On 2-absorbing commutative semigroups and their applications to rings

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Abstract A commutative ring R is called 2-absorbing (Badawi in Bull. Aust. Math. Soc. 75:417–429, 2007) if for arbitrary elements $a, b, c \in R$, $abc = 0$ if and only if $ab = 0$ or $bc = 0$ or $ac = 0$. In this paper we study this concept in a more general framework of commutative (multiplicative) semigroups with 0. The results obtained apply to many ring theoretic situations and make it possible to describe similarities and differences among some variants of the notion. We pay a particular attention to graded rings. We also show that a conjecture from (Anderson and Badawi in Commun. Algebra 39:1646–1672, 2011) concerning n -absorbing rings holds for rings with torsion-free additive groups.

Keywords Commutative rings · Commutative semigroups · 2-Absorbing rings · Prime ideals

1 Introduction and preliminaries

A commutative ring R is called *2-absorbing* (cf. [2]) if for arbitrary elements r_1, r_2, r_3 of R such that $r_1 r_2 r_3 = 0$ there are $1 \leq i \neq j \leq 3$ for which $r_i r_j = 0$. Obviously 2-absorbing rings generalize prime rings. In [2] the structure of such rings was described and it was applied to show that a ring R is 2-absorbing if and only if for

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arbitrary ideals I_1, I_2, I_3 of R such that $I_1 I_2 I_3 = 0$ there are $1 \leq i \neq j \leq 3$ for which $I_i I_j = 0$. Thus 2-absorbing rings can be defined in two equivalent ways, by elements or by ideals. It was quite intriguing for us that the proof of this result was rather non-trivial, whereas the respective characterization of prime rings by elements or by ideals is immediate. It became even more interesting when we observed that for graded rings the respective notions defined in terms of homogeneous elements and homogeneous ideals are already not equivalent. Trying to explain that phenomena we came to a conclusion that it is more appropriate for such studies to take the context of commutative (multiplicative) semigroups with 0. The definition of 2-absorbing rings as well as their ideal characterization are given in terms of the multiplication only, so they extend immediately to commutative semigroups with 0. One may expect that the results valid for rings extend to that more general situation. However it is not hard to construct examples showing that in this case the concepts defined by elements and ideals are already not equivalent. More important is that semigroups defined by the counterpart of the condition involving ideals can indeed be described similarly to 2-absorbing rings and the results obtained in this way can be applied to many ring theoretic situations, say, to all rings (giving the results obtained in [2]) or graded rings (which we do in this paper). This approach helped us also to specify the role of the additive structure of rings (or even semirings) in establishing that for rings the concepts defined by elements and ideals are equivalent.

One can define in an obvious way the concepts of n -absorbing rings for integers $n \geq 2$. It seems however that it might be hard to extend the results known for $n = 2$ to higher integers. Some studies concerning that more general situation were recently started in [1] where it was conjectured that also in this case the notions defined by elements and ideals coincide. We show that this conjecture holds for rings whose additive groups are torsion-free. Our arguments indicate also one of the reasons for which the cases $n = 2$ and $n \geq 3$ are substantially different.

Throughout the paper S denotes a commutative semigroup with 0.

We say that S is *2-absorbing* if for arbitrary elements $s_1, s_2, s_3 \in S$ satisfying $s_1 s_2 s_3 = 0$ there are $1 \leq i \neq j \leq 3$ such that $s_i s_j = 0$ and we say that S is *strongly 2-absorbing* if for arbitrary ideals I_1, I_2, I_3 of S satisfying $I_1 I_2 I_3 = 0$ there are $1 \leq i \neq j \leq 3$ such that $I_i I_j = 0$.

It is clear that strongly 2-absorbing semigroups are 2-absorbing. The following simple example shows that the converse does not hold.

Example 1.1 Let S be the Rees factor of the free commutative semigroup generated by x, y modulo the ideal I generated by x^2, y^2 . It is easy to see that $S^3 = 0$ but $S^2 \neq 0$. Hence S is not strongly 2-absorbing. If w_1, w_2, w_3 are arbitrary elements of the free semigroup, then two of them, say w_1, w_2 , both contain x or both contain y . Hence $w_1 w_2 \in I$, which shows that S is 2-absorbing.

We start our studies with describing the structure of strongly 2-absorbing semigroups and next apply the obtained results to rings and rings graded by abelian groups. It is clear that they can be applied to many other ring (or even semiring) situations. We show also that the multiplicative semigroups of commutative rings are 2-absorbing if and only they are strongly 2-absorbing. In the last section we show

that similar arguments can be applied to verify the above mentioned conjecture from [1] for rings whose additive groups are torsion-free.

2 Characterizations of strongly 2-absorbing semigroups

In what follows B denotes the prime radical of S , i.e., the intersection of all prime ideals of S . Since S is commutative, B coincides with the set of all nilpotent elements of S . For arbitrary elements $x, y \in S$ we denote by (x) and (x, y) the ideals of S generated by x and x, y , respectively.

Proposition 2.1 *If S is strongly 2-absorbing, then $B^2 = 0$.*

Proof For every $s \in B$ there is a positive integer n such that $(s)^n = 0$. Applying the assumption that S is strongly 2-absorbing one easily gets $(s)^2 = 0$, so $s^2 = 0$. This implies that for arbitrary $s, t \in B$, $(s)(s, t)(t) = 0$. Hence since S is strongly 2-absorbing, $(s)(t) = 0$ or $(s)(s, t) = 0$ or $(s, t)(t) = 0$. In all cases $st = 0$. Hence $B^2 = 0$. $\square$

Example 1.1 shows that there are nilpotent 2-absorbing semigroups which are not strongly 2-absorbing. There are also semiprime semigroups of that type.

Example 2.2 Let S be the Rees factor of the free commutative semigroup W generated by x, y, z modulo the ideal I generated by xy, yz, zx . It is evident that S contains no non-zero nilpotent element. If $w_1, w_2, w_3 \in W$ and $w_1 w_2 w_3 \in I$, then one of the words xy, yz, zx must be a subword of $w_i w_j$ for some $1 \leq i \neq j \leq 3$, so $w_i w_j \in I$. Hence S is 2-absorbing. Now $(x, y)(y, z)(z, x) \subseteq I$ but no product of the two ideals $(x, y), (y, z), (z, x)$ is contained in I , so S is not strongly 2-absorbing.

The following characterization of strongly 2-absorbing semigroups is quite useful as it allows one to apply induction arguments in some investigations.

Proposition 2.3 *S is strongly 2-absorbing if and only if for arbitrary ideals F_1, F_2, F_3 generated by ≤ 2 elements each and satisfying $F_1 F_2 F_3 = 0$ there are $1 \leq i \neq j \leq 3$ such that $F_i F_j = 0$.*

Proof Let X, Y, Z be ideals of S such that $XYZ = 0$. Suppose that $XY \neq 0, YZ \neq 0, XZ \neq 0$. Then there are $x_1, x_2 \in X, y_1, y_2 \in Y, z_1, z_2 \in Z$ such that $x_1 y_1 \neq 0, x_2 z_1 \neq 0, y_2 z_2 \neq 0$. Let $F_1 = (x_1, x_2), F_2 = (y_1, y_2), F_3 = (z_1, z_2)$. Clearly $F_1 F_2 F_3 = 0$ but $F_i F_j \neq 0$ for arbitrary $1 \leq i \neq j \leq 3$. This proves the “if” part. The “only if” part is clear. $\square$

For a given subset X of S we denote by $\text{ann}(X)$ the annihilator of X in S , i.e., $\text{ann}(X) = \{s \in S \mid Xs = 0\}$.

Now we pass to study the structure of strongly 2-absorbing semigroups.

Theorem 2.4 *S is strongly 2-absorbing if and only if*

- (a) If I, J are ideals of S such that $I \not\subseteq B$, $J \not\subseteq B$ and $IJ \subseteq B$, then $IJ = 0$ and $IB = 0 = JB$;
- (b) For every subset X of B , $\text{ann}(X)$ is a prime ideal of S ;
- (c) If I, J, K are ideals of S not contained in B , then $IJ \neq 0$ or $JK \neq 0$ or $IK \neq 0$.

Proof Suppose that S is strongly 2-absorbing.

- (a) By Proposition 2.1, $B^2 = 0$. Hence since $IJ \subseteq B$, we have $BIJ = 0$. Suppose that $IJ \neq 0$. Then $BI = 0$ or $BJ = 0$. If, say, $BI = 0$, then $I^2J = 0$. However $I \not\subseteq B$, so $I^2 \neq 0$. Since also $IJ \neq 0$, we get a contradiction with the assumption that S is strongly 2-absorbing. Hence $IJ = 0$. Now $(B \cup I)J \subseteq B$, so by the foregoing $(B \cup I)J = 0$ and further $BJ = 0$. Similarly $BI = 0$.
- (b) Suppose that I, J are ideals of S such that $XIJ = 0$. We can assume that X is an ideal of R . If $I \subseteq B$ or $J \subseteq B$, then since $B^2 = 0$, we get that $XI = 0$ or $XJ = 0$. Assume that both I, J are not contained in B . If $IJ \neq 0$, then $XI = 0$ or $XJ = 0$ as S is strongly 2-absorbing. If $IJ = 0$, then applying (a) we again get that $XI = 0$ or $XJ = 0$ and we are done.
- (c) Suppose that $IJ = JK = IK = 0$. Set $I_1 = I \cup J$, $I_2 = J \cup K$, $I_3 = I \cup K$. Clearly $I_1I_2I_3 = 0$ but since I, J, K are not contained in B , $I_iI_j \neq 0$ for all $1 \leq i, j \leq 3$, a contradiction.

Conversely, suppose that I_1, I_2, I_3 are ideals of S such that $I_1I_2I_3 = 0$. Set $I = I_1I_2$, $J = I_2I_3$, $K = I_1I_3$. Then $IJ = JK = IK = 0$. Hence by (c) one of I, J, K , say I , is contained in B . Now by (a), $I_1I_2 = 0$ or we can assume that, say, $I_1 \subseteq B$. However if $I_1 \subseteq B$, then by (b), $I_1I_2 = 0$ or $I_1I_3 = 0$. These show that S is strongly 2-absorbing. The result follows. $\square$

The following result is a semigroup counterpart of Theorem 2.4 in [2] describing the structure of 2-absorbing rings.

Theorem 2.5 S is strongly 2-absorbing if and only if

- (1) for every subset X of B , $\text{ann}(X)$ is a prime ideal of S ;
- (2) one of the following conditions holds
 - (a) B is a prime ideal of S
 - (b) S contains prime ideals P_1, P_2 such that $P_1P_2 = 0$.

Proof Suppose that S is strongly 2-absorbing. The condition (1) is precisely the condition (b) in Theorem 2.4. Suppose that B is not a prime ideal. Then there are $s, t \in S \setminus B$ such that $st \in B$. By Theorem 2.4(a), $st = 0$ and $sB = tB = 0$. Set $P_1 = \text{ann}(s)$, $P_2 = \text{ann}(t)$. Then $B \subseteq P_1$, $B \subseteq P_2$ and $t \in P_1$, $s \in P_2$. Let I, J be ideals of S not contained in P_1 . Then $sI \neq 0$, $sJ \neq 0$, so by Theorem 2.4(a), $sI \not\subseteq B$, $sJ \not\subseteq B$. Now $sI, sJ, (t)$ are ideals of S not contained in B and $(t)sI = 0$, $(t)sJ = 0$, so by Theorem 2.4(c), $sIsJ \neq 0$. Consequently $sIJ \neq 0$ and $IJ \not\subseteq P_1$, which shows that P_1 is a prime ideal of S . Similarly P_2 is a prime ideal of S . Note that $0 \neq s^2 \in P_1(s, t)$, $0 \neq t^2 \in P_2(s, t)$ but $P_1P_2(s, t) = 0$. Consequently $P_1P_2 = 0$. These show the “only if” part.

Conversely, suppose that S satisfies the listed conditions. Take ideals I_1, I_2, I_3 of S such that $I_1 I_2 I_3 = 0$. We have to show that $I_i I_j = 0$ for some $1 \leq j \neq i \leq 3$. If B is a prime ideal of S , then $I_i \subseteq B$ for some $1 \leq i \leq 3$ and we are done by (1). If B is not a prime ideal, then applying (2) we get that there are $1 \leq i, j \leq 3$ such that $I_i \subseteq P_1, I_j \subseteq P_2$ and $I_i I_j = 0$. If $i \neq j$ we are done. Thus suppose $i = j$ and, say, they are equal 1. Then $I_1 \subseteq B$ and $I_2 I_3 \subseteq \text{ann}(I_1)$. However then from (1) it follows that $I_1 I_2 = 0$ or $I_1 I_3 = 0$ and we are done again. $\square$

Remark 2.6 One easily gets that if P_1, P_2 are prime ideals of an arbitrary (even non-commutative) semigroup S such that $P_1 P_2 = 0$, then P_1, P_2 are unique minimal prime ideals of S .

3 Applications to rings

In what follows R is a commutative ring and $S(R)$ denotes the multiplicative semigroup of R .

We start with a simple lemma, which however plays a key role in the proof of Theorem 3.4 showing that $S(R)$ is 2-absorbing if and only if it is strongly 2-absorbing.

Lemma 3.1 *Suppose that A is a subsemigroup of the additive group of R and X, Y are subsets of R such that for each $a \in A$, $aX = 0$ or $aY = 0$. Then $AX = 0$ or $AY = 0$.*

Proof Suppose that there are $a, b \in A$ such that $aX \neq 0$ and $bY \neq 0$. Then by the assumption $aY = 0$ and $bX = 0$. Moreover $(a + b)X = 0$ or $(a + b)Y = 0$. In the former case $0 = (a + b)X = aX \neq 0$ and in the latter $0 = (a + b)Y = bY \neq 0$. This contradiction proves the result. $\square$

Remark 3.2 This proof applies also to the case when R is a semiring. One can check that in fact all the results below hold with the same proofs for semirings.

Corollary 3.3 *If A is a subsemigroup of the additive group of R and $\text{ann}(a)$ is a prime ideal of R for every $a \in A$, then $\text{ann}(A)$ is a prime ideal of R .*

Proof Let $x, y \in R$ and $Axy = 0$. Since for every $a \in A$, $\text{ann}(a)$ is a prime ideal, $ax = 0$ or $ay = 0$. Hence from Lemma 3.1 it follows that $Ax = 0$ or $Ay = 0$, so $\text{ann}(A)$ is a prime ideal. $\square$

Theorem 3.4 *The following conditions are equivalent*

- (a) R is 2-absorbing;
- (b) $S(R)$ is 2-absorbing;
- (c) $S(R)$ is strongly 2-absorbing;
- (d) for arbitrary ideals I_1, I_2, I_3 of R satisfying $I_1 I_2 I_3 = 0$ there are $1 \leq i \neq j \leq 3$ such that $I_i I_j = 0$.

Proof It is evident that (a) and (b) are equivalent, (c) implies (b) and (d) implies (a).

Every ideal of R is obviously an ideal of $S(R)$. For a given ideal I of $S(R)$ denote by $\bar{I}$ the ideal of R generated by I . It is clear that $\overline{IJ} = \bar{I}\bar{J}$ (where $\bar{I}\bar{J}$ denotes the product of ring ideals). These easily show that (c) and (d) are equivalent.

It remains to show that (a) implies (c). For that in view of Proposition 2.3 we have to show that if F_1, F_2, F_3 are ideals of $S(R)$ generated by ≤ 2 elements each and $F_1 F_2 F_3 = 0$, then $F_i F_j = 0$ for some $1 \leq i \neq j \leq 3$. Let n be the sum of the minimal numbers of generators of F_i . Obviously $3 \leq n \leq 6$. Since $S(R)$ is 2-absorbing the result holds for $n = 3$, i.e., when each F_i is a principal ideal of $S(R)$. Thus assume that $n \geq 4$, say, $F_1 = (s, t)$ and the result holds if the sum of the numbers of generators is $\leq n - 1$. If $F_2 F_3 = 0$, we are done. Otherwise, since for each $f \in \bar{F}_1$, $(f)F_2 F_3 = 0$, our assumptions on n implies that $f F_2 = 0$ or $f F_3 = 0$. Now applying Lemma 3.1 we get that $F_1 F_2 = 0$ or $F_1 F_3 = 0$. $\square$

Note that for every subset X of R the annihilators of X in R and in $S(R)$ coincide. Moreover if I is an ideal of R , then I is a prime ideal of R if and only if it is a prime ideal of $S(R)$. It is also clear that the prime radical of R coincides with the prime radical of $S(R)$. These, Theorem 2.5, Corollary 3.3 and Theorem 3.4 give the following

Theorem 3.5 ([2], Theorem) *R is 2-absorbing if and only if for every $x \in B$, $\text{ann}(x)$ is a prime ideal of R and one of the following conditions holds*

- (a) B is a prime ideal of R or
- (b) R contains prime ideals P_1, P_2 such that $P_1 P_2 = 0$.

Let G be an abelian group and let R be a G -graded ring. Recall that $R = \bigoplus_{g \in G} R_g$, the direct sum of additive subgroups R_g of R , with $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. Elements of $\bigcup_{g \in G} R_g$ are called *homogeneous*. Every $r \in R$ can be written as a finite sum $r = \sum_g r_g$, where $r_g \in R_g$ is called the *homogeneous component* of r . The set $\text{supp}(r) = \{g \in G \mid r_g \neq 0\}$ is called the *support* of r . An ideal I of R is called *homogeneous* if $I = \bigoplus_{g \in G} (I \cap R_g)$.

For more details concerning graded rings we refer to [4].

Denote by $S_h(R)$ the multiplicative semigroup of homogeneous elements of R . For a given ideal I of $S_h(R)$ denote by $\bar{I}$ the ideal of R generated by I . It is clear that $\bar{I}$ is a homogeneous ideal of R and $\overline{IJ} = \bar{I}\bar{J}$. For every homogeneous ideal I of R the set I^* of homogeneous components of elements from I is an ideal of $S_h(R)$ and I is a graded prime ideal of R if and only if I^* is a prime ideal of $S_h(R)$ as well as the prime radical $B_h(R)$ of $S_h(R)$ coincides with B_h^* , where B_h is the graded prime radical of R . Finally for every subset X of R consisting of homogeneous elements $\text{ann}(X)$ is a homogeneous ideal of R . These and Theorem 2.5 give

Theorem 3.6 *For a given G -graded ring R the following conditions are equivalent*

- (a) $S_h(R)$ is strongly 2-absorbing;
- (b) For arbitrary homogeneous ideals I_1, I_2, I_3 satisfying $I_1 I_2 I_3 = 0$ there are $1 \leq i \neq j \leq 3$ such that $I_i I_j = 0$;

- (c) For every subset X of B consisting of homogeneous elements, $\text{ann}(X)$ is a graded prime ideal and one of the following conditions holds
- B_h is a graded prime ideal of R or
 - R contains graded prime ideals P_1, P_2 such that $P_1 P_2 = 0$.

If R is a G -graded 2-absorbing ring then by Theorem 3.4, $S_h(R)$ is strongly 2-absorbing. The converse does not hold. For instance if G is a finite abelian group of order ≥ 3 and F is an algebraically closed field of characteristic 0 and R is the group algebra of G over F , then obviously $S_h(R)$ is strongly 2-absorbing but, since R is isomorphic to the direct sum of $|G|$ copies of F , R is not 2-absorbing. The following result, which generalizes Theorem 4.15 in [1], shows that the converse holds if G is torsion-free.

Theorem 3.7 *Suppose that R is a G -graded ring and G is torsion-free. If $S_h(R)$ is strongly 2-absorbing, then R is 2-absorbing (or, equivalently, $S(R)$ is 2-absorbing).*

Proof It is well known and easy to check that, since G is torsion-free, a homogeneous ideal of R is prime if and only if it is graded prime as well as $B = B_h$. Consequently if B_h is a graded prime ideal of R , then B is a prime ideal of R . If B is not a prime ideal, then by Theorems 3.6(c), R contains graded prime ideals P_1, P_2 such that $P_1 P_2 = 0$. Hence P_1, P_2 are also prime ideals of R . Hence in view of Theorem 3.5 to get the result it suffices to show that for each $0 \neq x \in B$, $\text{ann}(x)$ is a prime ideal of R . Suppose that for some $a, b \in R \setminus \text{ann}(x)$, $ab \in \text{ann}(x)$, i.e., $xab = 0$ but $xa \neq 0$ and $xb \neq 0$. Choose them in such a way that $n = \text{card}(\text{supp}(x)) + \text{card}(\text{supp}(a)) + \text{card}(\text{supp}(b))$ is minimal. Moreover, we can assume that G is generated by the union of the supports of x, a, b . Then G , being a finitely generated abelian torsion-free group, is linearly ordered. Obviously $n \geq 3$ and by Theorem 3.6(c) not all x, a, b are homogeneous, so $n > 3$. Let g, h, t be the largest elements in $\text{supp}(x), \text{supp}(a), \text{supp}(b)$, respectively. Then $x_g a_h b_t = 0$ and by Theorem 3.6(c) we can assume that $x_g a_h = 0$. If $x a_h = 0$, then $a_h \neq a$ and $x(a - a_h)b = 0$ and $\text{card}(x) + \text{card}(a - a_h) + \text{card}(b) < n$, so since $xb \neq 0$, we get $x(a - a_h) = 0$. These give $xa = 0$, a contradiction. Thus $x a_h \neq 0$. Now $(x a_h)ab = 0$ and $\text{card}(\text{supp}(x a_h)) + \text{card}(\text{supp}(a)) + \text{card}(\text{supp}(b)) < n$, so $x a_h a = 0$ or $x a_h b = 0$. If the former holds and x is a homogeneous element, then for each homogeneous component a_t of a we have $x a_h a_t = 0$. Since a is not homogeneous, $x a_h \neq 0$ and $\text{card}(\text{supp}(x)) + 2 < n$, we get that $x a_t = 0$, a contradiction. In the latter case $\text{card}(\text{supp}(x)) + \text{card}(\text{supp}(a_h)) + \text{card}(\text{supp}(b)) < n$ and $x a_h \neq 0$ as well as $xb \neq 0$, so we get a contradiction again. This finishes the proof. $\square$

The following example shows that in Theorem 3.7 it is not enough to assume merely that $S_h(R)$ is 2-absorbing.

Example 3.8 Let F be a field and $A = F[x, y]/I$, where I is the ideal of $F[x, y]$ generated by x^2, y^2 . Set $a = x + I$ and $b = y + I$. The F -subalgebra $R = F + Faz + Fbz^2 + Fabz^3$ of $A[z]$ is graded in a canonical way by the additive group of integers. Note that for $t = az + bz^2$ we have $t^3 = 0$ and $t^2 \neq 0$, so R is not 2-absorbing. However every non-invertible element of $S_h(R)$ is square-zero, so $S_h(R)$ is 2-absorbing.

4 On n -absorbing and strongly n -absorbing rings

A finite number of ideals of R (some ideals can appear several times) will be called a *collection of ideals* if their product is equal 0.

Let n be an integer ≥ 2 . A ring R is called *strongly n -absorbing* (c.f. [1]) if the following condition is satisfied:

(*) every collection of $n + 1$ ideals of R contains a collection of n ideals.

If the condition (*) is satisfied for collections of $n + 1$ principal ideals, then R is called *n -absorbing*.

It is evident that strongly n -absorbing rings are n -absorbing. In [1] it was conjectured that the converse holds as well. We will show that it is indeed the case if the additive group of R is torsion-free.

Arguments similar to those applied in the proof of Proposition 2.3 give the following characterization of strongly n -absorbing rings.

Proposition 4.1 *A ring R is strongly n -absorbing if and only if it satisfies the condition (*) for ideals generated by $\leq n$ generators each.*

Proof If R is not strongly n -absorbing, then there exists a collection $I_1, \dots, I_{n+1}$ of ideals of R such that no n of these ideals is a collection. Hence for every $1 \leq i \leq n + 1$ there are $f_{ij} \in I_j$, $1 \leq j \neq i \leq n + 1$ whose product is not equal 0. Let, for each $1 \leq j \leq n + 1$, F_j be the ideal of R generated by the set of all f_{ij} . It is clear that $F_1, F_2, \dots, F_{n+1}$ is a collection of ideals but no n of these ideals form a collection. This shows the “if” part. The “only if” part is evident. $\square$

Theorem 4.2 *If R is n -absorbing and the additive group of R is torsion-free, then R is strongly n -absorbing.*

Proof In view of Proposition 4.1 it suffices to show that if $F_1, F_2, \dots, F_{n+1}$ is a collection of ideals of R and each of them is generated by $\leq n$ elements, then for some $1 \leq j \leq n + 1$ the product of all F_i for $i \neq j$ is equal 0. Define the index of a collection of finitely generated ideals as the sum of minimal numbers of their generators. It is clear that if s is the index of the collection $F_1, \dots, F_{n+1}$, then $n + 1 \leq s \leq n(n + 1)$. If $s = n + 1$, then all ideals F_i are principal, so the claim holds as R is n -absorbing. Thus suppose that $s > n + 1$ and the result holds for collections of $n + 1$ ideals with index smaller than s . Then at least one of the F_i , say F_1 , is not principal. If the result does not hold for s , then $F_2 F_3 \cdots F_{n+1} \neq 0$. For each $f \in F_1$, $f F_2 F_3 \cdots F_{n+1} = 0$ and since F_1 is not principal, the index of the collection $(f), F_2, \dots, F_{n+1}$ is $< s$. Thus for some $2 \leq k \leq n + 1$, we have $f T_k = 0$, where T_k is the product of all ideals $F_2, \dots, F_{n+1}$ except F_k . Consequently $F_1 \subseteq \text{ann}(T_2) \cup \text{ann}(T_3) \cup \cdots \cup \text{ann}(T_{n+1})$. Obviously F_1 and all $\text{ann}(T_i)$ are subgroups of the additive group of R . It is well known (cf. [3]) that these imply that for a positive integer m , $m F_1$ is contained in $\text{ann}(T_i)$ for some $2 \leq i \leq n + 1$. Hence $m F_1 T_i = m \bar{F}_i = 0$, where $\bar{F}_i$ is the product of all $F_1, \dots, F_{n+1}$ except F_i . However $\bar{F}_i \neq 0$ and the additive group of R is torsion-free, so we get a contradiction. $\square$

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