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RESEARCH PAPER

Optimal estimates for the semigroup generated by the classical Volterra operator on L_p-spaces

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Abstract Optimal upper bounds are given for the norm of the semigroup $(e^{-tV})_{t\geq 0}$, where V is the classical Volterra operator acting on $L_p[0,1]$, $1 \leq p \leq \infty$. In particular, for every $p \in [1,\infty]$ we prove that

$$\overline{\lim_{t\to+\infty}} \left(t^{-|1/4-1/(2p)|}\|e^{-tV}\|_{L_p}\right) > 0.$$

Keywords Volterra operator · L_p-space · Operator semigroup

1 Introduction

We shall consider the classical integral Volterra operator V defined on the space $L_p = L_p[0,1]$, $1 \leq p \leq \infty$, by

$$(Vf)(x) = \int_0^x f(t)\,dt, \qquad f \in L_p.$$

The operator V acts boundedly on L_p and then the exponential function

$$e^{zV} = \sum_{m=0}^{\infty} \frac{z^m V^m}{m!}, \qquad z \in \mathbb{C}, \tag{1}$$

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is well defined.

It was proved in [1, Theorem 1.2] that

$$\lim_{t\to+\infty} \frac{\ln \|e^{te^{i\theta}V}\|_{L_p}}{t^{1/2}} = \sqrt{2}\cos(\theta/2), \quad \theta \in (-\pi, \pi], \tag{2}$$

where $1 \le p \le \infty$. For $|\theta| < \pi$ the right-hand side of the equality (2) is positive. On the other hand for $\theta = \pi$ the limit in (2) is zero. Thus, a natural and interesting case $\theta = \pi$ requires a further investigation.

As observed in [2], the semigroup $(e^{-tV})_{t\ge 0}$ is uniformly bounded on L_p if and only if $p = 2$. Indeed, suppose that for some $p \in [1, +\infty]$ one has $\|e^{-tV}\|_{L_p} \le M$, $t \ge 0$. Then by the Laplace transform representation for the generator's resolvent (see e.g. [3])

$$\|(I+V)^{-n}\|_{L_p} = \|(-V-I)^{-n}\|_{L_p} \le M, \quad n \in \mathbb{N}. \tag{3}$$

On the other hand, by [4, Theorem 1.1] (for $p = 1$ this is the result due to Hille [5]),

$$c_2 \le n^{-|1/4-1/(2p)|}\|(I+V)^{-n}\|_{L_p} \le c_1, \quad n \ge 1, \tag{4}$$

for some constants $c_1, c_2 > 0$, and then in view of (3) we obtain that $p = 2$. If $p = 2$ then the operator V is accretive:

$$\operatorname{Re}(Vf, f)_{L_2} = \frac{1}{2}\left|\int_0^1 f(x)\,dx\right|^2 \ge 0, \quad f \in L_2,$$

and therefore $(e^{-tV})_{t\ge 0}$ is a semigroup of contractions on L_2:

$$\|e^{-tV}\|_{L_2} \le 1, \quad t \ge 0.$$

In this note we shall prove that

$$t^{-|1/4-1/(2p)|}\|e^{-tV}\|_{L_p} \le \tilde{c}_1, \quad t \ge 1,$$

and

$$t_n^{-|1/4-1/(2p)|}\|e^{-t_n V}\|_{L_p} \ge \tilde{c}_2, \quad n \ge 1,$$

for some positive sequence $t_n \to \infty$, $n \to \infty$, thus providing a semigroup counterpart for the estimate (4) obtained in [4].

2 Main result

Let $J = -V$. We have

$$(J^n f)(x) = \frac{(-1)^n}{(n-1)!}\int_0^x (x-s)^{n-1} f(s)\,ds, \quad n \in \mathbb{N},$$

and then

$$(e^{tJ}f)(x) = f(x) + \sum_{n=1}^{\infty} \frac{t^n(-1)^n}{(n-1)!n!} \int_0^x (x-s)^{n-1} f(s)\,ds$$
$$= f(x) + \int_0^x K(x-s;t) f(s)\,ds,$$

where

$$K(z;t) := \sum_{n=1}^{\infty} \frac{z^{n-1}t^n(-1)^n}{(n-1)!n!} = -t\sum_{n=0}^{\infty} \frac{(-tz)^n}{n!(n+1)!} = -\frac{\sqrt{t}}{\sqrt{z}} J_1(2\sqrt{tz}),$$

and $J_1(\cdot)$ is the Bessel function of the first kind and of the first order. So,

$$(e^{tJ}f)(x) = f(x) - (S(t)f)(x), \quad t > 0, \tag{5}$$

where

$$(S(t)f)(x) = \sqrt{t}\int_0^x \frac{J_1(2\sqrt{ts})}{\sqrt{s}} f(x-s)ds$$
$$= \sqrt{t}\int_0^x \frac{J_1(2\sqrt{t(x-s)})}{\sqrt{x-s}} f(s)ds. \tag{6}$$

From (5) and (6), using Minkowski's inequality, we have

$$\|e^{tJ}f\|_{L_p} \le \|f\|_{L_p} + \|f\|_{L_p}\sqrt{t}\int_0^1 \frac{|J_1(2\sqrt{ts})|}{\sqrt{s}}ds$$
$$= \left(1 + \int_0^{2\sqrt{t}} |J_1(s)|ds\right) \cdot \|f\|_{L_p},$$

and by the well-known estimate for Bessel functions [6, Chap. 7]:

$$|J_1(s)| \le \frac{c}{\sqrt{s}}, \quad s > 0,$$

for some $c > 0$, so we obtain for some constant M

$$\|e^{tJ}\|_{L_p} \le Mt^{1/4}, \quad t \ge 1,\ 1 \le p \le \infty. \tag{7}$$

Moreover, since $(e^{tJ})_{t\ge 0}$ is a semigroup of contractions on L_2, by the Riesz-Thorin interpolation theorem [7, p. 97] and (7) we get

$$\|e^{tJ}\|_{L_p} \le \|e^{tJ}\|_{L_2}^{2-2/p}\|e^{tJ}\|_{L_1}^{2/p-1} \le M^{2/p-1}t^{1/(2p)-1/4}, \quad t \ge 1,\ p \in (1,2),$$
$$\|e^{tJ}\|_{L_p} \le \|e^{tJ}\|_{L_2}^{2/p}\|e^{tJ}\|_{L_\infty}^{1-2/p} \le M^{1-2/p}t^{1/4-1/(2p)}, \quad t \ge 1,\ p \in (2,\infty),$$

so that

$$\|e^{tJ}\|_{L_p} \le M^{|2/p-1|} t^{|1/(2p)-1/4|}, \quad t \ge 1,\ 1 \le p \le \infty. \tag{8}$$

The following theorem, which is the main result of the note, states that the estimate (8) is sharp for any $p \in [1, \infty]$.

Theorem *For every* $p \in [1, \infty]$:

$$\overline{\lim_{t\to+\infty}} \left(t^{-|1/4-1/(2p)|} \|e^{tJ}\|_{L_p} \right) > 0. \tag{9}$$

Proof Let q be the conjugate exponent of p, that is $1/p + 1/q = 1$. Then by [4, p. 765] we obtain

$$\|e^{tJ}\|_{L_q} = \|e^{tJ}\|_{L_p}, \quad 1 \le p \le \infty.$$

Thus is suffices to prove the estimates (9) for $p \in [1, 2]$.

Let $p \in [1, 2]$ be fixed. First of all, from (6) it follows that

$$\begin{aligned} \|S(t)f\|_{L_p} &\ge \|S(t)f\|_{L_p[1/t,1]} \\ &\ge \|S_0(t)f\|_{L_p[1/t,1]} - \|S_1(t)f\|_{L_p[1/t,1]}, \quad f \in L_p,\ t > 1, \end{aligned}$$

where the operator functions $S_j(t)$, $j = 0, 1$ are given by

$$(S_0(t)f)(x) = \sqrt{t} \int_0^{x-1/t} \frac{J_1(2\sqrt{t(x-s)})}{\sqrt{x-s}} f(s)ds,$$

$$(S_1(t)f)(x) = \sqrt{t} \int_{x-1/t}^{x} \frac{J_1(2\sqrt{t(x-s)})}{\sqrt{x-s}} f(s)ds.$$

By Minkowski's inequality,

$$\begin{aligned} \|S_1(t)f\|_{L_p[1/t,1]} &\le \sqrt{t} \int_0^{1/t} \frac{|J_1(2\sqrt{ts})|}{\sqrt{s}} \left(\int_{1/t}^1 |f(x-s)|^p dy \right)^{1/p} ds \\ &\le \|f\|_{L_p} \sqrt{t} \int_0^{1/t} \frac{|J_1(2\sqrt{ts})|}{\sqrt{s}} ds \\ &= \|f\|_{L_p} \int_0^2 |J_1(s)| ds = c_0 \|f\|_{L_p}, \end{aligned}$$

where the constant $c_0 > 0$ does not depend on $t > 1$. So, to prove (9) it suffices to show that

$$\overline{\lim_{t\to+\infty}} \left\{ t^{-|1/4-1/(2p)|} \left(\sup_{f \in L_p} \frac{\|S_0(t)f\|_{L_p[1/t,1]}}{\|f\|_{L_p[0,1-1/t]}} \right) \right\} > 0. \tag{10}$$

We will use the following asymptotic formula for the Bessel function [6, Sect. 7.4]:

$$J_1(2\sqrt{t(x-s)}) = -\frac{\cos(2\sqrt{t(x-s)} + \pi/4)}{\sqrt{\pi} t^{1/4} (x-s)^{1/4}} + \mathrm{O}(t^{-3/4}(x-s)^{-3/4}),$$

$$t(x-s) \to \infty.$$

Observe that $S_0(t)f$ can be decomposed as

$$(S_0(t)f)(x) = -(S_{0,1}f)(x) + (S_{0,2}(t)f)(x), \quad x \in (1/t, 1), \tag{11}$$

where

$$(S_{0,1}f)(x) = \frac{t^{1/4}}{\sqrt{\pi}} \int_0^{x-1/t} \frac{\cos(2\sqrt{t(x-s)} + \pi/4)}{(x-s)^{3/4}} f(s)ds,$$

and

$$|(S_{0,2}(t)f)(x)| \le ct^{-1/4} \int_0^{x-1/t} \frac{|f(s)|}{(x-s)^{5/4}} ds = c_1 t^{-1/4} \int_{1/t}^{x} \frac{|f(x-s)|}{s^{5/4}} ds.$$

Then, using Minkowski's inequality, we get the estimate

$$\begin{aligned} \|S_{0,2}(t)f\|_{L_p[1-1/t,1]} &\le c_1 t^{-1/4} \int_{1/t}^{1} \frac{1}{s^{5/4}} \left(\int_{1-1/t}^{1} \chi(x-s)|f(x-s)|^p dx \right)^{1/p} ds \\ &\le c_1 t^{-1/4} \int_{1/t}^{1} \frac{ds}{s^{5/4}} \|f\|_{L_p[0,1-1/t]} \le 4c_1 \|f\|_{L_p[0,1-1/t]}, \end{aligned}$$

where χ is the characteristic function of $[0, +\infty)$.

Furthermore

$$\|S_{0,1}(t)f\|_{L_p[1/t,1]} = \frac{t^{1/4}}{\sqrt{\pi}} \|\tilde{S}(t)f\|_{L_p[0,1-1/t]}, \tag{12}$$

where the operator function $\tilde{S}(t)$ is defined as

$$(\tilde{S}(t)f)(x) := \int_0^{y} \frac{\cos(2\sqrt{t(x-s+1/t)} + \pi/4)}{(x-s+1/t)^{3/4}} f(s)ds, \quad x \in (0, 1-1/t).$$

Thus, by (11) and (12), we conclude that for the proof of the inequality (10), and then for the proof of the theorem, it is enough to prove that

$$\varlimsup_{t\to+\infty} \left\{ t^{-|1/4-1/(2p)|} t^{1/4} \left(\sup_{f\in L_p[0,1-1/t]} \frac{\|\tilde{S}(t)f\|_{L_p[0,1-1/t]}}{\|f\|_{L_p[0,1-1/t]}} \right) \right\} > 0. \tag{13}$$

Let $t_n = \pi^2 n^2$, where $n = 2, 4, \ldots$. Then for $k = 1, 2, \ldots, n$ we have

$$\cos(2\sqrt{t_n z} + \pi/4) \ge \frac{\sqrt{2}}{2}, \quad z \in I_k := \left[\frac{(k-1/4)^2}{n^2}, \frac{k^2}{n^2} \right] \subset [0, 1]. \tag{14}$$

Note that

$$x - s + 1/t_n \in I_k, \quad s \in [1, 1/(4n)], \; x \in M_k, \tag{15}$$

where

$$M_k = \left[\frac{(k-1/4)^2}{n^2} - \frac{1}{t_n} + \frac{1}{4n}, \frac{k^2}{n^2} - \frac{1}{t_n}\right], \quad k = \frac{n}{2}+1, \dots, n.$$

The lengths $|M_k|$ of the segments M_k satisfy

$$|M_k| = \frac{k^2}{n^2} - \frac{(k-1/4)^2}{n^2} - \frac{1}{4n} = \frac{k-n/2-1/8}{2n^2},$$

so that

$$d_n := \sum_{k=n/2+1}^{n} |M_k| = \frac{1}{2n^2}\sum_{k=n/2+1}^{n}(k-n/2-1/8) = \frac{4n-9}{32n} \geq \frac{1}{9}, \quad n > 20.$$

Let now $n > 20$ be an even integer, let g be the characteristic function of

$$M = \bigcup_{k=n/2}^{n} M_k,$$

and let f be the characteristic function of the segment $[1, (4n)^{-1}]$. Set $\beta_n = 1 - 1/t_n$. Then

$$\|f\|_{L_p[0,\beta_n]} = \left(\frac{1}{4n}\right)^{1/p} \leq \frac{1}{n^{1/p}}, \qquad \|g\|_{L_q[0,1-1/t_n]} \leq 1,$$

$$\left|\int_0^{\beta_n} (\tilde{S}f)(x)g(x)dx\right| \leq \|\tilde{S}f\|_{L_p[0,\beta_n]}\|g\|_{L_q[0,\beta_n]} \leq \|\tilde{S}f\|_{L_p[0,\beta_n]}, \quad \frac{1}{p}+\frac{1}{q} = 1.$$

On the other hand, using properties of the segments M_k (see (14), (15)), we obtain

$$\begin{aligned}\left|\int_0^{\beta_n} (\tilde{S}f)(x)g(x)dx\right| &= \left|\int_{x\in M} (\tilde{S}f)(x)dx\right| \\ &= \int_{x\in M}\int_0^{1/4n} \frac{\cos(2\sqrt{t_n(x-s+1/t_n}+\pi/4)}{(x-s+1/t_n)^{3/4}} ds\, dx \\ &\geq \frac{1}{\sqrt{2}}\int_{x\in M}\int_0^{1/4n} \frac{ds\, dx}{(x-s+1/t_n)^{3/4}} \\ &\geq \frac{1}{\sqrt{2}}\int_{x\in M}\int_0^{1/4n} ds\, dx = \frac{1}{\sqrt{2}}\frac{d_n}{4n} \geq \frac{1}{72n}.\end{aligned}$$

Thus

$$t_n^{1/4}\frac{\|\tilde{S}f\|_{L_p[0,\beta_n]}}{\|f\|_{L_p[0,\beta_n]}} \geq \sqrt{\pi}\,\frac{n^{1/p}}{72n^{1/2}} = ct_n^{(1/p-1/2)/2}, \quad c > 0.$$

If $p \in [1,2]$, then the last inequality implies (13), and then (9) follows. The proof is complete. □

3 Comments and remarks

Define the differential operator A on L_p, $1 \le p < \infty$, by

$$(Af)(y) = -f^{'}(y), \quad \text{with the domain } D(A) = \{f \in W_p^1[0,1] : \ f(0) = 0\},$$

where W_p^1 stands for the Sobolev space. The operator A generates the nilpotent C_0-semigroup $(e^{tA})_{t \ge 0}$ given by

$$(e^{tA} f)(x) = f(x-t), \quad 0 < x - t \le 1, \qquad (e^{tA} f)(x) = 0, \quad 0 \le x < t,$$

and $A^{-1} = J$. So, the Theorem provides the example of a nilpotent C_0-semigroup $(e^{tA})_{t \ge 0}$ on L_p, $p \in [1, \infty)$, $p \ne 2$, such that the C_0-semigroup $(e^{tA^{-1}})_{t \ge 0}$ is not uniformly bounded (more precisely, the norm of $e^{tA^{-1}}$ grows at infinity as t^{α_p}, $\alpha_p = |1/4 - 1/(2p)|$).

Let us consider a more general situation. Let X be a Banach space with the norm $\| \cdot \|$. Denote by $\mathcal{G} = \mathcal{G}(X)$ the set of generators of uniformly bounded C_0-semigroups on X and by $\mathcal{G}_{exp} = \mathcal{G}_{exp}(X)$ the set of generators of exponentially stable C_0-semigroup acting on X.

If for $A \in \mathcal{G}_{exp}(X)$ one has for some $M \ge 1$, $\omega > 0$

$$\|e^{tA}\| \le Me^{-\omega t}, \quad t \ge 0,$$

then the inverse operator A^{-1} is bounded, so that A^{-1} generates the C_0-semigroup $(e^{tA^{-1}})_{t \ge 0}$ given by

$$e^{tA^{-1}} = \sum_{m=0}^{\infty} \frac{t^m A^{-m}}{m!}, \quad t \ge 0.$$

It can also be shown that in this case (see [8, 9]) the semigroup $(e^{tA^{-1}})_{t \ge 0}$ has the following integral representation:

$$e^{tA^{-1}} x = x - \sqrt{t} \int_0^{\infty} \frac{J_1(2\sqrt{ts})}{\sqrt{s}} e^{sA} x \, ds, \quad t > 0, \ x \in X \tag{16}$$

(compare with (5), (6)), and, in particular, we have the estimate

$$\|e^{tA^{-1}}\| \le 1 + M\sqrt{t} \int_0^{\infty} \frac{J_1(2\sqrt{ts})}{\sqrt{s}} e^{-\omega s} \, ds \le ct^{1/4}, \quad t \ge 1. \tag{17}$$

Consider now $A \in \mathcal{G}_{exp}(L_p(\Omega, d\mu))$, $1 \le p < \infty$, where (Ω, μ) is a σ-finite measure space. Suppose that operator $A \in \mathcal{G}_{exp}(L_p)$, $p \in [1, \infty)$ is such that

$$\|e^{tA}\|_{L_p} \le Me^{-\omega t}, \quad t \ge 0,$$

for some constants $M \ge 1$ and $\omega > 0$, which do not depend on p. Then from (17) and the Riesz-Thorin interpolation theorem, we obtain the estimates

$$\|e^{tA^{-1}}\|_{L_p} \le c_1 \|e^{tA^{-1}}\|_{L_2}^{2-2/p} t^{|1/(2p)-1/4|}, \quad t \ge 1, \ p \in [1, \infty). \tag{18}$$

Thus, if the semigroup $(e^{tA})_{t\geq 0}$ is uniformly bounded on L_2, then from (18) we have

$$\|e^{tA^{-1}}\|_{L_p} \leq c_2 t^{|1/(2p)-1/4|}, \quad t \geq 1, \; p \in [1, \infty), \tag{19}$$

and the above theorem shows the sharpness of (19) for $L_p(\Omega, \mu)$, $p \in [1,2) \cup (2, \infty)$.

The problem whether the inverse of the generator of a uniformly bounded C_0-semigroup is again a generator of a uniformly bounded C_0-semigroup was posed by deLaubenfels in [10]. If A is an injective linear operator on X with dense range generating a uniformly bounded analytic C_0-semigroup, then it is well known that A^{-1} also generates such a semigroup [10].

On the other hand, it was shown in [8, 9, 11, 12] that there exists a Banach space X and an injective linear operator on X with dense range generating a uniformly bounded C_0-semigroup whose inverse does not generate a C_0-semigroup. In [9] this was proved for $X = l_p$, $p \in (1, 2) \cup (2, \infty)$.

If $X = H$ is an infinite-dimensional Hilbert space then the question whether the implication

$$A \in \mathcal{G}(H), \ \ker A = \{0\} \Rightarrow A^{-1} \in \mathcal{G}(H)$$

holds is still open.

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