable,

$$g_3(x) = \frac{2\sigma_1}{\sigma_3(1+c)} f_1\left(-\frac{\sigma_1}{\sigma_3}x\right) + \frac{2\sigma_2}{\sigma_3(1+c)} f_2\left(-\frac{\sigma_2}{\sigma_3}x\right) + \frac{1-c}{1+c} f_3(x)$$

for almost all $x \geq 0$. This leads to the conclusion that g_1 , g_2 and g_3 must be given by the right-hand sides of (6.1). Checking that the corresponding $\tilde{f}$ satisfies the required condition is now straightforward.

Lemma 6.1 $\mathbb{Z}_0$ is invariant for C .

Proof Since $D(B_0)$ is dense in $\mathbb{Y}$, its image is dense in $\mathbb{Z}_0$. Hence, it suffices to show that for all $s \in \mathbb{R}$, $C(s)J^{-1}f$ is a member of this image, provided $f \in D(B_0)$. We will argue that

$$C(s)J^{-1}f = J^{-1}JC(s)J^{-1}f. \quad (6.4)$$

Thus, our aim is to show that $C(s)J^{-1}f$, which clearly is an extension of $JC(s)J^{-1}f$, is the extension given by (6.1) (we have proved that this extension is unique). In other words we need to show that for any $t \in \mathbb{R}$, $C(t)C(s)J^{-1}f$ belongs to $D(B_0)$. However, by definition of $C(t)C(s)J^{-1}f$, since $D(B_0)$ is linear,

$$C(t)C(s)J^{-1}f = \frac{1}{2}(C(t+s)J^{-1}f + C(t-s)J^{-1}f) \in D(B_0),$$

completing the proof. $\square$

Theorem 6.2 B_0 is the generator of a cosine family $C_{B_0} = \{C_{B_0}(t), t \in \mathbb{R}\}$ of equibounded operators given by the abstract Kelvin formula

$$C_{B_0}(t) = JC(t)J^{-1}f, \quad f \in \mathbb{Y}. \quad (6.5)$$

These operators preserve the integral, i.e., $FC_{B_0}(t)f = Ff$ for all $f \in \mathbb{Y}$, $t \in \mathbb{R}$, where $F \in \mathbb{Y}^*$ is defined by

$$F(f_1, f_2, f_3) = \int_{\mathbb{R}^-} (f_1 + f_2) + \int_{\mathbb{R}^+} f_3.$$

Proof The d'Alembert formula for C_{B_0} and strong continuity of $t \mapsto C_{B_0}(t)$ are direct consequences of the corresponding properties of C —use (6.4). The operators in the abstract Kelvin formula are equibounded because $C(t)$, $t \in \mathbb{R}$ are Markov operators in $\mathbb{Z}$ (and hence are contractions). Preservation of the integral is an immediate consequence of (6.2) coupled with fact that $C(t)$, $t \in \mathbb{R}$ preserve the integral in $\mathbb{Z}$.

Thus, all we need to show is that B_0 is the generator of the cosine family defined by (6.5). We note that the domain of the generator, say G , of C is $W^{2,1}(\mathbb{R}) \times W^{2,1}(\mathbb{R}) \times W^{2,1}(\mathbb{R})$, and $G(f_i)_{i=1,2,3} = (\sigma_i f_i'')_{i=1,2,3}$, and that the generator of C restricted to $\mathbb{Z}_0$ is the part G_p of G in $\mathbb{X}_0$. Since on the right-hand side of (6.5) instead of $C(t)$ we could write its restriction to $\mathbb{Z}_0$, we conclude that the generator of the cosine family given by the Kelvin formula is the isomorphic copy of G_p (via J).

Let $f \in D(G_p)$. Then, the coordinates f_1 , f_2 and f_3 of f are members of $W^{2,1}(\mathbb{R})$. Since the right-hand and left-hand limits of f_i 's at $x = 0$ must coincide, the last equation in (6.1) implies, after simple algebra, that

$$f_3(0) = \frac{1}{1+c} \left(\frac{\sigma_1}{\sigma_2} f_1(0) + \frac{\sigma_2}{\sigma_2} f_2(0) + f_3(0) \right),$$

and then the first two equations yield $f_1(0) = \alpha f_3(0)$ and $f_2(0) = \beta f_3(0)$. Similarly a short calculation based on the third equation in (6.1) shows that $a_a f_3'(0) = a_1 f_1'(0) + a_2 f_2'(0)$. This proves that the image of $D(G_p)$ via J is contained in $D(B_0)$. Since for f in this image $JG_p f = B_0 f$, B_0 extends the isomorphic image of G_p . The extension cannot be proper, because both operators are generators of semigroups. This completes the proof. $\square$

Remark 6.3 C_{B_0} is not a cosine family of Markov operators (even though the related semigroup is composed of Markov operators). For example, if $\alpha\sigma_1 - \beta\sigma_2 < \sigma_3$, taking $f_2 = 0$, $f_3 = 0$ and $f_1 = 1_{[-1,0]} \in L^1(\mathbb{R}^-)$, we obtain that $\tilde{f}_1(x) = \frac{2\alpha}{1+c} \frac{\sigma_1}{\sigma_3} - 1 < 0$, for $x \in [0, 1]$, and $\tilde{f}_1(x) = 0$ for $x > 1$. Then, the first coordinate of $C_{B_0}(2\sigma_1^{-1})f$ equals, for $x \in [-2, -1]$,

$$\frac{1}{2}(\tilde{f}_1(x+2) + \tilde{f}_1(x-2)) = \frac{1}{2} \left(\frac{2\alpha}{1+c} \frac{\sigma_1}{\sigma_3} - 1 \right) < 0.$$

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