Quasi-equational bases for graphs of semigroups, monoids and groups

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Abstract The graph of an algebra $\mathbf{A}$ is the relational structure $G(\mathbf{A})$ in which the relations are the graphs of the basic operations of $\mathbf{A}$. Let denote by $G(\mathcal{C})$ the class of all graphs of algebras from a class $\mathcal{C}$. We prove that if $\mathcal{C}$ is a class of semigroups possessing a nontrivial member with a neutral element, then $G(\mathcal{C})$ does not have finite quasi-equational basis. We deduce that, for a class $\mathcal{C}$ of monoids or groups with a nontrivial member, $G(\mathcal{C})$ also does not have finite quasi-equational basis.

Keywords Graphs of semigroups · Finite axiomatizability · Finite quasi-equational bases · Quasivarieties of relational structures

1 Introduction

The *graph* of an algebra $\mathbf{A} = (A, O)$ is the relational structure

$$G(\mathbf{A}) = (A, \{R_o \mid o \in O\}),$$

where each R_o is the graph of an operation o . This means that if o is an n -ary operation, then R_o is the $(n + 1)$ -ary relation consisting of those tuples $(a_1, \dots, a_{n+1})$ which satisfy $o(a_0, \dots, a_n) = a_{n+1}$. Most of the time we will work with groupoids

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(semigroups in fact) and, without any risk of ambiguity, we will omit the subscript o . For a class $\mathcal{C}$ of algebras by $\mathbf{G}(\mathcal{C})$ we denote the class of all graphs of algebras from $\mathcal{C}$.

We are mainly interested in quasi-equational theory of graphs of algebras. Following [8, 14] we call a sentence *quasi-identity* if it is of the form

$$(\forall \bar{x}) [\varphi_1(\bar{x}) \wedge \cdots \wedge \varphi_n(\bar{x}) \rightarrow \varphi(\bar{x})],$$

where $n \in \mathbb{N}$ and $\varphi_1, \dots, \varphi_n, \varphi$ are atomic formulas. We allow n to be zero, and in such case we call the sentence an *identity*. By a *quasi-equational theory* of a class $\mathcal{K}$ we mean the set of quasi-identities true in $\mathcal{K}$. A *quasivariety* is a class defined by quasi-identities or, equivalently, a class which is closed under taking substructures, direct products and ultraproducts (we tacitly assume that all considered classes are closed under taking isomorphic images). The smallest quasivariety containing a class $\mathcal{K}$, i.e. the class defined by the quasi-equational theory of $\mathcal{K}$, is denoted by $\mathbf{Q}(\mathcal{K})$. Any defining set of quasi-identities for $\mathbf{Q}(\mathcal{K})$ is called a *quasi-equational basis* of $\mathcal{K}$. By compactness theorem, $\mathbf{Q}(\mathcal{K})$ is finitely axiomatizable iff $\mathcal{K}$ has a finite quasi-equational basis.

In this paper we wish to attract the reader's attention to the following, elementary but curious, fact.

Theorem 1 *Let $\mathcal{C}$ be a class of semigroups possessing a nontrivial member with a neutral element. Then $\mathbf{G}(\mathcal{C})$ does not have a finite quasi-equational basis.*

Note that the existence of a nontrivial semigroup in $\mathcal{C}$ would not be a sufficient assumption in the above theorem. For example, if $\mathcal{Z}$ is the class of semigroups satisfying $(\forall x, y, x', y') [xy \approx x'y']$, then $\mathbf{QG}(\mathcal{Z})$ is axiomatized by

$$\begin{aligned} &(\forall x, y, z, z') [R(x, y, z) \wedge R(x, y, z') \rightarrow z \approx z'], \\ &(\forall x, y, x', y', z) [R(x, y, z) \rightarrow R(x', y', z)]. \end{aligned}$$

In the most interesting corollary among those we derive, we assert that, for a class $\mathcal{C}$ of monoids or groups possessing a nontrivial member, $\mathbf{G}(\mathcal{C})$ also lacks a finite quasi-equational basis (Corollary 3).

We present results from the literature that are related to our work.

O.M. Gornostaev proved that $\mathbf{QG}(\mathbb{Z}_2, +)$ and $\mathbf{QG}(\mathbb{Z}_2, \vee)$ are not finitely axiomatizable [10], see also [8, Sect. 6.2]. This sharply contrasts with the facts that all two element algebras have finite equational [2, 13] and quasi-equational [7, 16, 17] bases. Our motivation was to check whether Gornostaev's result is just a curious exception or there is a deeper reason for it.

The existence of definable graphs of semigroups may prevent the existence of finite quasi-equational basis for algebras. D. Casperson and J. Hyndman proved recently that if $\mathbf{A}$ is a finite unary algebra possessing a definable graph of a nontrivial group multiplication as a primitive positive formula, then $\mathbf{Q}(\mathbf{A})$ is not finitely axiomatizable [6].

Surprisingly, almost all finite semigroups generate non finitely axiomatizable quasivarieties [11] in the following sense: If $r(n)$ is the fraction of all semigroups on

the set $\{0, 1, \dots, n-1\}$ which generate finitely axiomatizable quasivarieties, then $\lim_{n \rightarrow \infty} r(n) = 0$. It stays in contrast with the situation for general algebras as almost all finite algebras, in a given finite language, have finite quasi-equational bases. This result was obtained in [11] by M. Jackson and M. Volkov by showing that all finite proper 3-nilpotent semigroups do not have finite quasi-equational bases and applying the result due to D.J. Kleitman, B.R. Rothschild and J.H. Spencer [12] that almost all finite semigroups are proper 3-nilpotent. We refer to [11] for more details. The reader may find there also a discussion about older results concerning finite axiomatizability of quasivarieties of semigroups.

Now we switch our attention to relational structures. There are only five finitely axiomatizable quasivarieties generated by a finite family of finite loopless graphs (relational structures with one binary relation which is symmetric and irreflexive): $Q(\emptyset)$, $Q(\bullet)$, $Q(\bullet \bullet)$, $Q(\bullet \bullet \bullet)$ and $Q(\bullet \bullet \bullet \bullet)$. As indicated by X. Caicedo [5, Corollary 3], this fact easily follows from J. Nešetřil and A. Pultr theorem [15, Theorem 3.2]: There are exactly five pseudoquasivarieties (classes of finite structures that are closed under taking substructures and finite direct products) finitely generated by loopless graphs that may be defined by a finite family of finite forbidden subgraphs. Indeed, by Characterization from the next section, a quasivariety generated by graphs is finitely axiomatizable iff it may be defined by a finite family of finite forbidden subgraphs, which in turn is equivalent to the fact that finite members of it form a pseudoquasivariety defined by a finite family of finite forbidden subgraphs.

Other examples of non finitely axiomatizable quasivarieties of relational structures come from studying constraint satisfaction problem. For a finite relational structure $\mathbf{M}$ in a finite language let $A(\mathbf{M})$ be the class of relational structures that admit a homomorphism into $\mathbf{M}$ (i.e. the *antivariety* generated by $\mathbf{M}$, see [8]). The class of finite members of $A(\mathbf{M})$ is traditionally denoted by $CSP(\mathbf{M})$. The class $A(\mathbf{M}) \cup \{\mathbf{Loop}\}$, where $\mathbf{Loop}$ is the one element relational structure in the language of $\mathbf{M}$ with all relations full, is a (finitely generated, see [8, Theorem 3.1.16]) quasivariety. Moreover, the finite axiomatizability of $A(\mathbf{M})$ is equivalent to the finite axiomatizability of $A(\mathbf{M}) \cup \{\mathbf{Loop}\}$. A relational structure $\mathbf{M}$ has *finite duality* if there is a finite number of finite relational structures $\mathbf{O}_1, \dots, \mathbf{O}_n$, called *obstructions*, such that each finite relational structure $\mathbf{N}$ admits a homomorphism into $\mathbf{M}$ iff $\mathbf{O}_1, \dots, \mathbf{O}_n$ do not admit a homomorphism into $\mathbf{N}$. We will sketch the proof that $\mathbf{M}$ has finite duality if and only if $A(\mathbf{M})$ is finitely axiomatizable. Most relational structures do not have finite duality [4], and hence we may infer that most quasivarieties of the form $A(\mathbf{M}) \cup \{\mathbf{Loop}\}$ are not finitely axiomatizable.

So assume that $A(\mathbf{M})$ is finitely axiomatizable. Because $A(\mathbf{M})$ is universal, by Characterization in the next section, $A(\mathbf{M})$ is defined by a finite family $\mathcal{F}$ of finite forbidden substructures. The family $\mathcal{F}$ may be also used as a set of obstructions witnessing finite duality for $\mathbf{M}$. Indeed: if $\mathbf{N} \notin A(\mathbf{M})$, then there is a homomorphism, an embedding, of some $\mathbf{O} \in \mathcal{F}$ into $\mathbf{N}$; if some $\mathbf{O}$ from $\mathcal{F}$ admits a homomorphism into $\mathbf{N}$, then $\mathbf{N} \notin A(\mathbf{M})$, for otherwise $\mathbf{O} \in A(\mathbf{M})$, which would give us a contradiction. For the converse assume that $\mathbf{M}$ has finite duality witnessed by obstructions $\mathbf{O}_1, \dots, \mathbf{O}_n$. Observe that there is a universal sentence φ such that $\mathbf{N}$ satisfies it iff none of $\mathbf{O}_1, \dots, \mathbf{O}_n$ admits a homomorphism into $\mathbf{N}$. Because $A(\mathbf{M})$ is universal, $\mathbf{N}$ belongs to $A(\mathbf{M})$ iff all finite substructures of $\mathbf{N}$ belong to $CSP(\mathbf{M})$ iff all finite

substructures of $\mathbf{N}$ satisfy φ iff $\mathbf{N}$ satisfies φ . It is worth mentioning that, as proved by A. Atserias [1] and B. Rossman [18], in fact having finite duality by $\mathbf{M}$ is equivalent to the finite axiomatizability of $\text{CSP}(\mathbf{M})$ relative to the class of all finite relational structures in the language of $\mathbf{M}$. But this is a sophisticated result, contrary to the equivalence described above.

Recalled examples together with ours may suggest that quasivarieties of relational structures “have a tendency” to be a non finitely axiomatizable. We do not know to what extent this supposition is true.

2 Proof

The proof is based on the following observation which goes back to R.L. Vaught [19].

Observation *Assume that $\mathcal{K}$ is a class of relational structures axiomatized by a finite set Φ of universal sentences. Let n be the maximum number of variables occurring in the sentences from Φ . Then for each relational structure $\mathbf{M}$*

$\mathbf{M}$ belongs to $\mathcal{K}$ if all its at most n -element substructures do. $(\mathcal{Q}^n)$

The converse is true in the following sense: Let $\mathcal{K}$ be a class of relational structures in a finite language. If there exists a natural number n such that $(\mathcal{Q}^n)$ holds for all relational structures, then $\mathcal{K}$ is finitely axiomatizable. Thus we obtain the following characterization.

Characterization *A universal class in a finite relational language is finitely axiomatizable if and only if it may be defined by a finite family of finite forbidden substructures.*

Let $\mathcal{C}$ be a class of semigroups possessing a nontrivial member $\mathbf{S}$ with a neutral element. By Observation, in order to prove that $\text{QG}(\mathcal{C})$ is not finitely axiomatizable it is enough to find for each positive integer n a relational structure $\mathbf{M}$ such that

- $\mathbf{M}$ does not belong to the quasivariety generated by all graphs of semigroups;
- all at most n -element substructures of $\mathbf{M}$ are substructures of powers of $\mathbf{G}(\mathbf{S})$.

The first condition yields $\mathbf{M} \notin \text{QG}(\mathcal{C})$, while the second condition guarantees that all at most n -element substructures of $\mathbf{M}$ belong to $\text{QG}(\mathcal{C})$. Hence $\mathbf{M}$ does not fulfill the condition $(\mathcal{Q}^n)$ for $\mathcal{K} = \text{QG}(\mathcal{C})$.

We start constructing $\mathbf{M}$ by defining its carrier set.

$$M = \{a_0, a_1, a'_0, a'_1, b, c_0, \dots, c_n, d_0, \dots, d_n, d'_0, \dots, d'_n, e\}.$$

For $k \leq n$ let

$$M_k = M - \{c_k\},$$

and $\varepsilon_k: M_k \rightarrow \mathbb{Z}_2^{n+6}$ be the 1–1 mapping given in Table 1.

Table 1 The mapping ε_k .

Elements of $\mathbb{Z}_2^{n+6}$ are represented as words over $\mathbb{Z}_2$. For the sake of clarity we divided these words into 3 segments of length 4, $n+1$ and 1 respectively. In the second segment $(k-1)$ th, k th and $(k+1)$ th digits, counting from 0, are placed between dots

Elements of M_k		Elements of $\mathbb{Z}_2^{n+6}$		
a_0		1100	000...000...000	0
a_1		0011	000...000...000	0
a'_0	$\mapsto$	1010	000...000...000	0
a'_1		0101	000...000...000	0
b	$\mapsto$	1111	000...000...000	0
c_0		0000	100...000...000	0
c_1		0000	010...000...000	0
...				
c_{k-1}	$\mapsto$	0000	000...100...000	0
c_{k+1}		0000	000...001...000	0
...				
c_n		0000	000...000...001	0
d_0		0011	100...000...000	0
d_1		0011	110...000...000	0
...				
d_{k-1}	$\mapsto$	0011	111...100...000	0
d_k		0011	111...110...000	1
d_{k+1}		0011	111...111...000	1
...				
d_n		0011	111...111...111	1
d'_0		0101	100...000...000	0
d'_1		0101	110...000...000	0
...				
d'_{k-1}	$\mapsto$	0101	111...100...000	0
d'_k		0101	111...110...000	0
d'_{k+1}		0101	111...111...000	0
...				
d'_n		0101	111...111...111	0
e	$\mapsto$	1111	111...111...111	1

Further we need to consider three cases:

- $(\mathbb{Z}_l, +)$ for some $l > 2$, or $(\mathbb{N}, +)$ is isomorphic to a subsemigroup of $\mathbf{S}$. Then define $\mathbf{U}^> = (\mathbb{Z}_l^{n+6}, +)$ or $\mathbf{U}^> = (\mathbb{N}^{n+6}, +)$ respectively;
- $(\mathbb{Z}_2, +)$ is isomorphic to a subsemigroup of $\mathbf{S}$. Then define $\mathbf{U}^2 = (\mathbb{Z}_2^{n+6}, +)$;
- $(\mathbb{Z}_2, \vee)$ is isomorphic to a subsemigroup of $\mathbf{S}$. Then define $\mathbf{U}^\vee = (\mathbb{Z}_2^{n+6}, \vee)$.

For $\diamond \in \{>, 2, \vee\}$, $k \leq n$, define

$$\begin{aligned}
 R_k^\diamond &= \{(x, y, z) \in M_k \mid \mathbf{U}^\diamond \models \varepsilon_k(x) \cdot \varepsilon_k(y) = \varepsilon_k(z)\} \\
 &= \{(x, y, z) \in M_k \mid \mathbf{G}(\mathbf{U}^\diamond) \models R(\varepsilon_k(x), \varepsilon_k(y), \varepsilon_k(z))\}
 \end{aligned}$$

and

$$\mathbf{M}_k^\diamond = (M_k, R_k^\diamond).$$

Then ε_k is an isomorphism of $\mathbf{M}_k^\diamond$ onto a substructure of $\mathbf{G}(\mathbf{U}^\diamond)$. The case $\diamond = >$ includes two possibilities $\mathbf{U}^> = (\mathbb{Z}_l^{n+6}, +)$ and $\mathbf{U}^> = (\mathbb{N}^{n+6}, +)$ but in either situation the relations $R_k^>$ are the same. Finally, we take

$$R^\diamond = \bigcup_{k=0}^n R_k^\diamond$$

and

$$\mathbf{M}^\diamond = (M, R^\diamond).$$

Claim $\mathbf{M}^\diamond$ does not belong to the quasivariety generated by all graphs of semigroups.

Proof In all semigroups the quasi-identity

$$(\forall \bar{x}, \bar{x}', \bar{z}) [x_0 x_1 \approx x'_0 x'_1 \rightarrow x_0 (\cdots (x_1 z_0) z_1 \cdots z_n) \approx x'_0 (\cdots (x'_1 z_0) z_1 \cdots z_n)] \quad (\mathfrak{O})$$

is valid. From this we infer that the quasi-identity

$$\begin{aligned} & (\forall \bar{x}, \bar{x}', y, \bar{z}, \bar{u}, \bar{u}', v) [R(x_0, x_1, y) \wedge R(x'_0, x'_1, y) \\ & \wedge R(x_1, z_0, u_0) \wedge R(u_0, z_1, u_1) \wedge \cdots \wedge R(u_{n-1}, z_n, u_n) \wedge R(x_0, u_n, v) \\ & \wedge R(x'_1, z_0, u'_0) \wedge R(u'_0, z_1, u'_1) \wedge \cdots \wedge R(u'_{n-1}, z_n, u'_n) \rightarrow R(x'_0, u'_n, v)] \quad (\mathfrak{Z}) \end{aligned}$$

holds in each graph (W, R) of a semigroup $\mathbf{W}$. Indeed, if the premises of $(\mathfrak{Z})$ are satisfied under a valuation $\rho: \text{Variable} \rightarrow W$, then in $\mathbf{W}$

$$\begin{aligned} \rho(y) &= \rho(x_0)\rho(x_1) = \rho(x'_0)\rho(x'_1), \\ \rho(u_n) &= (\cdots (\rho(x_1)\rho(z_0))\rho(z_1) \cdots)\rho(z_n), \\ \rho(u'_n) &= (\cdots (\rho(x'_1)\rho(z_0))\rho(z_1) \cdots)\rho(z_n) \end{aligned}$$

and by $(\mathfrak{O})$

$$\begin{aligned} \rho(v) &= \rho(x_0)\rho(u_n) = \rho(x_0)((\cdots (\rho(x_1)\rho(z_0))\rho(z_1) \cdots)\rho(z_n)) \\ &= \rho(x'_0)((\cdots (\rho(x'_1)\rho(z_0))\rho(z_1) \cdots)\rho(z_n)) = \rho(x'_0)\rho(u'_n). \end{aligned}$$

Now consider the valuation $\delta: \text{Variable} \rightarrow M$;

$$\begin{aligned} x_0 &\mapsto a_0, & x_1 &\mapsto a_1, & x'_0 &\mapsto a'_0, & x'_1 &\mapsto a'_1, \\ y &\mapsto b, & z_k &\mapsto c_k, & u_k &\mapsto d_k, & u'_k &\mapsto d'_k, & v &\mapsto e \end{aligned}$$

for $k \leq n$. The straightforward verification shows that

$$\begin{aligned} (a_0, a_1, b), (a'_0, a'_1, b) &\in R_0^\diamond, \\ (a_1, c_0, d_0) &\in R_1^\diamond, & (d_0, c_1, d_1), \dots, (d_{n-1}, c_n, d_n), (a_0, d_n, e) &\in R_0^\diamond, \\ (a'_1, c_0, d'_0) &\in R_1^\diamond, & (d'_0, c_1, d'_1), \dots, (d'_{n-1}, c_n, d'_n) &\in R_0^\diamond. \end{aligned}$$

Thus, under the valuation δ , the premises in the quasi-identity $(\mathfrak{Z})$ are satisfied. However (a'_0, d'_n, e) does not belong to any $R_k^\diamond$, and hence $(\mathfrak{Z})$ fails in $\mathbf{M}^\diamond$. $\square$

Lemma 2 $R^\diamond \cap M_k^3 = R_k^\diamond$.

Proof Let $x, y, z \in M_k$ and $\varepsilon_l(x) \cdot \varepsilon_l(y) = \varepsilon_l(z)$, i.e. $(x, y, z) \in R_l^\diamond$, for some $l \leq n$. Our aim is to prove that $(x, y, z) \in R_k^\diamond$.

Suppose first that at least two of x, y, z are equal. Then $\diamond = \vee$. Indeed, if $x = y$, then 2 would have to occur in $\varepsilon_l(z)$ when $\diamond = >$, or $\varepsilon_l(z)$ would be the tuple of zeros when $\diamond = 2$. If $x = z$ or $y = z$, one of $\varepsilon_l(x), \varepsilon_l(y)$ would be the tuple of zeros in both cases. Thus, say, $\varepsilon_l(x) \leq \varepsilon_l(y) = \varepsilon_l(z)$ in the order of the semilattice $\mathbf{U}^\vee$. For any u , it is the case that $\varepsilon_k(u)$ and $\varepsilon_l(u)$ are the same except possibly for the last digit. This is why the uncertainty may occur only when $\varepsilon_k(x)$ ends with 1. This means $x = e$, which forces $y = z = e$, or $x = d_i$ for some $i \geq k$. In the latter case, by the definition of ε_k , we must have $y = z \in \{d_i, d_{i+1}, \dots, d_n, e\}$, and hence $\varepsilon_k(y)$ ends with 1, too. Therefore, in both cases, $(x, y, z) \in R_k^\vee$.

Now suppose that x, y, z are distinct. We may assume that $\varepsilon_l|_{\{x, y, z\}} \neq \varepsilon_k|_{\{x, y, z\}}$, which implies that $d_i \in \{x, y, z\}$ for some $i < n$. Then either $\{x, y, z\} = \{a_1, c_0, d_0\}$ and $k, l > 0$, or $\{x, y, z\} = \{d_j, c_{j+1}, d_{j+1}\}$, for some j , and $k, l \neq j + 1$. In both cases we again conclude that $\varepsilon_k(x) \cdot \varepsilon_k(y) = \varepsilon_k(z)$. $\square$

Claim Each substructure $\mathbf{N}$ of $\mathbf{M}^\diamond$ having at most n elements is isomorphic to a substructure of a power of $\mathbf{G}(\mathbf{S})$.

Proof By Lemma 2, $\mathbf{N}$ is a substructure of some $\mathbf{M}_k^\diamond$. We defined $\mathbf{M}_k^\diamond$ in such a way that it is isomorphic to a substructure of $\mathbf{G}(\mathbf{U}^\diamond) \leq \mathbf{G}(\mathbf{S}^{n+6}) = \mathbf{G}(\mathbf{S})^{n+6}$. $\square$

3 Related results

We start by demonstrating how small modifications in the proof of Theorem 1 give the following result.

Corollary 3 Let $\mathcal{C}$ be a class of monoids or groups possessing a nontrivial member. Then $\mathbf{QG}(\mathcal{C})$ is not finitely axiomatizable.

Proof In the monoid case let $\mathbf{U}_m^\diamond$ be the monoid with the semigroup reduct $\mathbf{U}^\diamond$. As previously, we define $\mathbf{M}_{m,k}^\diamond = (M_k, R_{\cdot,k}^\diamond, R_{1,k})$, $k \leq n$, in such a way that ε_k is an isomorphism of $\mathbf{M}_{m,k}^\diamond$ onto a substructure of $\mathbf{G}(\mathbf{U}_m^\diamond)$. Note that $R_{1,k}^\diamond = \emptyset$, and thus the reasoning from the proof of Theorem 1 remains correct here.

In the group case a slightly larger modification is needed. We define $\mathbf{U}_g^>$ to be the group $(\mathbb{Z}_l^{n+6}, +, -, 0)$, $l > 2$, or $(\mathbb{Z}^{n+6}, +, -, 0)$, and $\mathbf{U}_g^2$ to be the group $(\mathbb{Z}_2^{n+6}, +, -, 0)$. Then, in analogically constructed $\mathbf{M}_g^\diamond = (M, R_\cdot, R_{(-1)}^\diamond, R_1)$, we have $R_1^\diamond = \emptyset$, $R_{(-1)}^> = \emptyset$ and $R_{(-1)}^2 = \{(a, a) \mid a \in M\}$. $\square$

Now we consider fragments of quasi-equational theories.

Recall that by an identity we mean a sentence of the form $(\forall \bar{x}) \varphi(\bar{x})$, where φ is an atomic formula. A *variety* is a class defined by identities. The following simple statement shows that the problem of finite axiomatizability of varieties of relational structures is not interesting.

Proposition 4 *Let $\mathcal{K}$ be a variety in a finite relational language. Then $\mathcal{K}$ is finitely axiomatizable.*

Proof It follows from the fact that there is, up to renaming of variables, only finite number of identities in a finite relational language. $\square$

Here is a more peculiar result. For a class $\mathcal{K}$ let $\mathbf{Q}_{\approx}(\mathcal{K})$ be the class defined by all quasi-identities true in $\mathcal{K}$ in which the equality symbol $\approx$ does not occur. Quasi-identities without the equality symbol form a fragment of first order logic particularly important in abstract algebraic logic [3].

Corollary 5 *Let $\mathcal{C}$ be a class of monoids or groups possessing a nontrivial member, or of semigroups possessing a nontrivial member with a neutral element. Then $\mathbf{Q}_{\approx}\mathbf{G}(\mathcal{C})$ is not finitely axiomatizable.*

Proof The quasi-identity $(\mathfrak{F})$ was chosen in such a way that it also serves here. The proof of Theorem 1 may be used without any change. $\square$

After obtaining Theorem 1 the author learned that Gornostaev proved a similar fact. We formulate it as follows. By a *reduction* of a relational structure (M, R) we mean any relational structure (M, R') , where $R' \subseteq R$. Let $\mathbf{R}(\mathcal{K})$ stand for the class of all reductions of relational structures from a class $\mathcal{K}$. We will prove in Appendix that $\mathbf{RQ}(\mathcal{K})$ is the quasivariety defined by the quasi-identities true in $\mathcal{K}$ whose conclusions are of the form $x \approx y$.

Theorem 6 (Gornostaev [9]) *Let $\mathcal{C}$ be a class of semigroups possessing a nontrivial member with a neutral element. Then $\mathbf{RQG}(\mathcal{C})$ is not finitely axiomatizable.*

We will provide a sketch of the proof for the publication [9] is scarcely available.

Proof The main idea is again the same, but details are simpler. We cannot use the quasi-identity $(\mathfrak{F})$ for in the conclusion of it the symbol R was used. But we may use the following one

$$\begin{aligned} &(\forall \bar{x}, \bar{x}', y, \bar{z}, \bar{u}, \bar{u}', v, v')[R(x_0, x_1, y) \wedge R(x'_0, x'_1, y) \\ &\quad \wedge R(x_1, z_0, u_0) \wedge R(u_0, z_1, u_1) \wedge \cdots \wedge R(u_{n-1}, z_n, u_n) \wedge R(x_0, u_n, y) \\ &\quad \wedge R(x'_1, z_0, u'_0) \wedge R(u'_0, z_1, u'_1) \wedge \cdots \wedge R(u'_{n-1}, z_n, u'_n) \wedge R(x'_0, u'_n, v') \\ &\quad \rightarrow v \approx v']. \end{aligned} \quad (\mathfrak{F}')$$

Instead of three $\mathbf{M}^\diamond$ it is enough to define one relational structure $\mathbf{M}' = (M', R')$, where $M' = M \cup \{e'\}$ and

$$\begin{aligned} R' = & \{(a_0, a_1, b), (a'_0, a'_1, b), \\ & (a_1, c_0, d_0), (d_0, c_1, d_1), \dots (d_{n-1}, c_n, d_n), \\ & (a'_1, c_0, d'_0), (d'_0, c_1, d'_1), \dots (d'_{n-1}, c_n, d'_n), \\ & (a_0, d_n, e), (a'_0, d_n, e')\}. \end{aligned}$$

Then $\mathbf{M}' = (M', R')$ does not model $(\mathfrak{F}')$. For $k < n$ let $M'_k = M' - \{c_k\}$, $\mathbf{M}'_k$ be the substructure of $\mathbf{M}'$ with the carrier set M'_k , and ε'_k be the extension of ε_k mapping e' onto the tuple of $n + 5$ ones and one zero at the end. Then $\varepsilon'_k: \mathbf{M}'_k \rightarrow \mathbf{G}(\mathbf{U}^\diamond)$ is a 1–1 homomorphism. The original proof is similar. $\square$

It was not indicated by Gornostaev, but this result clearly holds for classes of monoids or groups containing a nontrivial member. Just, in the proof above, define the relations corresponding to graphs of operations 1 and $()^{-1}$ to be the empty set.

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Appendix

Here we take a closer look at the reduction class operator. First we extend the definition to models in an arbitrary language. Let $\mathbf{M} = (M, R, O)$ be a model, where $R = \{R_\gamma \mid \gamma \in \Gamma\}$ is an indexed set of relations, and O is a set of operations. A *reduction* of $\mathbf{M}$ is any model of the form (M, R', O) , where $R' = \{R'_\gamma \mid \gamma \in \Gamma\}$ and $R'_\gamma \subseteq R_\gamma$ for all $\gamma \in \Gamma$. As previously, $\mathbf{R}(\mathcal{K})$ stands for the class of all reductions of models from $\mathcal{K}$.

Proposition 7 *For an arbitrary quasivariety $\mathcal{Q}$ the class $\mathbf{R}(\mathcal{Q})$ is the quasivariety defined by the quasi-identities true in $\mathcal{Q}$ whose conclusions are equalities of terms.*

Proof We divide the set Λ of all quasi-identities satisfied in $\mathcal{Q}$ into two sets: Λ_R of those with conclusions containing a relational symbol different from $\approx$, and Λ_O of those with conclusions being equalities of terms. That Λ_O holds in $\mathbf{R}(\mathcal{Q})$ is fairly straightforward.

Take an arbitrary model $M = (M, R', O)$ satisfying Λ_O . We define relations R_γ , $\gamma \in \Gamma$, on the set M in such a way that each R_γ contains R'_γ and $(M, R, O) \in \mathcal{Q}$.

Precisely, let R_γ be the set of all tuples $\bar{a}$ of elements in M for which there exist a quasi-identity

$$(\forall \bar{y}) [\psi_1(\bar{y}) \wedge \cdots \wedge \psi_m(\bar{y}) \rightarrow \psi(\bar{y})] \in \Lambda_R$$

and a tuple $\bar{b}$ such that $(M, R', O) \models \psi_i(\bar{b})$ for $i \leq m$, and $\psi(\bar{b}) = R_\gamma(\bar{a})$. The fact that $(\forall \bar{y}) [R_\gamma(\bar{y}) \rightarrow R_\gamma(\bar{y})]$ belongs to Λ_R yields $R'_\gamma \subseteq R_\gamma$. Now take

$$q = (\forall \bar{x}) [\varphi_1(\bar{x}) \wedge \cdots \wedge \varphi_n(\bar{x}) \rightarrow \varphi(\bar{x})] \in \Lambda$$

and a tuple $\bar{a}$ such that $(M, R, O) \models \varphi_i(\bar{a})$ for all $i \leq n$. Our aim is to verify that (M, R, O) satisfies $\varphi(\bar{a})$. Assume that the formulas $\varphi_1, \dots, \varphi_k$ contain relational symbols different from $\approx$, and $\varphi_{k+1}, \dots, \varphi_n$ contain $\approx$. Then for every $i \leq k$ there are a quasi-identity

$$q_i = (\forall \bar{y}_i) [\psi_{i,1}(\bar{y}_i) \wedge \cdots \wedge \psi_{i,m_i}(\bar{y}_i) \rightarrow \psi_i(\bar{y}_i)] \in \Lambda_R$$

and a tuple $\bar{b}_i$ such that $\psi_i(\bar{b}_i) = \varphi_i(\bar{a})$ and $(M, R', O) \models \psi_{i,j}(\bar{b}_i)$ for $j \leq m_i$. At this stage a bit care about variables is needed. First, we may assume that all valuations $\bar{x} \mapsto \bar{a}$, $\bar{y}_i \mapsto \bar{b}_i$, $i \leq k$, are 1–1 (identify variables if needed). Then we may rename variables in $\bar{y}_i$ in such a way that $\psi_i(\bar{y}_i) = \varphi_i(\bar{x})$. Finally, assume that if a variable from $\bar{y}_i$ does not occur in $\varphi_i(\bar{x})$, then it does not occur in $q, q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_k$. Let q' be the quasi-identity

$$(\forall \bar{x}, \bar{y}_1, \dots, \bar{y}_k) [\psi_{1,1}(\bar{y}_1) \wedge \cdots \wedge \psi_{k,m_k}(\bar{y}_k) \wedge \varphi_{k+1}(\bar{x}) \wedge \cdots \wedge \varphi_n(\bar{x}) \rightarrow \varphi(\bar{x})].$$

Then $q' \in \Lambda$ and

$$(M, R', O) \models \psi_{1,1}(\bar{b}_1) \wedge \cdots \wedge \psi_{k,m_k}(\bar{b}_k) \wedge \varphi_{k+1}(\bar{a}) \wedge \cdots \wedge \varphi_n(\bar{a}).$$

If $q' \in \Lambda_R$, then the definition of the relations in R yields $(M, R, O) \models \varphi(\bar{a})$. If $q' \in \Lambda_O$, then $(M, R', O) \models q'$, and hence $(M, R', O) \models \varphi(\bar{a})$, which implies $(M, R, O) \models \varphi(\bar{a})$. We conclude that (M, R, O) satisfies q . $\square$

Corollary 8 *A class of models in a given language is definable by quasi-identities whose conclusions are equalities of terms if and only if it is closed under R and Q class operators.*

Note that there are quasi-identities without $\approx$ in the conclusion that may hold in $RQ(\mathcal{K})$. Take for instance the tautology $(\forall \bar{x}) [R(\bar{x}) \rightarrow R(\bar{x})]$.

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