Wild ramification and $K(\pi, 1)$ spaces

Piotr Achinger^{1,2}

Received: 28 February 2017 / Accepted: 4 May 2017 / Published online: 17 May 2017
© The Author(s) 2017. This article is an open access publication

Abstract We prove that every connected affine scheme of positive characteristic is a $K(\pi, 1)$ space for the étale topology. The main ingredient is the special case of the affine space $\mathbb{A}_k^n$ over a field k . This is dealt with by induction on n , using a key “Bertini-type” statement regarding the wild ramification of ℓ -adic local systems on affine spaces, which might be of independent interest. Its proof uses in an essential way recent advances in higher ramification theory due to T. Saito. We also give rigid analytic and mixed characteristic versions of the main result.

Mathematics Subject Classification Primary 14F35; Secondary 14F20 · 14R10

Contents

1	Introduction	454
1.1	Artin neighborhoods	455
1.2	The Bertini theorem for lcc sheaves	457
1.3	Higher ramification theory and proof of the Bertini theorem	458
1.4	Mixed characteristic and rigid analytic variants	459
1.5	The naive étale topology	459

✉ Piotr Achinger
achinger@ihes.fr; pachinger@impan.pl

¹ Instytut Matematyczny PAN, ul. Śniadeckich 8, 00-656 Warsaw, Poland

² Institut des Hautes Études Scientifiques, Le Bois-Marie 35, route de Chartres, 91440 Bures-sur-Yvette, France

1.6	Implications in étale homotopy	460
1.7	Examples and complements	460
1.8	Outline	461
2	Review of wild ramification	461
2.1	The Swan conductor	461
2.2	The Deligne–Laumon theorem	463
2.3	The characteristic cycle of a constructible sheaf	463
3	Proof of the Bertini theorem	466
3.1	The non-fierce case of the Bertini theorem	466
3.2	The general case of the Bertini theorem	469
4	Review of $K(\pi, 1)$ schemes	474
5	Affine $\mathbb{F}_p$ -schemes are $K(\pi, 1)$	476
5.1	The affine space is a $K(\pi, 1)$	476
5.2	Étale schemes over the affine space	477
5.3	Henselian pairs and Gabber’s theorem	478
5.4	The general case	479
6	Mixed characteristic and rigid analytic variants	480
6.1	Review of rigid geometry	480
6.2	The Gabber–Fujiwara theorem	481
6.3	Affinoid rigid spaces in characteristic p	481
6.4	Affinoid rigid spaces in mixed characteristic	482
6.5	Application to p -adic Milnor fibers	483
6.6	Étale affinoids over the polydisc	484
7	Examples and complements	485
7.1	Linear projections do not suffice	485
7.2	Complements of hyperplane arrangements	486
7.3	Fundamental groups of affine spaces	488
7.4	Pro- p completion and p -Sylow subgroups	488
7.5	Relation to $K(\pi, 1)$ pro- ℓ	490
7.6	Anabelian geometry speculations	490
	Appendix A: The rank one case of the Bertini theorem	492
	A.1: Ramification of Artin–Schreier–Witt sheaves	492
	A.2: Artin–Schreier–Witt sheaves on $\mathbb{A}_k^1$	495
	References	497

1 Introduction

The étale homotopy theory of schemes of positive characteristic is quite poorly understood. For example, already the étale fundamental group $\pi_1(\mathbb{A}_k^1)$ of the affine line over an algebraically closed field k of characteristic p is incredibly complicated [36]. One of our main results is the realization that the étale homotopy theory of characteristic p schemes is in a certain way controlled by the étale fundamental group.

The notion of a $K(\pi, 1)$ space for the étale topology plays a central role in this paper. In algebraic topology, a connected topological space X (with a basepoint x) satisfying suitable technical assumptions is called a $K(\pi, 1)$ space if its higher homotopy groups $\pi_q(X, x)$ ($q > 1$) are zero. The homotopy type of such a space is completely determined by its fundamental group $\pi_1(X, x)$,

and in particular the cohomology of any local system on X agrees with the group cohomology of the corresponding representation of $\pi_1(X, x)$. Similarly, we call a connected scheme X with a geometric point $\bar{x}$ a $K(\pi, 1)$ *scheme* if for every locally constant étale sheaf of finite abelian groups $\mathcal{F}$ on X , the natural maps

$$H^*(\pi_1(X, \bar{x}), \mathcal{F}_{\bar{x}}) \longrightarrow H^*(X, \mathcal{F}) \quad (1.1)$$

are isomorphisms (cf. Definition 4.1). If X is noetherian and geometrically unibranch, this is equivalent to the vanishing of the higher étale homotopy groups $\pi_q(X, \bar{x})$ for $q > 1$ (cf. Proposition 4.4).

One of our main results is the following.

Theorem 1.1 *Every connected affine $\mathbf{F}_p$ -scheme is a $K(\pi, 1)$ scheme.*

This of course stands in stark contrast with the characteristic zero case. One might interpret this by saying that the étale fundamental group of a connected affine $\mathbf{F}_p$ -scheme is so large that it ‘absorbs’ the higher homotopy groups. To go one step beyond the affine case, Theorem 1.1 implies that the étale homotopy type of a normal quasi-compact and separated $\mathbf{F}_p$ -scheme can be described as the homotopy colimit of a finite diagram of classifying spaces of profinite groups.

1.1 Artin neighborhoods

To explain both our initial motivation and the idea of proof of Theorem 1.1, we will start by discussing Artin’s construction of $K(\pi, 1)$ neighborhoods on smooth complex algebraic varieties.

The notion of a $K(\pi, 1)$ scheme first appeared in algebraic geometry in Artin’s proof of the comparison theorem between singular and étale cohomology of a smooth scheme X over $\mathbf{C}$, which states that

$$H^*(X, \mathcal{F}) \xrightarrow{\sim} H^*(X(\mathbf{C}), \mathcal{F}) \quad (1.2)$$

for every locally constant étale sheaf of finite abelian groups $\mathcal{F}$ on X [6, Exp. XI].

Recall that an *elementary fibration* is a morphism of schemes $\pi : X \longrightarrow S$ for which there exists a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{j} & \bar{X} & \xleftarrow{i} & Y \\ & \searrow \pi & \downarrow \bar{\pi} & \swarrow & \\ & & S & & \end{array}$$

where $j: X \rightarrow \bar{X}$ is an open immersion, $i: Y \rightarrow \bar{X}$ the complementary closed immersion, $\bar{\pi}$ is projective with geometrically connected fibers, smooth of dimension 1, and its restriction $\bar{\pi}|_Y: Y \rightarrow S$ is finite étale surjective. Thus the geometric fibers of π are smooth affine curves. Moreover, if S is of finite type over $\mathbf{C}$, then the associated morphism $\pi: X(\mathbf{C}) \rightarrow S(\mathbf{C})$ is a locally trivial fibration. Artin showed that a smooth variety X over an infinite field K can be covered by Zariski open subsets U for which there exists a chain of elementary fibrations

$$U = U_{\dim X} \rightarrow \cdots \rightarrow U_1 \rightarrow U_0 = \operatorname{Spec} K.$$

We call a scheme U admitting a chain as above an *Artin neighborhood*. Thus U is an ‘iterated fibration in affine curves,’ and it follows that if K has characteristic zero, an Artin neighborhood is a $K(\pi, 1)$ scheme. Moreover, if $K = \mathbf{C}$, then the associated analytic space $U(\mathbf{C})$ is a $K(\pi, 1)$ space, and $\pi_1(U(\mathbf{C}))$ is an iterated extension of free groups. Consequently, in the natural commutative square

$$\begin{array}{ccc} H^*(\pi_1(U, u), \mathcal{F}_u) & \longrightarrow & H^*(\pi_1(U(\mathbf{C}), u), \mathcal{F}_u) \\ \downarrow & & \downarrow \\ H^*(U, \mathcal{F}) & \longrightarrow & H^*(U(\mathbf{C}), \mathcal{F}) \end{array}$$

the vertical and top arrows are isomorphisms, which implies (1.2) for $X = U$ an Artin neighborhood. The general case of (1.2) follows then easily by cohomological descent.

Our initial goal in this project was to generalize Artin’s theorem by showing that a smooth scheme over an infinite field of positive characteristic admits a covering by $K(\pi, 1)$ open subschemes; we dared not hope that something as striking as Theorem 1.1 can be true. To this end, a good understanding of the problem with extending Artin’s characteristic zero proof put us on the right track.

Namely, the reason why the argument that an Artin neighborhood is a $K(\pi, 1)$ scheme works only in characteristic zero is related to wild ramification. Suppose that $\pi: X \rightarrow S$ is an elementary fibration, and we know that S is a $K(\pi, 1)$. To show that X has to be a $K(\pi, 1)$ as well, it is easily seen that it is enough to prove that for a locally constant constructible sheaf $\mathcal{F}$ on X , the following condition is satisfied:

$$\begin{aligned} &\text{The higher direct images } R^q \pi_* \mathcal{F} \text{ are locally constant,} \\ &\text{with formation commuting with base change for all } q \geq 0. \end{aligned} \tag{1.3}$$

This last assertion is true in characteristic zero e.g. by comparison with the complex case, where π is topologically a locally trivial fibration, but in positive characteristic this can fail due to phenomena related to the wild ramification of the restriction of $\mathcal{F}$ to the fibers of f at the points of Y , as the following basic example shows.

Example 1.2 Let $\text{pr}: \mathbf{A}_k^2 \rightarrow \mathbf{A}_k^1$ be the projection to the second coordinate, where k is algebraically closed of characteristic $p > 0$, $\mathcal{F}$ the rank one sheaf of $\mathbf{F}_\ell$ -vector spaces ($\ell \neq p$) on $\mathbf{A}_k^2$ associated to the Artin–Schreier covering

$$t^p - t = x_1 x_2$$

and a nontrivial character $\psi: \mathbf{F}_p \rightarrow \mathbf{F}_\ell^\times$. Then the restriction of $\mathcal{F}$ to $\pi^{-1}(0)$ is constant, while for $x_2 \neq 0$, $\mathcal{F}|_{\pi^{-1}(x_2)}$ is not constant. Consequently, $\pi_* \mathcal{F} = 0$ does not have the required compatibility with base change, and (1.3) does not hold.

1.2 The Bertini theorem for lcc sheaves

Our main technical result below states that one can make problems as in Example 1.2 go away by applying a non-linear automorphism of the affine space. Consequently, one can make the above inductive argument work for sheaves on $\mathbf{A}_k^{n+1}$ if one is allowed to choose the fibration $\pi: \mathbf{A}_k^{n+1} \rightarrow \mathbf{A}_k^n$ after being given the locally constant sheaf $\mathcal{F}$. For brevity, let us call a sheaf $\mathcal{F}$ *well-aligned* with respect to a map π if condition (1.3) holds.

Theorem 1.3 (Bertini theorem for lcc sheaves) *Let k be an infinite field of characteristic $p > 0$. Let $\ell \neq p$ be a prime and let $\mathcal{F}$ be a locally constant constructible sheaf of $\mathbf{F}_\ell$ -vector spaces on $\mathbf{A}_k^{n+1}$ ($n \geq 0$). Let $\pi: \mathbf{A}_k^{n+1} \rightarrow \mathbf{A}_k^n$ be the projection to the first n coordinates. Then there exists an automorphism φ of $\mathbf{A}_k^{n+1}$ such that $\varphi^* \mathcal{F}$ is well-aligned with respect to π (equivalently, $\mathcal{F}$ is well-aligned with respect to $\pi \circ \varphi^{-1}$).*

In fact, we prove that the assertion holds for a (not necessarily linear!) automorphism φ which is general in a suitable sense. This justifies the name ‘Bertini theorem.’ See Theorem 3.6 for a precise formulation.

We shall now explain how Theorem 1.3 implies Theorem 1.1. First, Theorem 1.3 enables us to prove by induction on n that the affine space $\mathbf{A}_k^n$ is a $K(\pi, 1)$, along the lines sketched in Sect. 1.1. This turns out to be the key case of Theorem 1.1. To deduce the general case, one first treats affine étale schemes over $\mathbf{A}_k^n$ as an intermediate step. To this end, one uses the following result.

Proposition (5.2) *Let U be an affine scheme of finite type over k admitting an étale map $g: U \rightarrow \mathbf{A}_k^n$. Then there exists a finite étale map $f: U \rightarrow \mathbf{A}_k^n$.*

The proof of this assertion is surprisingly easy, and is based on Nagata's proof of the Noether normalization lemma [32, I §1]. A similar result has been obtained by Kedlaya [27], and both were inspired by a trick used in [26] in the one-dimensional case. The following example illustrates the general idea.

Example 1.4 Let $U = \mathbb{A}_k^1 \setminus \{x_1, \dots, x_r\}$ and let $g: U \rightarrow \mathbb{A}_k^1$ be the inclusion. Then g is of course étale, but it is not finite because the points x_i have empty preimages. To remedy this, we can send these points off to infinity by adding to g functions with poles at the x_i . This will make the map g finite, but might destroy étaleness — unless we are in characteristic p , in which case the added functions can be taken to be p -th powers. Concretely, we might take

$$f(x) = x + \sum_{i=1}^r \frac{1}{(x - x_i)^p}.$$

One can apply a similar reasoning in a mixed characteristic situation and prove that an affinoid rigid space which is étale over a polydisc is also finite étale over a polydisc, cf. Proposition 6.10.

Combining Proposition 5.2 with the fact that $\mathbb{A}_k^n$ is a $K(\pi, 1)$, we see that if U is an affine scheme of finite type over k , admitting an étale map $U \rightarrow \mathbb{A}_k^n$, then U is a $K(\pi, 1)$. Finally, using limit arguments and Gabber's affine analog of the proper base change theorem [20], we deduce Theorem 1.1.

1.3 Higher ramification theory and proof of the Bertini theorem

Theorem 1.3 is where higher ramification theory enters the picture. A key ingredient in the proof is the Deligne–Laumon theorem (cf. Corollary 2.3), which yields a condition for the higher direct images $R^q \pi_* \mathcal{F}$ ($q \geq 0$) being locally constant in terms of the Swan conductors at infinity of the restriction of $\mathcal{F}$ to the fibers of π . The ‘baby case’ is when $\mathcal{F}$ is non-fierce at infinity (cf. Definition 3.1), in which case we can take φ to be a general linear automorphism (cf. Proposition 3.4). In the general case, we use the recent work of Takeshi Saito on the characteristic cycle associated to a locally constant $\mathbf{F}_\ell$ -sheaf [38, 39]. It turns out that we can take the automorphism φ to be quadratic.

As pointed out to us by Maxim Kontsevich, it makes sense to ask whether a variant of Theorem 1.3 holds for irregular connections on $\mathbb{A}_{\mathbb{C}}^{n+1}$. Our method of proof, employing the characteristic cycle, seems to suggest that the answer should be positive. We plan to address this question in a future paper.

In the course of our work on Theorem 1.3, we started by solving its rank one case first. In this situation, the calculations are very explicit, and we include them in an “Appendix”. One good feature of this proof is that, unlike our treatment of the general case, the arguments work over a finite field. It could be interesting to obtain a general proof of our Bertini theorem over finite fields.

1.4 Mixed characteristic and rigid analytic variants

If X is a connected affine $\mathbf{F}_p$ -scheme and $\mathcal{F}$ a locally constant constructible $\mathbf{F}_p$ -sheaf on X , then the maps (1.1) are isomorphisms. This can be seen easily using the Artin–Schreier sequence, cf. Example 4.5. Scholze [41, Theorem 4.9] observed that using perfectoid spaces, one can deduce that every mixed characteristic noetherian affinoid adic space is a $K(\pi, 1)$ space for p -adic coefficients. Using a similar argument and Theorem 1.1, one can give the following strengthening of Scholze’s result.

Theorem (6.6) *Every noetherian affinoid adic space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ is a $K(\pi, 1)$ space.*

This in turn allows us to give a mixed characteristic variant of Theorem 1.1.

Theorem (6.7) *Let A be a noetherian $\mathbf{Z}_{(p)}$ -algebra such that (A, pA) is a henselian pair. Then $\mathrm{Spec} A$ and $\mathrm{Spec} A[1/p]$ are $K(\pi, 1)$ schemes.*

This has a natural application to Milnor fibers and Faltings’ topos, allowing us to remove the log smoothness hypothesis of the main result of [3], cf. Corollary 6.9.

1.5 The naive étale topology

Let X be a scheme. By the *naive étale topology* we mean the topology on the category of étale X -schemes generated by Zariski coverings and finite étale surjective maps. The corresponding topos, which we denote by $X_{\mathrm{n.ét}}$, is related to the étale topos by a natural map

$$\varepsilon: X_{\mathrm{ét}} \longrightarrow X_{\mathrm{n.ét}}.$$

Theorem 1.1 implies that for $\mathbf{F}_p$ -schemes we can compute étale cohomology of locally constant constructible sheaves using the naive étale topology.

Corollary 1.5 *Let X be an $\mathbf{F}_p$ -scheme and let $\mathcal{F}$ be locally constant constructible sheaf on $X_{\mathrm{ét}}$. Then the maps*

$$\varepsilon^*: H^*(X_{\mathrm{n.ét}}, \varepsilon_* \mathcal{F}) \longrightarrow H^*(X_{\mathrm{ét}}, \mathcal{F})$$

are isomorphisms.

(Presumably the same assertion holds for general constructible sheaves under suitable finiteness conditions on X .)

Suppose that X is a normal noetherian $\mathbf{F}_p$ -scheme. Then the naive étale site of X is locally connected, and applying the Verdier functor one can associate to it the ‘naive étale’ homotopy type $\Pi(X_{n.\text{ét}})$. It follows that the natural map

$$\Pi(\varepsilon): \Pi(X_{n.\text{ét}}) \longrightarrow \Pi(X_{\text{ét}})$$

is a $\natural$ -isomorphism, and hence a weak equivalence.

By the results of Sect. 6, analogous results hold for rigid spaces in positive and mixed characteristic, in which case the naive étale topology is generated by *admissible* open coverings and finite étale covers of affinoids.

1.6 Implications in étale homotopy

Theorem 5.1 yields a “finite” description of the homotopy type of a smooth n -dimensional variety in characteristic p in terms of the single profinite group $\pi_1(\mathbf{A}_k^\eta)$.

Corollary 1.6 *Let $(X, \bar{x})$ be a pointed connected quasi-projective scheme over k , smooth of dimension n . Then there exist $n + 1$ affine open subsets $U_0, \dots, U_n \subseteq X$ containing $\bar{x}$ and covering X , and for every non-empty subset $I \subseteq \{0, \dots, n\}$, a finite étale map*

$$f_I: U_I \longrightarrow \mathbf{A}_k^\eta \quad \text{where} \quad U_I = \bigcap_{i \in I} U_i$$

with $f_I(\bar{x}) = 0$. Thus each U_I is a $K(\pi, 1)$, and f_I induces an isomorphism of $\pi_1(U_I, \bar{x})$ with an open subgroup $\Pi_I \subseteq \pi_1(\mathbf{A}_k^\eta, 0)$. For $I \subseteq J$, let $h_{IJ}: \Pi_J \longrightarrow \Pi_I$ be the homomorphism induced by the inclusion $U_J \subseteq U_I$. Then the étale homotopy type of X is the homotopy colimit of the diagram $(\{\mathbf{B}\Pi_I\}, \{h_{IJ}\})$.

In principle, this tells us that a good understanding of the group $\pi_1(\mathbf{A}_k^\eta)$ would shed light on the étale homotopy types of smooth k -schemes. Unfortunately, this group is too complicated for us to derive any concrete corollaries from the above presentation.

In any case, the results seem to suggest that a very strong form of Grothendieck’s anabelian conjectures could be true in positive characteristic. We allow ourselves to put forth some ambitious-looking questions in this direction in Sect. 7.6.

1.7 Examples and complements

We finish the paper with a few examples:

- Example 7.1, showing that in the presence of fierce ramification, linear projections are not enough in general in the context of Theorem 1.3.
- Example 7.2 of a smooth affine variety X (the complement of a hyperplane arrangement) over $\mathbf{Z}$ such that $X_{\overline{\mathbf{F}}_p}$ is a $K(\pi, 1)$ for every $p > 0$ while $X_{\overline{\mathbf{Q}}}$ is not.
- Example 7.3 showing that $\pi_1(\mathbf{A}_k^n)$ and $\pi_1(\mathbf{A}_k^m)$ are not isomorphic as pro-finite groups for $n \neq m$, even though they have the same finite quotients.

We also study some abstract properties of fundamental groups of affine schemes in Sect. 7.4, comment on the relationship between our work and the ‘ $K(\pi, 1)$ pro- ℓ ’ neighborhoods of Friedlander and Gabber in Sect. 7.5, and state some open questions in the spirit of anabelian geometry in Sect. 7.6.

1.8 Outline

In Sects. 2, 3 and “Appendix”, we deal with the proof of the Bertini theorem. We start with a review of relevant ramification theory in Sect. 2. Then we prove the easy case of the theorem when the sheaf is non-fiercely ramified at infinity in Sect. 3.1, and proceed to the general case in Sect. 3.2. “Appendix” contains an alternative proof of the rank one case of Theorem 1.3.

In Sects. 4–6, we deal with $K(\pi, 1)$ schemes and rigid analytic spaces. In Sect. 4, we review the notion of a $K(\pi, 1)$ scheme. Then in Sect. 5, we prove Theorem 1.1. The subsequent Sect. 6 treats the mixed characteristic and rigid geometry analogues of Theorem 1.1.

In the last Sect. 7, we provide relevant examples and further discussion as listed above.

2 Review of wild ramification

Let k be an algebraically closed field of characteristic $p > 0$.

2.1 The Swan conductor

Let C be a smooth curve over k , $x \in C(k)$ a point, $\mathcal{F}$ a locally constant constructible $\mathbf{F}_\ell$ -sheaf on $C \setminus \{x\}$. The Swan conductor $\mathrm{Sw}_x(\mathcal{F})$ is an integer measuring the wild ramification of $\mathcal{F}$ at x . It depends only on the restriction of $\mathcal{F}$ to $\mathrm{Spec} K$ where K is the fraction field of the henselization of $\mathcal{O}_{C,x}$. It appears in the Grothendieck–Ogg–Shafarevich formula [45, Exp. X, formula 7.2]

$$\chi_c(C, \mathcal{F}) = \mathrm{rk}(\mathcal{F}) \cdot \chi_c(C, \mathbf{F}_\ell) - \sum_{x \in S} \mathrm{Sw}_x(\mathcal{F}). \quad (2.1)$$

Here C is a smooth geometrically integral curve with a smooth projective model $\overline{C}$, $S = \overline{C} \setminus C$, and $\mathcal{F}$ is a locally constant constructible $\mathbf{F}_\ell$ -sheaf on C . It is often more convenient to use the *total dimension*, defined as

$$\dim_{\text{tot}}(\mathcal{F}) = \text{rk}(\mathcal{F}) + \text{Sw}_x(\mathcal{F})$$

instead of the Swan conductor. For example, the version of the above formula without compact supports is

$$\chi(C, \mathcal{F}) = \text{rk}(\mathcal{F}) \cdot \chi(C, \mathbf{F}_\ell) - \sum_{x \in S} \dim_{\text{tot}} \mathcal{F}. \quad (2.2)$$

Let $\eta = \text{Spec } K$ where K is a henselian discrete valuation field containing k whose residue field κ is finitely generated over k . Let $\ell \neq p$ be a prime and let $\mathcal{F}$ be a locally constant constructible $\mathbf{F}_\ell$ -sheaf on η . Choose a separable closure K^{sep} of K , and let $\overline{\eta} = \text{Spec } K^{\text{sep}}$. Then $\mathcal{F}$ corresponds to the continuous $\text{Gal}(K^{\text{sep}}/K)$ -module $M = \mathcal{F}(\overline{\eta})$.

Definition 2.1 (1) We call $\mathcal{F}$ *non-fiercely ramified* if there exists a finite separable Galois extension K'/K such that the pullback of $\mathcal{F}$ to $\text{Spec } K'$ is constant and such that the residue field extension κ'/κ is separable.
 (2) Let K'/K be as in (1), and let $x \in \mathcal{O}_{K'}$ be a generator of $\mathcal{O}_{K'}$ as an $\mathcal{O}_K$ -algebra [44, Chap. III, Prop. 12]. The *ramification groups* $G_i \subseteq G = \text{Gal}(K'/K)$ (cf. [44, Chapter IV, §1]) are defined as

$$G_i = \{\sigma \in G : v_{K'}(\sigma(x) - x) \geq i + 1\}.$$

They are independent of the choice of x . The group G_0 is the inertia subgroup of G , and G_1 is called the wild inertia subgroup.

(3) Suppose that $\mathcal{F}$ is non-fiercely ramified. Let K'/K be as in (1), let $G = \text{Gal}(K'/K)$ (so that $\text{Gal}(K^{\text{sep}}/K)$ acts on M through its quotient G), and let $G_i \subseteq G$ ($i \geq 0$) be the ramification groups. The *Swan conductor* of $\mathcal{F}$ is defined as (cf. [43, §19.3], [29, §1.1])

$$\text{Sw}(\mathcal{F}) = \sum_{i=1}^{\infty} \frac{1}{[G_0 : G_i]} \dim_{\mathbf{F}_\ell}(M/M^{G_i}).$$

It is an integer, and is independent of the choice of K' . We also define the *total dimension*

$$\dim_{\text{tot}}(\mathcal{F}) = \dim_{\mathbf{F}_\ell} M + \text{Sw}(\mathcal{F}).$$

2.2 The Deligne–Laumon theorem

In the context of Theorem 1.3, the utility of the Swan conductor comes from the following result of Deligne and Laumon.

Theorem 2.2 [29, Théorème 2.1.1] *Let S be a noetherian excellent scheme, and let $f: X \rightarrow S$ be a separated morphism, smooth of relative dimension 1. Let $Y \subseteq X$ be a closed subscheme which is finite and flat over S . Let $U = X \setminus Y$, let ℓ be a prime invertible on S , and let $\mathcal{F}$ be a locally constant constructible $\mathbf{F}_\ell$ -sheaf on U of constant rank r . Consider the function $\varphi: S \rightarrow \mathbf{Z}$ defined as follows*

$$\varphi(s) = \sum_{y \in Y_{\bar{s}}} (\mathrm{Sw}_y(\mathcal{F}|_{U_{\bar{s}}}) + r) = \sum_{y \in Y_{\bar{s}}} \dim_{\mathrm{tot}_y}(\mathcal{F}|_{U_{\bar{s}}})$$

(here $\bar{s}$ is any geometric point over s , and the value of the function does not depend on the choice of $\bar{s}$). Then

- (i) *The function φ is a constructible and lower-semicontinuous,*
- (ii) *If φ is locally constant on S , then the triple $(X, \mathcal{F}, f)$ is universally locally acyclic.*

Corollary 2.3 *Let $f: X \rightarrow S$ be a projective morphism with geometrically connected fibers, smooth of relative dimension 1, $i: S \rightarrow X$ a section, $\mathcal{F}$ a locally constant constructible $\mathbf{F}_\ell$ -sheaf on $U = X \setminus i(S)$. Suppose that the number $\mathrm{Sw}_{i(\bar{s})}(\mathcal{F}|_{U_{\bar{s}}})$ is independent of the geometric point $\bar{s}$ of S . Then the sheaves $R^q f_* \mathcal{F}$ and $R^q f_! \mathcal{F}$ are locally constant with formation commuting with base change for all $q \geq 0$. In particular, we have $R^q f_* \mathcal{F} = 0$ for $q > 1$.*

Proof We note first that for a locally constant constructible $\mathbf{F}_\ell$ -sheaf on $\eta = \mathrm{Spec} K$ where K is a henselian discrete valuation field with perfect residue field, we have $\mathrm{Sw}(\mathcal{F}) = \mathrm{Sw}(\mathcal{F}^\vee)$. Indeed, it is clear from the fact that (using the notation of Definition 2.1) the G_i are p -groups for $i \geq 1$, and hence M is semisimple as a G_i -representation (by Maschke's theorem), so $\dim_{\mathbf{F}_\ell}(M/M^{G_i}) = \dim_{\mathbf{F}_\ell}(M^\vee/(M^\vee)^{G_i})$ and $\mathrm{Sw}(M) = \mathrm{Sw}(M^\vee)$.

It follows that we can apply [29, Corollaire 2.1.2] (together with [29, Remarque 2.1.3]) to both $\mathcal{F}^\vee$ and $\mathcal{F}$ to see that $R^q f_! \mathcal{F}^\vee$ and $R^q f_! \mathcal{F}$ are locally constant for $q \geq 0$. By Poincaré–Verdier duality, the sheaves $R^q f_* \mathcal{F}$ are then locally constant with formation commuting with base change. $\square$

2.3 The characteristic cycle of a constructible sheaf

The recent work of Beilinson [10] and Saito [39] provides an analogue of the classical theory of the singular support and the characteristic cycle [28, Chapter

IX] for constructible étale sheaves, fulfilling an expectation of Deligne. Let us review the relevant points briefly, following [39].

Let X be a smooth scheme over k which is everywhere of dimension n , let $\ell \neq p$ be a prime, and let $\mathcal{F}$ be a constructible complex of $\mathbf{F}_\ell$ -vector spaces on X . In [10], Beilinson defines the singular support $SS\mathcal{F}$ inside the cotangent bundle T^*X . It is the smallest closed conical subset $C \subseteq T^*X$ such that $\mathcal{F}$ is micro-supported on C (cf. Definition 2.4(4) below). He proves that all of its irreducible components have dimension n .

Saito [39] employed Deligne's ideas to define the characteristic cycle $CC\mathcal{F}$. It is an integral combination of the irreducible components of $SS\mathcal{F}$. It is uniquely determined by the property of being compatible with étale base change and by the Milnor formula

$$-\dim_{\text{tot}} \varphi_x(\mathcal{F}, f) = (CC\mathcal{F}, df)_{T^*X, x} \quad (\text{cf. [39, 5.15]})$$

for the total dimension of the vanishing cycles of a morphism $f: X \rightarrow Y$ is to a smooth curve which is $SS\mathcal{F}$ -transversal away from x [cf. Definition 2.4(2)].

If X is a curve, $j: U \subseteq X$ a dense open subset, and $\mathcal{F}$ is a locally constant constructible $\mathbf{F}_\ell$ -sheaf on U , then

$$CC(j_!\mathcal{F}) = - \left(\text{rk } \mathcal{F} \cdot [T_X^*X] + \sum_{x \in X \setminus U} \dim_{\text{tot}_x} \mathcal{F} \cdot [T_x^*X] \right)$$

(cf. [39, Lemma 5.11.3]). Here T_X^*X denotes the zero section in T^*X and T_x^*X the fiber at x .

Definition 2.4 Let $C \subseteq T^*X$ be a conical (i.e., stable under the $\mathbf{G}_m$ -action) closed subset.

1. A morphism $h: W \rightarrow X$ from a smooth k -scheme W is called *C-transversal* if for every $w \in W(k)$,

$$C \cap \ker \left(h^*: T_{h(w)}^*X \rightarrow T_w^*W \right) = \{0\} \text{ or } \emptyset.$$

In this case, we define

$$h^\circ C = \text{im} \left(W \times_X C \rightarrow W \times_X T^*X \xrightarrow{dh} T^*W \right).$$

2. A morphism $f: X \rightarrow Y$ to a smooth k -scheme Y is called *C-transversal* if for every $x \in X(k)$,

$$\text{the preimage of } C \text{ under } f^*: T_{f(x)}^*Y \rightarrow T_x^*X = \{0\} \text{ or } \emptyset.$$

3. A pair of morphisms $h: W \rightarrow X, f: W \rightarrow Y$ of smooth k -schemes is called *C-transversal* if h is C -transversal and f is h^*C -transversal.
4. We say that a constructible complex of $\mathbf{F}_\ell$ -sheaves $\mathcal{F}$ on X is *micro-supported on C* if for every C -transversal pair of morphisms $h: W \rightarrow X, f: W \rightarrow Y, f$ is locally acyclic with respect to $h^*\mathcal{F}$ (cf. [15, Th.Finitude, Definition 2.12]).

In our proof of the Bertini theorem, we have to control the wild ramification of the restrictions to curves of a given sheaf $\mathcal{F}$. To this end, we need some compatibility of the characteristic cycle with pull-back.

Definition 2.5 In the situation of Definition 2.4, suppose that every irreducible component of C has dimension n . A morphism $h: W \rightarrow X$ from a smooth k -scheme W which is everywhere of dimension m is called *properly C-transversal* if it is C -transversal and if every irreducible component of $W \times_X C$ has dimension m . In this situation, let $A = \sum m_a [C_a]$ be an integral combination of the irreducible components of C . We define

$$h^!A = (-1)^{n-m}(dh)_*h^*A.$$

Here h^* denotes pull-back along $W \times_X T^*X \rightarrow T^*X$, and $(dh)_*$ means the push-forward along $dh: W \times_X T^*X \rightarrow T^*W$ in the sense of intersection theory.

Theorem 2.6 [39, Theorem 7.6] *Suppose that $h: W \rightarrow X$ is properly $SS\mathcal{F}$ -transversal. Then*

$$CCh^*\mathcal{F} = h^!CC\mathcal{F}.$$

Corollary 2.7 *Let X be a smooth k -scheme, $D \subseteq X$ a divisor, $\mathcal{F}$ a locally constant constructible $\mathbf{F}_\ell$ -sheaf on $U = X \setminus D$. Then there exists a dense open subset $T^\circ \in D \times_X \mathbf{P}(TX)$ with the following property: if $x \in D$ and $L \subseteq T_xX$ is a line such that the corresponding point (x, L) lies in T° , and if $C, C' \subseteq X$ are smooth locally closed curves with $C \cap D = \{x\} = C' \cap D$ and $T_xC = L = T_xC'$, then*

$$\mathrm{Sw}_x(\mathcal{F}|_{C \setminus \{x\}}) = \mathrm{Sw}_x(\mathcal{F}|_{C' \setminus \{x\}}).$$

Here we follow the convention that $\mathbf{P}(TX)$ parametrizes lines in the tangent bundle TX (this is consistent with [39]). Thus points $D \times_X \mathbf{P}(TX)$ are identified with pairs (x, L) of a point $x \in D$ and a tangent direction $L \subseteq T_xX$.

Proof We can assume that X is everywhere of dimension n , and that $\mathcal{F}$ is of constant rank. Let $j: U \rightarrow X$ be the inclusion. Since every irreducible component of $SS(j_!\mathcal{F})$ has dimension n , while D has dimension $n - 1$, there

exists a dense open subset $D^\circ \subseteq D$ such that $\dim(SS(j_! \mathcal{F}) \cap T_x^* X) \leq 1$ for $x \in D(k)$. Replace D with D° and set

$$T^\circ = \{(x, L) \in D \times_X \mathbf{P}(TX) : \omega|_L \neq 0 \text{ for all nonzero } \omega \in SS(j_! \mathcal{F}) \times_X \{x\}\}.$$

Then T° is a dense open subset of $D \times_X \mathbf{P}(TX)$.

Suppose that $C \subseteq X$ is a locally closed curve with $C \cap D = \{x\}$ and $(x, T_x C) \in T^\circ$, as in the statement. We check that the inclusion $i : C \rightarrow X$ is properly $SS(j_! \mathcal{F})$ -transversal. The condition that $(x, T_x C) \in T^\circ$ is equivalent to the fact that

$$SS(j_! \mathcal{F}) \cap \ker(T_x^* X \rightarrow T_x^* C) = \{0\}.$$

This means that i is $SS(j_! \mathcal{F})$ -transversal. The additional condition on the dimensions of the irreducible components of $C \times_X SS(j_! \mathcal{F})$ is satisfied automatically.

Theorem 2.6 implies now that

$$CC(i^* j_! \mathcal{F}) = i^! CC(j_! \mathcal{F}),$$

while

$$CC(i^* j_! \mathcal{F}) = -\operatorname{rk} \mathcal{F} \cdot [T_C^* C] + (\operatorname{rk} \mathcal{F} + \operatorname{Sw}_x(\mathcal{F}|_{C \setminus \{x\}})) \cdot [T_x^* C].$$

By definition of $i^!$, the coefficient of $[T_x^* C]$ in $i^! CC(j_! \mathcal{F})$ depends only on the map $T_x^* X \rightarrow T_x^* C$. $\square$

Remark 2.8 In his slightly earlier paper [38], predating Beilinson's ideas, Saito defined the characteristic cycle of a locally constant constructible sheaf in a neighborhood of the generic point of the boundary divisor D using a different method, and studied its behavior upon restrictions to curves. Our Corollary 2.7 can also be deduced from [38, Corollary 3.9.2].

3 Proof of the Bertini theorem

In this section, k remains to denote a fixed algebraically closed field of characteristic $p > 0$. The assertions of Proposition 3.4 and Theorem 3.6 remain valid over any infinite characteristic p field.

3.1 The non-fierce case of the Bertini theorem

As a warm-up, we show that a variant of Theorem 1.3 holds for sheaves with non-fierce ramification at infinity. In contrast with the general case,

it is possible to choose the automorphism φ to be a general linear automorphism.

Let X be an integral smooth scheme over k , $D \subseteq X$ an irreducible smooth divisor, $U = X \setminus D$ its complement. Let $\mathcal{F}$ be a locally constant constructible $\mathbf{F}_\ell$ -sheaf on U . Let $X_{(\eta_D)}$ denote the localization of X at the generic point η_D of D for the étale topology. Then $X_{(\eta_D)} \times_X U$ is the spectrum of the henselization of the fraction field of X with respect to the discrete valuation given by D .

Definition 3.1 We call $\mathcal{F}$ *non-fiercely ramified along D* if the restriction of $\mathcal{F}$ to $X_{(\eta_D)} \times_X U$ is non-fiercely ramified in the sense of Definition 2.1(1), and if this is the case we write $\mathrm{Sw}_D(\mathcal{F}) = \mathrm{Sw}(\mathcal{F}|_{X_{(\eta_D)} \times_X U})$.

Proposition 3.2 (cf. [30, §2.2]) *In the above situation, suppose that $\mathcal{F}$ is non-fiercely ramified along D . Then there exists a dense open $D^\circ \subseteq D$ with the property that for any $x \in D^\circ(k)$ and any smooth locally closed curve $C \subseteq X$ with $C \cap D = \{x\}$ and transverse to D at x , we have $\mathrm{Sw}_x(\mathcal{F}|_{C \setminus \{x\}}) = \mathrm{Sw}_D(\mathcal{F})$.*

Proof We include a direct proof (surely standard) because we were unable to find one in the literature (but see Remark 3.3 below). Let $\bar{\eta}_D$ be a geometric point above η_D . Setting $\eta = X_{(\bar{\eta}_D)} \times_X U$ puts us in the henselian situation described in Sect. 2. Let K'/K be as in Definition 2.1. Since the residue field κ of K is separably closed, while κ'/κ is separable because of the non-fierceness assumption, we have $\kappa' = \kappa$. Thus K' is a totally ramified extension of K . By [44, I §6], there exists an Eisenstein polynomial $P \in \mathcal{O}_K[T]$ such that $\mathcal{O}_{K'} \simeq \mathcal{O}_K[T]/(P)$, and the image of the variable T is a uniformizer of $\mathcal{O}_{K'}$ under this isomorphism.

Both the non-fierceness assumption on $\mathcal{F}$ and the assertion of the proposition are étale local in a neighborhood of $\bar{\eta}_D$. Spreading out the data described in the first paragraph to an étale neighborhood, we can assume that

- (1) $X = \mathrm{Spec} A$ is affine,
- (2) D is principal, its ideal generated by an element $\pi \in A$,
- (3) there exists a polynomial $P \in A[T]$ whose image in $\hat{A}[T]$ ($\hat{A} = \varprojlim A/\pi^{n+1}$) is an Eisenstein polynomial, such that, setting $B = \hat{A}[T]/(P)$, $Y = \mathrm{Spec} B$, then Y is normal and finite over X , and $V = \mathrm{Spec} B[1/\pi]$ is an étale torsor under a finite group G over $U = \mathrm{Spec} A[1/\pi]$,
- (4) the pull-back of $\mathcal{F}$ to V is constant,
- (5) the polynomial P has the form

$$P = T^r + a_1 T^{r-1} + \cdots + a_r, \quad a_i = u_i \cdot \pi^{m_i}, \quad u_i \in A^\times, \quad m_i \geq 1,$$

(6) for all $\sigma \in G$, there exists an integer $m(\sigma)$ and a unit $u(\sigma) \in B^\times$ such that

$$\sigma(T) - T = u(\sigma) \cdot T^{m(\sigma)}.$$

Let $C \subseteq X$ be a smooth locally closed curve through a point $x \in D$ and transverse to D at that point. Transversality means that π maps to a uniformizer of $\mathcal{O}_{C,x}$. Let $C' = Y \times_X C = \operatorname{Spec} \mathcal{O}_C[T]/(P)$. Since the image of P in $\mathcal{O}_{C,x}[T]$ is Eisenstein (by condition (5) above), T gives a uniformizer of C' at the unique point x' over x . Moreover, condition (6) implies that

$$v_{C'}(\sigma(T) - T) = v_D(\sigma(T) - T), \quad (\sigma \in G)$$

We deduce that the G -Galois extensions $\mathcal{O}_{C',x'}/\mathcal{O}_{C,x}$ and B/A induce the same ramification filtration on G . This implies the required assertion. $\square$

Remark 3.3 The following argument employing the characteristic cycle was pointed out to us by an anonymous referee. Observe that by Theorem 2.6 the assertion of Proposition 3.2 holds under the weaker assumption that

$$SS(j_! \mathcal{F}) \subseteq T_D^* X \cup T_X^* X \quad (3.1)$$

holds generically along D . If $\pi : Y \rightarrow X$ is as in the above proof the normalization of X in a finite étale Galois cover $V \rightarrow U$ trivializing $\mathcal{F}$, then

$$SS(j_! \mathcal{F}) \subseteq SS(\pi_* \underline{\mathbf{F}}_{\ell,Y}). \quad (3.2)$$

On the other hand, [39, Lemma 4.2.6] implies that

$$SS(\pi_* \underline{\mathbf{F}}_{\ell,Y}) \subseteq \pi_o SS(\underline{\mathbf{F}}_{\ell,Y}) = \pi_o T_Y^* Y \quad (3.3)$$

(cf. [39, Definition 3.7] for the definition of π_o). Now if $\mathcal{F}$ is non-fiercely ramified along D , there exists $V \rightarrow U$ as above such that $\pi : E := \pi^{-1}(D)_{\text{red}} \rightarrow D$ is separable. Passing to a neighborhood of D , we can assume that $\pi : E \rightarrow D$ is étale, in which case the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\pi|_E)^* N_{D/X} & \longrightarrow & (\pi|_E)^* (\Omega_{X/k}^1|_D) & \longrightarrow & (\pi|_E)^* \Omega_{D/k}^1 \longrightarrow 0 \\ & & \downarrow & & \downarrow \pi^* & & \downarrow \simeq \\ 0 & \longrightarrow & N_{E/Y} & \longrightarrow & \Omega_{Y/k}^1|_E & \longrightarrow & \Omega_{E/k}^1 \longrightarrow 0 \end{array}$$

shows that for $x \in D$,

$$\pi_*(T_Y^*Y) \times_X \{x\} = \bigcup_{y: \pi(y)=x} \ker(\pi^*: T_x^*X \longrightarrow T_y^*Y) \subseteq T_D^*X \times_X \{x\},$$

and hence

$$\pi_*(T_Y^*Y) \subseteq T_D^*X \cup T_X^*X. \quad (3.4)$$

Combining (3.2)–(3.4) we get (3.1).

Note that in the classical complex analytic setting (3.1) is always satisfied because the irreducible components of the singular support of a holonomic $\mathcal{D}$ -module are Lagrangian subvarieties of T^*X , and hence are the closures in T^*X of conormal bundles T_Z^*X of smooth locally closed subschemes $Z \subseteq X$.

Proposition 3.4 *The assertion of Theorem 1.3 holds for a general linear automorphism φ of $\mathbb{A}_k^{\eta+1}$ if the sheaf $\mathcal{F}$ is non-fiercely ramified along the hyperplane at infinity.*

Proof By Proposition 3.2, there exists a dense open subset $H^\circ \subseteq H$ of the hyperplane at infinity with the property that for any line $L \subseteq \mathbb{A}_k^{\eta+1}$ which meets H° at infinity the Swan conductor $\mathrm{Sw}_\infty(\mathcal{F}|_L)$ is independent of L . Therefore if we take for $\pi: \mathbb{A}_k^{\eta+1} \longrightarrow \mathbb{A}_k^\eta$ a linear projection along a line L which meets H° at infinity, then function $y \mapsto \mathrm{Sw}_\infty(\mathcal{F}|_{\pi^{-1}(y)})$ will be constant. Hence the sheaves $R^q\pi_*\mathcal{F}$ will be locally constant with formation commuting with base change by the Deligne–Laumon theorem (Corollary 2.3). $\square$

Remark 3.5 We expect that for a locally constant constructible $\mathbf{F}_\ell$ -sheaf $\mathcal{F}$ on $\mathbb{A}_k^{\eta+1}$, there exists an automorphism φ of the form

$$\varphi(x_1, \dots, x_{n+1}) = (x_1 + x_{n+1}^{d_1}, \dots, x_n + x_{n+1}^{d_n}, x_{n+1})$$

such that $\varphi^*\mathcal{F}$ is non-fierce at infinity.

3.2 The general case of the Bertini theorem

Before going into the proof, let us explain its main idea. In the non-fierce situation in the previous section, the Swan conductor $\mathrm{Sw}_x(\mathcal{F}|_{C \setminus \{x\}})$ of the restriction of a sheaf $\mathcal{F}$ to a curve meeting the boundary divisor D transversally at a single point x was independent of C for x in a dense open $D^\circ \subseteq D$.

In the general case, the theory of the characteristic cycle (Corollary 2.7) shows that the same assertion holds if the tangent space $T_x C \subseteq T_x X$ is a fixed element of a dense open subset T° of $D \times_X \mathbf{P}(TX)$. This implies that for $x \in D$ and a tangent direction $L \subseteq T_x X$ such that $(x, L) \in T^\circ$, the number $\mathrm{Sw}_x(\mathcal{F}|_{C \setminus \{x\}})$ is independent of C as long as $T_x C = L$.

It would therefore suffice to construct an $\mathbf{A}^1$ -fibration $\mathbf{A}_k^{\eta+1} \rightarrow \mathbf{A}_k^\eta$ whose fibers meet the hyperplane at infinity transversally with the same tangent direction. This is probably impossible, but we can produce such a fibration whose fibers are tangent to order two to the hyperplane at infinity and agree to sufficiently high order at that point. If $p > 2$, taking the normalization of their preimages in a cyclic covering of degree two ramified along the hyperplane at infinity makes them transverse to the boundary, with the same tangent direction (this is another idea due to T. Saito). This allows one to apply Corollary 2.7 to the cyclic covering.

Theorem 3.6 *Let k be a field of characteristic p . Let $\ell \neq p$ be a prime and let $\mathcal{F}$ be a locally constant constructible sheaf of $\mathbf{F}_\ell$ -vector spaces on $\mathbf{A}_k^{\eta+1}$ ($n \geq 0$). Consider the map*

$$\varpi: \mathbf{A}_k^{\eta+1} \rightarrow \mathbf{A}_k^\eta, \quad \varpi(x_1, \dots, x_{n+1}) = (x_1 - x_{n+1}^2, x_2, \dots, x_n).$$

Let G be the group of affine automorphisms ψ of $\mathbf{A}_k^{\eta+1}$ satisfying $\psi^*x_1 = x_1$. Then there exists a dense open $G^\circ(\mathcal{F}) \subseteq G$ such that for all $\psi \in G^\circ(\mathcal{F})(\bar{k})$, $\psi^*\mathcal{F}$ is well-aligned with respect to ϖ .

(We note that this implies Theorem 1.3: since G has an open subset isomorphic to an open subset of the affine space, $G(k)$ is dense in k as long as k is infinite, and hence $G^\circ(\mathcal{F})(k) \neq \emptyset$. Let

$$\varphi_0: \mathbf{A}_k^{\eta+1} \rightarrow \mathbf{A}_k^{\eta+1}, \quad \varphi_0(x_1, \dots, x_{n+1}) = (x_1 - x_{n+1}^2, x_2, \dots, x_n, x_{n+1}).$$

Then $\varpi = \pi \circ \varphi_0$, so if $\psi^*\mathcal{F}$ is well-aligned with respect to ϖ then $\varphi^*\mathcal{F}$ is well-aligned with respect to π where $\varphi = \psi \circ \varphi_0^{-1}$.)

Proof Let the coordinates on $\mathbf{A}_k^{\eta+1}$ be $x_1, \dots, x_{n+1}$. We put $\mathbf{A}_k^{\eta+1}$ inside $\mathbf{P}_k^{n+1}$ in the usual way, by adding an additional homogeneous coordinate x_0 . We will look at what happens ‘at infinity’ by considering the open subset

$$X = D^+(x_1) \subseteq \mathbf{P}_k^{n+1}.$$

It is isomorphic to $\mathbf{A}_k^{\eta+1}$, with coordinates $z = x_0/x_1, z_i = x_i/x_1$ ($i = 2, \dots, n+1$). Let $D = V(z) \subseteq X$ be the intersection of X with the hyperplane at infinity and let $U = X \setminus D$.

Consider the cyclic cover of degree two of X ramified along D :

$$\sigma: X' = \operatorname{Spec} \mathcal{O}_X[w]/(w^2 - z) \rightarrow X,$$

and let

$$U' = \sigma^{-1}(U), \quad D' = X' \setminus U' = \sigma^{-1}(D)_{\text{red}}, \quad \mathcal{F}' = (\sigma|_{U'})^*\mathcal{F}.$$

We apply Corollary 2.7 to the tuple $(X', D', U', \mathcal{F}')$, obtaining the subset $T^\circ(\mathcal{F}) \subseteq D' \times_{X'} \mathbf{P}(TX')$. We have $X' \simeq \mathbf{A}_k^{\eta+1}$ with coordinates $w, z_2, \dots, z_{n+1}$, and $\sigma(w, z_2, \dots, z_{n+1}) = (w^2, z_2, \dots, z_{n+1})$. The commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\sigma|_{D'})^* N_{D'/X} & \longrightarrow & (\sigma|_{D'})^* (\Omega_{X'/k}^1|_{D'}) & \longrightarrow & (\sigma|_{D'})^* \Omega_{D'/k}^1 \longrightarrow 0 \\ & & \downarrow 0 & & \downarrow & & \downarrow \simeq \\ 0 & \longrightarrow & N_{D'/X'} & \longrightarrow & \Omega_{X'/k}^1|_{D'} & \longrightarrow & \Omega_{D'/k}^1 \longrightarrow 0 \end{array}$$

shows that there is a natural splitting

$$\Omega_{X'/D'}^1 \simeq N_{D'/X'} \oplus \Omega_{D'/k}^1,$$

using which we can regard $\mathbf{P}(T_{D'}X')$ as a subset of $D' \times_{X'} \mathbf{P}(TX')$. We set

$$T^{\circ\circ} = (D' \times_{X'} \mathbf{P}(TX')) \setminus (\mathbf{P}(T_{D'}X') \cup \mathbf{P}(TD')).$$

Note that the group G acts naturally on $\mathbf{P}_k^{n+1}$ as the group of automorphisms fixing both x_0 and x_1 . Consequently, the action of G preserves not only $\mathbf{A}_k^{\eta+1}$ but also X , on which it acts as the group of affine transformations fixing the coordinate $z = x_0/x_1$. Furthermore, since z is fixed, the action of G on X lifts to an action on X' fixing D' . We claim that $T^{\circ\circ}$ is the open orbit of the action of G on $D' \times_{X'} \mathbf{P}(TX')$. Writing $X = \mathbf{A}_z^1 \times V$, $X' = \mathbf{A}_w^1 \times V$ where V is the affine space $\mathbf{A}_{z_2, \dots, z_{n+1}}^n$, the action of an element $g \in G$ can be presented as

$$g(z, v) = (z, Av + v_0 + zv_1), \quad g(w, v) = (w, Av + v_0 + w^2v_1)$$

for some $A \in GL(V)$, $v_0, v_1 \in V$. Present an element ξ of $D' \times_{X'} TX'$ in the form $\xi = (0, v) + \varepsilon \cdot (w, \tilde{v})$ where $\varepsilon^2 = 0$, then

$$\begin{aligned} g(\xi) &= (0, Av + v_0) + \varepsilon \cdot (w, A\tilde{v}) + \varepsilon^2 \cdot (0, w^2v_1) \\ &= (0, Av + v_0) + \varepsilon \cdot (w, A\tilde{v}). \end{aligned}$$

On the other hand, the class of ξ lies in $T^{\circ\circ}$ if and only if $w \neq 0$ and $\tilde{v} \neq 0$. We can thus assume $w = 1$. For $\xi' = (0, v') + \varepsilon \cdot (1, \tilde{v}')$ whose class lies in $T^{\circ\circ}$, take $A \in GL(V)$ satisfying $A\tilde{v} = \tilde{v}'$ and set $v_0 = v' - Av$, $v_1 = 0$, then g as above sends ξ to ξ' .

We define $G^\circ(\mathcal{F}) \subseteq G$ as the subscheme of all $\psi \in G$ which map the point

$$(x', L') \in D' \times_{X'} \mathbf{P}(TX'): \quad x' = (0, \dots, 0), \quad L' = k \cdot (1, 0, \dots, 0, 1)$$

to a point in $T^\circ(\mathcal{F})$. (This particular choice will be explained shortly by the calculations of the fibers of ϖ .) Since $T^\circ(\mathcal{F})$ is a dense open and (x', L') lies in the open orbit T° , $G^\circ(\mathcal{F})$ is a dense open subscheme of G . Thus $\psi \in G^\circ(\mathcal{F})(k)$ means that the point (x', L') defined above lies in $T^\circ(\psi^*\mathcal{F})$.

We analyze the fibers of ϖ (cf. Fig. 1). The fiber over $y = (y_1, \dots, y_n) \in \mathbf{A}_k^n(\bar{k})$ can be parametrized by

$$\gamma(t) = (y_1 + t^2, y_2, \dots, y_n, t) : \mathbf{A}_k^1 \longrightarrow \mathbf{A}_k^{n+1}.$$

This extends to a map $\bar{\gamma} : \mathbf{P}_k^1 \longrightarrow \mathbf{P}_k^{n+1}$ as follows:

$$\bar{\gamma}(t : s) = (s^2 : y_1 s^2 + t^2 : y_2 s^2 : \dots : y_n s^2 : ts).$$

We want to see what happens at infinity, so we look at $\beta(s) = \bar{\gamma}(1 : s)$:

$$\beta(s) = (s^2 : y_1 s^2 + 1 : y_2 s^2 : \dots : y_n s^2 : s).$$

So $\beta(0) = (0 : 1 : 0 : \dots : 0) \in X$, which corresponds to the point $(z, z_2, \dots, z_{n+1}) = (0, \dots, 0) \in X$, which we denote henceforth by x . Thus in the z -coordinates, β takes the form

$$\beta(s) = \left(\frac{s^2}{y_1 s^2 + 1}, \frac{y_2 s^2}{y_1 s^2 + 1}, \dots, \frac{y_n s^2}{y_1 s^2 + 1}, \frac{s}{y_1 s^2 + 1} \right)$$

(defined for s such that $y_1 s^2 + 1 \neq 0$).

Let C be the closure of the image of β_y in X . Then $C \cap D = \{x\}$, C meets D with multiplicity two at x and is smooth at x . Let

$$\tilde{C} = \mathbf{A}^1 \setminus \{y_1 s^2 + 1 = 0\} \longrightarrow C$$

be the normalization of C , $\tilde{x} = 0$ the unique point of $\tilde{C}$ above x .

Case $p > 2$: The preimage of C in X' has two branches at $x' = (0, \dots, 0)$, each one smooth and transverse to D' (cf. top-right of Fig. 1). This can be seen formally locally: look at

$$\beta'_\pm(s) = \left(\frac{\pm s}{\sqrt{y_1 s^2 + 1}}, \frac{y_2 s^2}{y_1 s^2 + 1}, \dots, \frac{y_n s^2}{y_1 s^2 + 1}, \frac{s}{y_1 s^2 + 1} \right)$$

Fig. 1 Analysis of the fibers of w ($n = 1, k = \mathbf{R}$)

where $\sqrt{y_1 s^2 + 1} = \sum \binom{1/2}{i} (y_1 s^2)^i$. So $\beta'_\pm: \text{Spf } k[[s]] \rightarrow X'$ are two formal curve germs at x' , smooth and transverse to D' at x' . Moreover, they map isomorphically to the normalization of the germ of C at x — in other words, the diagram

$$\begin{array}{ccc}
 \text{Spf } k[[s]] & \xrightarrow{\beta'_\pm} & \text{Spf } \hat{\mathcal{O}}_{X',x'} \\
 \cong \downarrow & & \downarrow \sigma \\
 \text{Spf } \hat{\mathcal{O}}_{\tilde{C},0} & \xrightarrow{\beta} \text{Spf } \hat{\mathcal{O}}_{C,x} \hookrightarrow & \text{Spf } \hat{\mathcal{O}}_{X,x}
 \end{array}$$

commutes.

Case $p = 2$: In this case, since $(y_1 s^2 + 1) = (y_1^{1/2} s + 1)^2$, β lifts (globally) along σ :

$$\beta'_1(s) = \left(\frac{s}{y_1^{1/2}s + 1}, \frac{y_2 s^2}{y_1 s^2 + 1}, \dots, \frac{y_n s^2}{y_1 s^2 + 1}, \frac{s}{y_1 s^2 + 1} \right), \quad \beta = \sigma \circ \beta'_1.$$

In either case, the tangent vector of β'_1 at x' is

$$\frac{d}{ds} \beta'_1(s)|_{s=0} = (1, 0, \dots, 0, 1) \in L'.$$

Shrinking X' to an étale neighborhood of x' , we can assume the existence of a locally closed curve C' through x' with germ β'_1 at x' (more precisely, if $p > 2$, the étale cover comes from adjoining $\sqrt{y_1 s^2 + 1}$). Then $C' \rightarrow \tilde{C}$ is étale at $x' \mapsto \tilde{x}$, so by the fact that the Swan conductor depends only on the complete local field at x , we have

$$\mathrm{Sw}_\infty(\mathcal{F}|_{\varpi^{-1}(y)}) = \mathrm{Sw}_x(\mathcal{F}|_{C \setminus \{x\}}) = \mathrm{Sw}_{x'}(\mathcal{F}|_{C' \setminus \{x'\}}).$$

The tangent direction (x', L') of C' (which does not depend on y) lies in $T^\circ(\psi^* \mathcal{F})$. The defining property of $T^\circ(\psi^* \mathcal{F})$ implies thus that $\mathrm{Sw}_\infty(\mathcal{F}|_{\varpi^{-1}(y)})$ is independent of y . By the Deligne–Laumon theorem (Corollary 2.3), this implies that the sheaves $R^q \varpi_*(\psi^* \mathcal{F})$ are locally constant, with formation commuting with base change. $\square$

4 Review of $K(\pi, 1)$ schemes

Following Abbes and Gros [4, §9], we will only consider schemes X satisfying the following condition:

$$X \text{ is coherent and has finitely many connected components.} \quad (4.1)$$

By definition, a scheme is coherent if it is quasi-compact and quasi-separated. Note that if A is a ring with finitely many idempotent elements, then $X = \mathrm{Spec} A$ satisfies (4.1), and that if $X' \rightarrow X$ is a finite étale morphism and X satisfies (4.1), then so does X' .

Definition 4.1 (cf. [4, Definition 9.20]) A pointed connected scheme $(X, \bar{x})$ satisfying (4.1) is called a $K(\pi, 1)$ if for every locally constant constructible abelian sheaf $\mathcal{F}$ on X , the natural maps

$$\rho^q : H^q(\pi_1(X, \bar{x}), \mathcal{F}_{\bar{x}}) \rightarrow H^q(X, \mathcal{F}) \quad (4.2)$$

are isomorphisms for all $q \geq 0$. This notion is independent of the choice of the geometric base point $\bar{x}$. We call a scheme X satisfying (4.1) a $K(\pi, 1)$ if its connected components are $K(\pi, 1)$ schemes.

Note that this is stronger than the notion used in *op.cit.*, as we do not require that $\mathcal{F}$ be of torsion order invertible on X . The reference [4, §9] contains the most detailed discussion of this and related notions.

Proposition 4.2 *Let X be a scheme satisfying (4.1).*

- (a) *X is a $K(\pi, 1)$ if and only if for every locally constant constructible abelian sheaf $\mathcal{F}$ on X , and every class $\zeta \in H^q(X, \mathcal{F})$ with $q > 0$, there exists a finite étale surjective map $f: X' \rightarrow X$ such that $f^*(\zeta) = 0 \in H^q(X', f^*\mathcal{F})$.*
- (b) *Let $f: X' \rightarrow X$ be a finite étale surjective map. Then X is a $K(\pi, 1)$ if and only if X' is.*
- (c) *X is a $K(\pi, 1)$ if and only if for every prime ℓ , every locally constant constructible $\mathbf{F}_\ell$ -sheaf $\mathcal{F}$ on X , and every class $\zeta \in H^q(X, \mathcal{F})$ with $q > 0$, there exists a finite étale surjective map $f: X' \rightarrow X$ such that $f^*(\zeta) = 0 \in H^q(X', f^*\mathcal{F})$.*

Proof Assertions (ab) are [3, Proposition 3.2(ab)]. For (c), note that functoriality of the long cohomology exact sequence implies that if

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

is a short exact sequence of locally constant constructible sheaves and $\mathcal{F}'$ and $\mathcal{F}''$ satisfy the assertion of (a) after pulling back along every finite étale $f: X' \rightarrow X$, then so does $\mathcal{F}$. Since every locally constant constructible sheaf $\mathcal{F}$ has a finite filtration whose quotients are $\mathbf{F}_\ell$ -sheaves for various primes ℓ , and every X' finite étale over X is a $K(\pi, 1)$ if X is [by (b)], the ‘if’ part of (c) follows, and the ‘only if’ part is obvious. $\square$

Lemma 4.3 *The maps ρ^q are isomorphisms for $q \leq 1$. (In fact, this holds for $q = 0$ and sheaves of sets, and for $q = 1$ and sheaves of groups as well.) Therefore schemes of cohomological dimension ≤ 1 (in particular, affine schemes of finite type of dimension ≤ 1 over a separably closed field) are $K(\pi, 1)$.*

Proof For $q = 0$, cf. [4, Proposition 9.17]. The statement for $q = 1$ follows from the torsor interpretation of H^1 (cf. [33, Remark 5.2]): a class $\zeta \in H^1(X, \mathcal{F})$ corresponds to an isomorphism class of an $\mathcal{F}$ -torsor $f: X' \rightarrow X$. The pullback $X' \times_X X' \rightarrow X'$ has a section, and hence is a trivial $f^*\mathcal{F}$ -torsor, thus the corresponding class $f^*\zeta \in H^1(X', f^*\mathcal{F})$ is zero. $\square$

Proposition 4.4 *Let $(X, \bar{x})$ be a pointed connected noetherian scheme. Assume moreover that X is geometrically unibranch ([21, 6.15.1], e.g. X normal). Then X is a $K(\pi, 1)$ if and only if $\pi_q(X, x) = 0$ for $q > 1$, where $\pi_q(X, x)$ is the étale homotopy group of Artin–Mazur [5].*

Proof Consider the natural map of sites $\rho: X_{\text{ét}} \longrightarrow B\pi_1(X, x)$ and the associated map $\Pi\rho$ where Π is the Verdier functor [5, §9]. On the one hand, X is a $K(\pi, 1)$ if and only if $\Pi\rho$ is a $\natural$ -isomorphism (cf. [5, Theorem 4.3]). On the other hand, $\Pi\rho$ induces an isomorphism on π_1 and $\pi_q(B\pi_1(X, x)) = 0$ for $q > 1$, so $\pi_q(X, x) = 0$ for $q > 1$ if and only if $\Pi\rho$ is a weak equivalence. Both source and target of $\Pi\rho$ being pro-finite (thanks to X being geometrically unibranch, [5, Theorem 11.1]), we conclude by [5, Corollary 4.4]. $\square$

Example 4.5 Let p be a prime. Then every connected affine $\mathbf{F}_p$ -scheme X is a $K(\pi, 1)$ for p -torsion coefficients, that is, condition (c) of Proposition 4.2 holds for $\ell = p$. Let $\mathcal{F}$ be a locally constant constructible $\mathbf{F}_\ell$ -sheaf on X , and let $\zeta \in H^q(X, \mathcal{F})$ ($q > 0$). We need to find a finite étale cover $X' \longrightarrow X$ killing ζ . First, we can assume that $\mathcal{F}$ is constant, as there exists a finite étale cover $X' \longrightarrow X$ such that the pullback of $\mathcal{F}$ to X' is constant, with X' affine and connected. Second, we can reduce to the case $\mathcal{F} = \mathbf{F}_p$. In this case, the Artin–Schreier sequence on $X_{\text{ét}}$

$$0 \longrightarrow \mathbf{F}_p \longrightarrow \mathcal{O}_X \xrightarrow{1-F} \mathcal{O}_X \longrightarrow 0$$

together with Serre vanishing ($H^q(X_{\text{ét}}, \mathcal{O}_X) = H^q(X, \mathcal{O}_X) = 0$ for $q > 0$) shows that $H^q(X, \mathbf{F}_p) = 0$ for $q > 1$. Thus if $q > 1$, we are done. If $q = 1$, then ζ corresponds to an $\mathbf{F}_p$ -torsor on $X_{\text{ét}}$, which again can be made trivial by a finite étale $X' \longrightarrow X$.

This example has been recently used by Scholze [41, Theorem 4.9] to show that any Noetherian affinoid adic space over $\text{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ is a $K(\pi, 1)$ for p -adic coefficients. We will follow Scholze’s argument to prove that such spaces are in fact $K(\pi, 1)$ for all coefficients in Sect. 6.

5 Affine $\mathbf{F}_p$ -schemes are $K(\pi, 1)$

5.1 The affine space is a $K(\pi, 1)$

We start by showing that $\mathbf{A}_k^n$ over a field k of characteristic $p > 0$ is a $K(\pi, 1)$ by induction on n , using Theorem 1.3 in the induction step. Let us sketch the idea of the proof. By the characterization of Proposition 4.2(b), being a $K(\pi, 1)$ means being able to kill nonzero degree cohomology classes of locally constant constructible $\mathbf{F}_\ell$ -sheaves (ℓ an arbitrary prime) using finite étale covers. The case $\ell = p$ follows easily from Artin–Schreier theory, so suppose $\ell \neq p$. If $\mathcal{F}$ is such a sheaf, then Theorem 1.3 for $\mathcal{F}$ implies that for a certain fibration $\pi: \mathbf{A}_k^{n+1} \longrightarrow \mathbf{A}_k^n$, the higher direct image sheaves $R^i\pi_*\mathcal{F}$ are locally constant, and hence one can kill their cohomology using finite étale covers of $\mathbf{A}_k^n$. We derive the corresponding statement for $\mathcal{F}$ using the Leray spectral sequence of π .

Theorem 5.1 *Let k be a field. Then the affine space $\mathbf{A}_k^\eta$ is a $K(\pi, 1)$ scheme.*

Proof We prove this by induction on $n \geq 0$. We can assume that k is an infinite field of characteristic $p > 0$. Let $\mathcal{F}$ be a locally constant constructible abelian sheaf on $\mathbf{A}_k^{\eta+1}$. We want to show that for every class $\zeta \in H^q(\mathbf{A}_k^{\eta+1}, \mathcal{F})$ ($q > 0$) there exists a finite étale surjective $f: X' \rightarrow \mathbf{A}_k^{\eta+1}$ such that $f^*(\zeta) = 0 \in H^q(X', f^*\mathcal{F})$. This is automatic for $q = 1$ (Lemma 4.3), so we can assume $q > 1$. Moreover, by Proposition 4.2(c), we can assume that $\mathcal{F}$ is an $\mathbf{F}_\ell$ -sheaf for a certain prime ℓ . The case $\ell = p$ is handled by Example 4.5, so we can assume $\ell \neq p$.

By Theorem 1.3, there exists an $\mathbf{A}^1$ -bundle $\pi: \mathbf{A}_k^{\eta+1} \rightarrow \mathbf{A}_k^\eta$ such that the sheaves $R^i\pi_*\mathcal{F}$ are locally constant for $i \geq 0$, with formation commuting with base change. Since the fibers are $\mathbf{A}_k^1$, we have $R^i\pi_*\mathcal{F} = 0$ for $i > 1$, so the Leray spectral sequence for π has only two nonzero rows, yielding an exact sequence

$$\dots \rightarrow H^q(\mathbf{A}_k^\eta, \pi_*\mathcal{F}) \rightarrow H^q(\mathbf{A}_k^{\eta+1}, \mathcal{F}) \rightarrow H^{q-1}(\mathbf{A}_k^\eta, R^1\pi_*\mathcal{F}) \rightarrow \dots$$

Let ζ_0 be the image of ζ in $H^{q-1}(\mathbf{A}_k^\eta, R^1\pi_*\mathcal{F})$. Since $q > 1$ and $R^1\pi_*\mathcal{F}$ is locally constant, the $K(\pi, 1)$ property of $\mathbf{A}_k^\eta$ implies that there exists a finite étale surjective $X \rightarrow \mathbf{A}_k^\eta$ killing ζ_0 .

Replace $\mathbf{A}_k^{\eta+1}$ with $X' = X \times_{\mathbf{A}_k^\eta} \mathbf{A}_k^{\eta+1}$, $\pi: \mathbf{A}_k^{\eta+1} \rightarrow \mathbf{A}_k^\eta$ with its pullback $\pi': X' \rightarrow X$, $\mathcal{F}$ and ζ with their pullbacks $\mathcal{F}'$, ζ' to X' . We again have an exact sequence as above, but now since ζ' maps to 0 in $H^{q-1}(X, R^1\pi'_*\mathcal{F}')$, it is the pullback of a class $\zeta_1 \in H^q(X, \pi'_*\mathcal{F}')$. Again, since X is a $K(\pi, 1)$ and $\pi'_*\mathcal{F}'$ is locally constant, we conclude that there is a finite étale surjective $Y \rightarrow X$ killing ζ_1 , and then $Y' = Y \times_X X' \rightarrow \mathbf{A}_k^{\eta+1}$ kills ζ , as desired. $\square$

5.2 Étale schemes over the affine space

Next, we deal with affine schemes endowed with an étale map to $\mathbf{A}_k^\eta$. To this end, we employ the following result.

Proposition 5.2 *Let k be a field of characteristic $p > 0$. Let U be an affine scheme of finite type over k , and let $g: U \rightarrow \mathbf{A}_k^\eta$ be an étale map. Then there exists a finite étale map $f: U \rightarrow \mathbf{A}_k^\eta$.*

Proof This is a variant of Nagata's proof of Noether normalization (cf. [32, I §1]). Write $U = \operatorname{Spec} R$, $R = k[x_1, \dots, x_n, x_{n+1}, \dots, x_r]/I$, where $x_1, \dots, x_n$ are the pull-backs of the coordinates on $\mathbf{A}_k^\eta$ via g . We shall prove a slightly stronger statement: *given any $x_1, \dots, x_r \in R$ such that $r \geq n = \dim R$ and $x_1, \dots, x_n$ are algebraically independent over k , there exist $y_1, \dots, y_n \in R$ such that R is finite over $k[x_1 + y_1^p, \dots, x_n + y_n^p]$.* This implies what we

want to prove because $dy_i^p = 0$, so $f = (x_1 + y_1^p, \dots, x_n + y_n^p)$ is étale if and only if g is.

The proof of this assertion is by induction on $r - n$: if $r = n$, g is a closed immersion, and we take $f = g$. For the induction step, pick positive integers $a_1, \dots, a_{r-1}$ and consider the elements

$$x'_i = x_i - x_r^{pa_i}, \quad i = 1, \dots, nr - 1.$$

Pick a nonzero $g \in I$, so that we have the relation

$$g(x'_1 + x_r^{pa_1}, \dots, x'_{r-1} + x_r^{pa_{r-1}}, x_r) = 0 \text{ in } R.$$

By the usual argument, for $0 \ll a_1 \ll a_2 \ll \dots \ll a_{r-1}$, this will be a monic polynomial in x_r with coefficients in $k[x'_1, \dots, x'_{r-1}]$. This means that if $R' \subseteq R$ is the subring generated by $x'_1, \dots, x'_{r-1}$, then x_r is integral over R' . Since $x_i = x'_i + x_r^{pa_i}$, the other x_i are also integral over R' , hence R is integral over R' . As R' is generated by $r - 1$ elements, with $x'_1, \dots, x'_n$ among them, we can apply the induction assumption to R' , $x'_1, \dots, x'_n$ to find $y'_i \in R'$ such that R' is finite over $k[x'_1 + (y'_1)^p, \dots, x'_n + (y'_n)^p]$. Thus R is finite over $k[x_1 + y_1^p, \dots, x_n + y_n^p]$ where $y_i = y'_i - x_r^{ai}$.

Remark 5.3 A related result has been obtained by Kedlaya [27]. As far as the author can tell, the trick of adding p -th powers to make a given map finite while preserving étaleness goes back to Abhyankar [2]. The author learned this technique from Katz's lectures [26]. We have previously used a variant of this fact in a mixed characteristic situation [3, Proposition 5.4]. In Sect. 6.6, we will give a rigid analytic variant of Proposition 5.2.

Corollary 5.4 *Let U be an affine scheme of finite type over k , and let $g: U \rightarrow \mathbf{A}_k^n$ be an étale map. Then U is a $K(\pi, 1)$ and for every geometric point $\bar{u}$ of U , $\pi_1(U, \bar{u})$ is isomorphic to an open subgroup of $\pi_1(\mathbf{A}_k^n, 0)$.*

Proof This follows directly from Proposition 5.2, Theorem 5.1, and Proposition 4.2(b). $\square$

5.3 Henselian pairs and Gabber's theorem

Recall [35, Chapitre XI] that a *henselian pair* is a pair (A, I) consisting of a ring A and an ideal $I \subseteq A$ such that for every étale A -algebra A' , the restriction map

$$\mathrm{Hom}_A(A', A) \longrightarrow \mathrm{Hom}_A(A', A/I)$$

is a bijection. If (A, I) is any pair, the *henselization* $(A, I) \rightarrow (A^h, I^h)$ of (A, I) is the initial map of pairs to a henselian pair. Henselization exists, and can be constructed as the inductive limit of pairs (B, J) indexed by étale A -algebras B endowed with a map of A -algebras $\sigma: B \rightarrow A/I$, and $J = \ker(\sigma)$ (cf. [20, §0]). Moreover $I^h = I \cdot A^h$.

Let (A, I) be a henselian pair. We set $X = \operatorname{Spec} A$, $X_0 = \operatorname{Spec} A/I$, $i: X_0 \rightarrow X$ the closed immersion. By [20, §1], the restriction functors

$$\begin{aligned} i^*: (\text{finite étale covers of } X) &\rightarrow (\text{finite étale covers of } X_0), \\ i^*: (\text{lcc sheaves on } X) &\rightarrow (\text{lcc sheaves on } X_0) \end{aligned}$$

are equivalences. For higher cohomology, we have the following.

Theorem 5.5 [20, Theorem 1] *In the above situation, for every torsion abelian sheaf $\mathcal{F}$ on X the restriction maps*

$$i^*: H^q(X, \mathcal{F}) \rightarrow H^q(X_0, i^* \mathcal{F})$$

are isomorphisms for all $q \geq 0$.

Corollary 5.6 *In the above situation, X is a $K(\pi, 1)$ if and only if X_0 is a $K(\pi, 1)$.*

Proof If $Y = \operatorname{Spec} B$ is a finite étale X -scheme and $J = IB$, then (B, J) is a henselian pair, and hence the above statements hold for (B, J) . Let $\mathcal{F}$ be an lcc sheaf on X , $\zeta \in H^q(X, \mathcal{F}) \simeq H^q(X_0, i^* \mathcal{F})$ ($q > 0$). To show that X (resp. X_0) is a $K(\pi, 1)$ means that for every such pair $(\mathcal{F}, \zeta)$, there exists a finite étale surjective $Y \rightarrow X$ (resp. $Y_0 \rightarrow X_0$) killing ζ . By the above remarks, X is thus a $K(\pi, 1)$ if and only if X_0 is. $\square$

5.4 The general case

Finally, we deal with general connected affine $\mathbf{F}_p$ -schemes.

Lemma 5.7 *Suppose that A_∞ is a ring with no nontrivial idempotents which is the union of a filtered family of subrings $A_i \subseteq A_\infty$ such that $X_i = \operatorname{Spec} A_i$ is a $K(\pi, 1)$. Then $X = \operatorname{Spec} A_\infty$ is a $K(\pi, 1)$.*

Proof Let $\mathcal{F}$ be a locally constant constructible sheaf on X , $\zeta \in H^q(X, \mathcal{F})$ where $q > 0$. We need to find a finite étale surjective $f: X' \rightarrow X$ such that $f^*(\zeta) = 0 \in H^q(X', f^* \mathcal{F})$. There exists a sheaf $\mathcal{F}_i$ on X_i for some i such that $\mathcal{F} \simeq \mathcal{F}_i \otimes_{A_i} A_\infty$. Since the limit is filtered, we can restrict ourselves to A_j for $j \geq i$; let $\mathcal{F}_j = \mathcal{F} \otimes_{A_i} A_j$. The natural map

$$\varinjlim_{j \geq i} H^q(X_j, \mathcal{F}_j) \rightarrow H^q(X, \mathcal{F})$$

is an isomorphism, and hence there exists a j such that ζ is the image of a class $\zeta_j \in H^q(X_j, \mathcal{F}_j)$. Since X_j is a $K(\pi, 1)$, there exists a finite étale surjective $f_j: X'_j \rightarrow X_j$, but then the base change $f: X' = X'_j \times_{X_j} X \rightarrow X$ kills ζ , as desired. $\square$

Theorem (1.1) *Every connected affine scheme over $\mathbf{F}_p$ is a $K(\pi, 1)$ scheme.*

Proof Let $X = \operatorname{Spec} A$ be a connected affine scheme over $\mathbf{F}_p$. For a finite subset $S \subseteq A$, let $\mathbf{F}_p[S]$ be the subring of A generated by S . Then A is the union of all such rings $\mathbf{F}_p[S]$. Therefore by Lemma 5.7 it suffices to treat the case when A is generated over $\mathbf{F}_p$ by a finite number of elements $a_1, \dots, a_n$. These elements exhibit X as a closed subscheme of $\mathbf{A}_k^n = \operatorname{Spec} P$, $P = k[x_1, \dots, x_n]$ ($k = \mathbf{F}_p$) defined by an ideal $I \subseteq P$. Let $X^h = \operatorname{Spec} P^h$ be the henselization of $\mathbf{A}_k^n$ along X . By definition, P^h is the inductive limit of étale P -algebras B endowed with a section over X , i.e. a P -algebra homomorphism $B/IB \rightarrow P/I = A$. By Corollary 5.4, each $\operatorname{Spec} B$ is a $K(\pi, 1)$. By Lemma 5.7, X^h is a $K(\pi, 1)$. But (P^h, IP^h) is a henselian pair and $P^h/IP^h = P/I = A$. Thus $X = \operatorname{Spec} A$ is a $K(\pi, 1)$ by Corollary 5.6. $\square$

6 Mixed characteristic and rigid analytic variants

6.1 Review of rigid geometry

We recall the setup for rigid geometry in the sense of Raynaud, for which we follow [1] (albeit we will only need the noetherian case, as opposed to the more general idyllic case treated in that book). An *adic ring* is a complete and separated topological ring A admitting a finitely generated ideal $I \subseteq A$ such that the ideals I^n are open and form a basis of neighborhoods of $0 \in A$. Such an I is called an *ideal of definition*. A formal scheme $\mathcal{X}$ is called *adic* if it is locally of the form $\operatorname{Spf} A$ for an adic ring A . An *admissible blow-up* is a morphism $\varphi: \mathcal{X}' \rightarrow \mathcal{X}$ of finite type between adic formal schemes which is isomorphic to the blow-up of $\mathcal{X}$ of a finitely generated open ideal. The category $\mathbf{R}$ of coherent rigid spaces is the localization of the category $\mathbf{S}$ of noetherian quasi-compact formal schemes and morphisms of finite type with respect to admissible blow-ups. If $\mathcal{X}$ is an object of $\mathbf{S}$, we denote the associated object of $\mathbf{R}$ by $\mathcal{X}^{\text{rig}}$. We call $\mathcal{X}^{\text{rig}}$ the *associated rigid space* of $\mathcal{X}$. A *formal model* of a coherent rigid space X is an object $\mathcal{X}$ of $\mathbf{S}$ together with an isomorphism $X \simeq \mathcal{X}^{\text{rig}}$. An *affinoid rigid space* is a coherent rigid space admitting a formal model of the form $\operatorname{Spf}(A)$ for a noetherian adic ring A which locally admits a principal ideal of definition.

One has natural notions of finite and étale morphisms in $\mathbf{R}$, which allow one to define the rig-étale topos $X_{\text{ét}}$ of a rigid space X .

6.2 The Gabber–Fujiwara theorem

Let (A, I) be a noetherian henselian pair, $\hat{A}$ the I -adic completion of A . Our goal is to compare the cohomology of the affinoid rigid space $X = \mathrm{Spf}(\hat{A})^{\mathrm{rig}}$ to the cohomology of the scheme $U = \mathrm{Spec} A \setminus V(I)$. The GAGA functor [1, Chap. VII] produces a morphism of topoi

$$\varepsilon: X_{\mathrm{\acute{e}t}} \longrightarrow U_{\mathrm{\acute{e}t}},$$

inducing equivalences as in Sect. 5.3

$$\begin{aligned} \varepsilon^*: (\text{finite \acute{e}tale covers of } U) &\xrightarrow{\sim} (\text{finite \acute{e}tale covers of } X), \\ \varepsilon^*: (\text{lcc sheaves on } U) &\xrightarrow{\sim} (\text{lcc sheaves on } X). \end{aligned}$$

Theorem 6.1 (Gabber–Fujiwara, [19, Corollary 6.6.3]) *In the above situation, for every torsion abelian sheaf $\mathcal{F}$ on U the maps*

$$\varepsilon^*: H^q(U, \mathcal{F}) \longrightarrow H^q(X, \varepsilon^* \mathcal{F})$$

are isomorphisms for $q \geq 0$.

6.3 Affinoid rigid spaces in characteristic p

With the Gabber–Fujiwara theorem in place, we can easily deduce from Theorem 1.1 its rigid analytic variant.

Definition 6.2 We call a rigid space X a $K(\pi, 1)$ space if for every locally constant constructible \acute{e}tale sheaf $\mathcal{F}$ on X , and every class $\zeta \in H^q(X, \mathcal{F})$ for $q > 0$, there exists a finite \acute{e}tale surjective map $f: X' \longrightarrow X$ such that $f^*(\zeta) = 0 \in H^q(X', f^* \mathcal{F})$.

Lemma 6.3 *Let (A, I) be a noetherian henselian pair, $\hat{A}$ the I -adic completion of A . Then $\mathrm{Spf}(\hat{A})^{\mathrm{rig}}$ is a $K(\pi, 1)$ rigid space if and only if $\mathrm{Spec}(A) \setminus V(I)$ is a $K(\pi, 1)$ scheme.*

Proof This is analogous to the proof of Corollary 5.6. Let $f: V \longrightarrow U$ be a finite \acute{e}tale cover, B the normalization of A in V , $J = IB$. Then (B, J) is a noetherian henselian pair, and $V = \mathrm{Spec} B \setminus V(J)$. Moreover, if $Y = \mathrm{Spf}(\hat{B})^{\mathrm{rig}}$, then $g: Y \longrightarrow X$ is the finite \acute{e}tale cover corresponding to f via the equivalence ε^* , and the square

$$\begin{array}{ccc}
 V_{\text{ét}} & \xrightarrow{\varepsilon} & Y_{\text{ét}} \\
 f \downarrow & & \downarrow g \\
 U_{\text{ét}} & \xrightarrow{\varepsilon} & X_{\text{ét}}
 \end{array}$$

2-commutes. Consequently, if $\mathcal{F}$ is an abelian sheaf on $U_{\text{ét}}$, then the square

$$\begin{array}{ccc}
 H^*(V, f^* \mathcal{F}) & \xrightarrow{\varepsilon^*} & H^*(Y, \varepsilon^*(f^* \mathcal{F})) \\
 f^* \uparrow & & \uparrow g^* \\
 H^*(U, \mathcal{F}) & \xrightarrow{\varepsilon^*} & H^*(X, \varepsilon^* \mathcal{F})
 \end{array}$$

commutes. By Theorem 6.1, the horizontal arrows are isomorphisms if $\mathcal{F}$ is a torsion sheaf.

Let $\mathcal{F}$ be a locally constant constructible sheaf on U , $\zeta \in H^q(U, \mathcal{F}) \cong H^q(X, \varepsilon^* \mathcal{F})$. If U (resp. X) is a $K(\pi, 1)$ then there exists a finite étale surjective $f: V \rightarrow U$ (resp., $g: Y \rightarrow X$) such that $f^*(\zeta) = 0$ (resp. $g^*(\zeta) = 0$). The equivalence is thus clear in view of the preceding discussion. $\square$

Theorem 6.4 *Let $X = \text{Spf}(A)^{\text{rig}}$ be an affinoid rigid space such that $pA = 0$. Then X is a $K(\pi, 1)$ space.*

Proof Let $I \subseteq A$ be an ideal of definition, and let $U = \text{Spec } A \setminus V(I)$. Since A locally admits a principal ideal of definition, the map $U \rightarrow \text{Spec } A$ is affine, and hence U is an affine scheme. Thus U is a $K(\pi, 1)$ scheme by Theorem 1.1. Lemma 6.3 implies that X is a $K(\pi, 1)$ rigid analytic space. $\square$

6.4 Affinoid rigid spaces in mixed characteristic

To deduce the mixed characteristic case, we use perfectoid spaces. Since they are by definition adic spaces, not rigid spaces, we have to consider $K(\pi, 1)$ adic spaces: we will simply call an adic space a $K(\pi, 1)$ if the condition of Definition 6.2 holds. If X is a rigid space, X^{ad} the associated adic space, then the étale topoi $X_{\text{ét}}$ and $X_{\text{ét}}^{\text{ad}}$ are equivalent (cf. [24, Proposition 2.1.4]), and X is a $K(\pi, 1)$ rigid space if and only if X^{ad} is a $K(\pi, 1)$ adic space. Our argument is analogous to [41, Theorem 4.9].

Proposition 6.5 *Let (A, A^+) be a perfectoid algebra over a perfectoid field K . Then $\text{Spa}(A, A^+)$ is a $K(\pi, 1)$ adic space.*

Proof Suppose first that K has characteristic p . Without loss of generality, we can assume that K is the completed perfection of $\mathbf{F}_p((t))$. In this case, A^+ is a

K° -algebra. By [40, Lemma 6.13 (i)], (A, A^+) is the completion of a filtered direct limit of p -finite perfectoid affinoid K -algebras $(A_i, A_i^+ = A_i^\circ)$. If we set $X_i = \mathrm{Spa}(A_i, A_i^+)$, then every finite étale cover of X comes by base change from a finite étale cover of X_i . Thus if $\mathcal{F}$ is a locally constant constructible sheaf on X , we can assume that there exists an i_0 such that $\mathcal{F}$ is the base change of a locally constant constructible sheaf $\mathcal{F}_{i_0}$ on X_{i_0} . For $i \geq i_0$, let $\mathcal{F}_i$ be the pullback of $\mathcal{F}_{i_0}$ to X_i . The natural map

$$\lim_{\substack{\longrightarrow \\ i \geq i_0}} H^q(X_i, \mathcal{F}_i) \longrightarrow H^q(X, \mathcal{F})$$

is an isomorphism ([40, Corollary 7.18] combined with [40, Corollary 7.19]). These remarks show that if all X_i are $K(\pi, 1)$, so is X . Therefore it suffices to treat the case when X is p -finite.

By definition, being p -finite means being the completed perfection of an affinoid algebra $(B, B^+ = B^\circ)$ topologically of finite type over K . For such algebras, $\mathrm{Spa}(B, B^+)$ is a $K(\pi, 1)$ by the noetherian case (Theorem 6.4). But $\mathrm{Spa}(A, A^+) \longrightarrow \mathrm{Spa}(B, B^+)$ induces an equivalence of the étale topoi ([40, Corollary 7.19]), and hence $\mathrm{Spa}(A, A^+)$ is a $K(\pi, 1)$.

Finally, we treat the case when K has characteristic zero. Let X^b/K^b be the tilt. Then by [40, Theorem 7.12], the étale topoi of X and X^b are equivalent, and the same holds for all finite étale covers of X and X^b is a compatible way. Thus X is a $K(\pi, 1)$ if and only if X^b is. $\square$

Theorem 6.6 *Let X be (1) an affinoid noetherian adic space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ or (2) $X = \mathrm{Spf}(A)^{\mathrm{rig}}$ for a noetherian p -adic ring A . Then X is a $K(\pi, 1)$ space.*

Proof By [24, Proposition 2.1.4], (2) follows from (1). By the proof of [41, Theorem 4.9], there exists a system of finite étale covers X_i of X and an affinoid perfectoid X_∞ over $\mathbf{C}_p$ such that

$$X_\infty \sim \varprojlim X_i \quad (\text{cf. [40, Definition 7.14]}).$$

Then X_∞ is a $K(\pi, 1)$ by Proposition 6.5, and we conclude that X is a $K(\pi, 1)$ by [40, Corollary 7.18]. $\square$

6.5 Application to p -adic Milnor fibers

We can apply the Gabber–Fujiwara theorem once again, now in mixed characteristic, to go back to the henselian case.

Theorem 6.7 *Let A be a noetherian $\mathbf{Z}_{(p)}$ -algebra such that (A, pA) is a henselian pair. Then $\mathrm{Spec} A[1/p]$ is a $K(\pi, 1)$ scheme.*

Proof By Lemma 6.3, $\mathrm{Spec} A[1/p]$ is a $K(\pi, 1)$ if and only if the rigid space $\mathrm{Spf}(\hat{A})^{\mathrm{rig}}$ is a $K(\pi, 1)$, where $\hat{A}$ is the p -adic completion of A . The latter is a $K(\pi, 1)$ by Theorem 6.6. $\square$

Fix a strictly henselian discrete valuation ring V with residue field k of characteristic $p > 0$ and fraction field K of characteristic zero. Let $s = \mathrm{Spec} k$, $\eta = \mathrm{Spec} K$, $\bar{\eta} = \mathrm{Spec} \bar{K}$ where $\bar{K}$ is an algebraic closure of K . For a scheme X of finite type over $S = \mathrm{Spec} V$ and a geometric point $\bar{x}$ of X_s , the *Milnor fiber* of X at $\bar{x}$ is the scheme

$$M_{\bar{x}} = X_{(\bar{x})} \times_S \bar{\eta}.$$

Corollary 6.8 *Let X be a V -scheme of finite type. Then for every geometric point $\bar{x}$ of X_s , the Milnor fiber $M_{\bar{x}}$ is a $K(\pi, 1)$ scheme.*

This allows us to strengthen the main result of [3], removing the log smoothness hypothesis (but only in the case of $X^\circ = X$).

Corollary 6.9 *Let X be a V -scheme of finite type. Consider the Faltings' topos $\tilde{E}$ of $X_{\bar{\eta}} \rightarrow X$ and the morphism of topoi*

$$\Psi : X_{\bar{\eta}, \text{ét}} \longrightarrow \tilde{E}.$$

Let $\mathcal{F}$ be a locally constant constructible abelian sheaf on $X_{\bar{\eta}, \text{ét}}$. Then $R^i \Psi_ \mathcal{F} = 0$ for $i > 0$, and the natural maps*

$$H^*(\tilde{E}, \Psi_* \mathcal{F}) \longrightarrow H^*(X_{\bar{\eta}, \text{ét}}, \mathcal{F})$$

are isomorphisms.

Proof Corollary 6.8 shows that condition (B) of [3, §1.2] holds. This implies condition (D) of *op.cit.* which is exactly our assertion. $\square$

6.6 Étale affinoids over the polydisc

Interestingly, we have a rigid analytic variant of Proposition 5.2 (cf. [3, Proposition 5.10]).

Proposition 6.10 *Let K be a complete discretely valued field whose residue field k is of characteristic $p > 0$, U an affinoid rigid analytic space of finite type over K , $\mathbf{B}^n$ the formal n -polydisc over K (the rigid-analytic generic fiber of $\mathbf{A}_{\mathcal{O}_K^n}$), $g : U \rightarrow \mathbf{B}^n$ an étale morphism. Then there exists a finite étale morphism $f : U \rightarrow \mathbf{B}^n$.*

Proof Let V be the valuation ring of K . Let R be the Tate algebra corresponding to U , let $R^\circ \subseteq R$ be the integral subring, and let $x_1, \dots, x_n \in R^\circ$ be the pull-backs of the coordinates on $\mathbf{B}^n$ via g . Pick $x_{n+1}, \dots, x_r \in R^\circ$ such that $x_1, \dots, x_r$ form a set of topological generators of R° over V . This gives us a presentation

$$R^\circ \cong V\langle x_1, \dots, x_n, x_{n+1}, \dots, x_r \rangle / I, \quad R \cong R^\circ \left[\frac{1}{\pi} \right]$$

where $\pi \in V$ is a uniformizer. Let $\Omega_g = \hat{\Omega}_{R^\circ/V\langle x_1, \dots, x_n \rangle}$. Since g is étale, we have $\mathcal{H}_{\text{rig}}^0(\Omega_g) = 0$, and hence Ω_g is killed by a power of π , say $\pi^m \cdot \Omega_g = 0$. By [3, Proposition 5.4] applied to the reduction of g modulo π and $N = p^{m+1}$, there exist elements $\bar{y}_1, \dots, \bar{y}_n \in \bar{R}$ which are polynomials in p^{m+1} -th powers of elements of $\bar{R}$, such that the map $\text{Spec } \bar{R} \rightarrow \mathbf{A}_k^\eta$ given by $\bar{x}_1 + \bar{y}_1, \dots, \bar{x}_n + \bar{y}_n$ is finite. Let y_i be any lifts of $\bar{y}_i$ to R° which are polynomials in p^{m+1} -powers of elements of R° . Arguing as in [3, Lemma 5.9], we see that the map $f: U \rightarrow \mathbf{B}^n$ given by $x_1 + y_1, \dots, x_n + y_n$ is étale. It is also finite, because its reduction modulo π is. $\square$

7 Examples and complements

7.1 Linear projections do not suffice

Let k be an algebraically closed field of characteristic $p > 0$, and let $m = p^e > 2$ for some integer $e \geq 1$. Let x and y be coordinates on $\mathbf{A}_k^2$ and let $\mathcal{F} = \mathcal{L}_{\psi, x^{m-1}y}$ be the Artin–Schreier sheaf associated to a nontrivial character $\psi: \mathbf{Z}/p\mathbf{Z} \rightarrow \mathbf{F}_\ell^\times$ and the function $x^{m-1}y$. Let $\pi: \mathbf{A}_k^2 \rightarrow \mathbf{A}_k^1$ be a surjective linear map. The following lemma shows that $R^1\pi_!\mathcal{F}$ is not locally constant. Consequently, the assertion of Proposition 3.4 is false for sheaves with fierce ramification at infinity.

Lemma 7.1 $\text{Sw}_\infty(\mathcal{F}|_{\pi^{-1}(0)}) \neq \text{Sw}_\infty(\mathcal{F}|_{\pi^{-1}(1)})$.

Proof Say $\pi(x, y) = ax + by$ with $a, b \in k$ not both zero. Suppose first that $b \neq 0$, then x is a coordinate on every fiber of π , and $y = -\frac{a}{b}x + \frac{1}{b}\pi(x, y)$. If $\pi(x, y) = 0$, then $x^{m-1}y = -\frac{a}{b}x^m$, and hence $\text{Sw}_\infty(\mathcal{F}|_{\pi^{-1}(0)}) = 1$ if $a \neq 0$, 0 if $a = 0$. If $\pi(x, y) = 1$, then $x^{m-1}y = -\frac{a}{b}x^m + \frac{1}{b}x^{m-1}$, so $\text{Sw}_\infty(\mathcal{F}|_{\pi^{-1}(1)}) = m - 1 > 1$. It remains to consider the case $b = 0$. Then y is a coordinate on every fiber and $x = \frac{1}{a}\pi(x, y)$, so $x^{m-1}y = 0$ if $\pi(x, y) = 0$, and $\text{Sw}_\infty(\mathcal{F}|_{\pi^{-1}(0)}) = 0$, and if $\pi(x, y) = 1$ then $x^{m-1}y = \frac{1}{a^{m-1}}y$ and $\text{Sw}_\infty(\mathcal{F}|_{\pi^{-1}(1)}) = 1$. In each case we have $\text{Sw}_\infty(\mathcal{F}|_{\pi^{-1}(0)}) \neq \text{Sw}_\infty(\mathcal{F}|_{\pi^{-1}(1)})$. $\square$

Corollary 7.2 *In the above situation, $R^q \pi_! \mathcal{F}$ is not locally constant for some $q \geq 0$.*

Proof For $t \in \mathbf{A}_k^1(k)$, let $\mathcal{F}_t = \mathcal{F}|_{\pi^{-1}(t)}$. By the Grothendieck–Ogg–Shafarevich formula (2.1), we have

$$\chi_c(\mathbf{A}_k^1, \mathcal{F}_t) = \mathrm{rk}(\mathcal{F}_t) \cdot \chi_c(\mathbf{A}_k^1, \mathbf{F}_\ell) - \mathrm{Sw}_\infty(\mathcal{F}_t) = 1 - \mathrm{Sw}_\infty(\mathcal{F}_t).$$

Thus Lemma 7.1 implies that the function $t \mapsto \chi_c(\mathbf{A}_k^1, \mathcal{F}_t)$ is not constant. On the other hand,

$$\chi_c(\mathbf{A}_k^1, \mathcal{F}_t) = \dim_{\mathbf{F}_\ell}(R^0 f_! \mathcal{F})_t - \dim_{\mathbf{F}_\ell}(R^1 f_! \mathcal{F})_t + \dim_{\mathbf{F}_\ell}(R^2 f_! \mathcal{F})_t,$$

and hence one of the sheaves $R^q f_! \mathcal{F}$ is not locally constant. $\square$

7.2 Complements of hyperplane arrangements

Theorem 1.1 implies that every complement of a hyperplane arrangement in $\mathbf{A}_k^n$ is a $K(\pi, 1)$. This is of course false over $\mathbf{C}$, and this contrast yields examples of interesting arithmetic behavior, the failure of the Lefschetz principle or the question of the existence of a finite étale cover killing a given étale cohomology class.

Remark 7.3 The question whether certain *complex* complements of hyperplane arrangements are $K(\pi, 1)$ (in the topological sense) has been extensively studied, cf. e.g. [11, 12, 14] or [34, §5.1]. Of course, the fundamental group of the complement of a hyperplane arrangement loses its link to combinatorics (or representation theory) when one passes to positive characteristic.

Proposition 7.4 *Let $X = \mathbf{A}_{\mathbf{Z}}^2 \setminus \{xy(x + y - 1) = 0\}$. Then*

- (1) $\pi_2(X(\mathbf{C})) \neq 0$ and $\pi_1(X(\mathbf{C})) \simeq \mathbf{Z}^3$. In particular, $X(\mathbf{C})$ is not a $K(\pi, 1)$ space, and its fundamental group is a good group in the sense of Serre [42, §2.6].
- (2) $X_{\mathbf{C}}$ is not a $K(\pi, 1)$ scheme. In particular X_K is not a $K(\pi, 1)$ scheme for every field K of characteristic zero.
- (3) X_k is a $K(\pi, 1)$ scheme for every field k of characteristic $p > 0$.

Proof (1) These statements follow from A. Hattori’s work [23] on the topology complements of generic hyperplane arrangements. See [34, Example 5.24] for a detailed discussion of this space, whose homotopy type is the same as that of the image Q of the boundary $\partial[0, 1]^3$ of the unit cube in $\mathbf{R}^3$ under the quotient map $\mathbf{R}^3 \rightarrow \mathbf{R}^3/\mathbf{Z}^3$. The inclusion $Q \hookrightarrow \mathbf{R}^3/\mathbf{Z}^3$ induces an isomorphism on fundamental groups, and the surjection $S^2 \simeq \partial[0, 1]^3 \rightarrow$

Q induces an injection on π_2 . (In [34, Example 5.24], the authors claim that this map is an isomorphism on π_2 , which seems to be incorrect.)

- (2) For a subgroup $\Lambda \subseteq \pi_1(X(\mathbf{C})) \simeq \mathbf{Z}^3$ of finite index, let us denote by $X_\Lambda \rightarrow X_{\mathbf{C}}$ the induced finite étale covering. Using the description of (1), $X_\Lambda(\mathbf{C})$ can be identified with the image in $\mathbf{R}^3/\Lambda$ of the union of the boundaries of all unit cubes in $\mathbf{R}^3$ with vertices in $\mathbf{Z}^3$. Let C be the boundary of one of these cubes, then for every finite abelian group M the composition

$$H^2(X(\mathbf{C}), M) \longrightarrow H^2(C, M) \simeq H^2(S^2, M) \simeq M$$

is easily seen to be surjective. In the commutative diagram

$$\begin{array}{ccccc} H^2(C, M) & \longleftarrow & \varinjlim_{\Lambda} H^2(X_{\Lambda}(\mathbf{C}), M) & \xleftarrow{\sim} & \varinjlim_{\Lambda} H^2(X_{\Lambda}, M) \\ & \nwarrow & \uparrow & & \uparrow \\ & & H^2(X(\mathbf{C}), M) & \xleftarrow{\sim} & H^2(X_{\mathbf{C}}, M), \end{array}$$

the slant arrow is surjective, and hence the left vertical arrow is nonzero. Consequently, the top right term is nonzero, which shows that $X_{\mathbf{C}}$ is not a $K(\pi, 1)$ by the characterization of Proposition 4.2 and the fact that $\pi_1(X_{\mathbf{C}})$ is the profinite completion of $\pi_1(X(\mathbf{C}))$. The case of arbitrary characteristic zero fields follows from [3, Proposition 3.2(c)]. (In fact, the term $\varinjlim_{\Lambda} H^2(X_{\Lambda}, M)$ equals $\mathrm{Hom}(\pi_2(X_{\mathbf{C}}), M)$ by [5, Proposition 6.3].)

- (3) This is a direct consequence of Theorem 1.1. $\square$

Remark 7.5 An easier example of a scheme presenting such behavior is the ‘2-sphere’

$$X = \{x^2 + y^2 + z^2 = 1\} \subseteq \mathbf{A}_{\mathbf{Z}[1/2]}^3.$$

After base change to a field k containing $\sqrt{-1}$, we can transform the equation to $xy = (z+1)(z-1)$. Since X is affine, X_k is a $K(\pi, 1)$ for every field k of odd characteristic. However, the map $\pi : X \rightarrow \mathbf{P}^1$ defined as

$$\pi(x, y, z) = \begin{cases} (x : z-1) & z \neq 1 \\ (z+1 : y) & z \neq -1 \end{cases}$$

makes X into an $\mathbf{A}^1$ -bundle over $\mathbf{P}^1$. Since $\mathbf{A}_K^1$ is simply connected when K is algebraically closed of characteristic zero, X_K is simply connected as well. Moreover, the induced map

$$\pi^* : \mathbf{Z}/n\mathbf{Z} \simeq H^2(\mathbf{P}_K^1, \mu_n) \longrightarrow H^2(X_K, \mu_n)$$

is an isomorphism, and hence X_K is not a $K(\pi, 1)$.

7.3 Fundamental groups of affine spaces

Let k be an algebraically closed field of characteristic $p > 0$. Recall that by the work of Raynaud on Abhyankar's conjecture [36], a finite group G arises as a quotient of $\pi_1(\mathbf{A}_k^1)$ if and only if G has no nontrivial quotient of order prime to p . It follows that $\pi_1(\mathbf{A}_k^n)$ has the same property for all $n \geq 1$, and hence the profinite groups $\pi_1(\mathbf{A}_k^n)$ and $\pi_1(\mathbf{A}_k^1)$ have the same finite quotients. One can ask naively whether $\pi_1(\mathbf{A}_k^n) \simeq \pi_1(\mathbf{A}_k^1)$ as profinite groups (we could deduce this from the previous statement if the groups were topologically finitely generated, cf. [16, Proposition 15.4]). It is easy to see that such an isomorphism cannot be induced by an algebraic morphism $\mathbf{A}_k^n \rightarrow \mathbf{A}_k^1$.

Proposition 7.6 *If $n \neq m$, then $\pi_1(\mathbf{A}_k^n)$ and $\pi_1(\mathbf{A}_k^m)$ are not isomorphic as profinite groups.*

Proof Theorem 5.1 implies that the cohomological dimension of $\pi_1(\mathbf{A}_k^n)$ equals the largest q for which there exists a locally constant constructible sheaf $\mathcal{F}$ on $\mathbf{A}_k^n$ with $H^q(\mathbf{A}_k^n, \mathcal{F}) \neq 0$. Thus it suffices to show that the latter equals n . This is easy and well-known, but we include a quick proof.

Let $\mathcal{F}_1$ be any $\mathbf{F}_\ell$ -sheaf on $\mathbf{A}_k^1$ (for some $\ell \neq p$) with $H^1(\mathbf{A}_k^1, \mathcal{F}_1) \neq 0$ (for example, the Artin–Schreier sheaf $\mathcal{L}_{\psi, x^m}$ where $m > 1$ is an integer prime to p and $\psi: \mathbf{Z}/p\mathbf{Z} \rightarrow \mathbf{F}_\ell^\times$ is a nontrivial character, cf. (2.2) and Proposition A.1.3). Let $\mathcal{F}_n = \mathcal{F}_1 \boxtimes \dots \boxtimes \mathcal{F}_1$ (n times). By the Künneth formula (cf. [15, Th. finitude, Corollaire 1.11]), $H^n(\mathbf{A}_k^n, \mathcal{F}_n) \simeq H^1(\mathbf{A}_k^1, \mathcal{F}_1)^{\otimes n} \neq 0$. On the other hand, $H^q(\mathbf{A}_k^n, \mathcal{F})$ vanishes for $q > n$ for all constructible sheaves $\mathcal{F}$, by Artin's theorem on the cohomological dimension of affine schemes [7, Exp. XIV, Corollaire 3.2]. $\square$

7.4 Pro- p completion and p -Sylow subgroups

Let ℓ be a prime. In analogy with the notion of a good group [42, §2.6], let us call a topological group G ℓ -good, if for every continuous representation of the pro- ℓ completion $G^{\text{pro-}\ell}$ on a finite dimensional $\mathbf{F}_\ell$ -vector space M , the natural maps

$$H^*(G^{\text{pro-}\ell}, M) \longrightarrow H^*(G, M)$$

are isomorphisms.

Proposition 7.7 *Let $(X, \bar{x})$ be connected noetherian pointed affine $\mathbf{F}_p$ -scheme. Then the following hold.*

(1) *The group $\pi_1(X, \bar{x})$ is a p -good group,*

- (2) The p -completion $\pi_1(X, \bar{x})^{\text{pro-}p}$ is a free p -group of rank equal to $\dim_{\mathbf{F}_p} \Gamma(X, \mathcal{O}_X)/(1 - F)$,
- (3) Every p -Sylow subgroup of $\pi_1(X, \bar{x})$ is a free p -group. In particular, $\pi_1(X, \bar{x})$ is p -torsion free.

Proof (1) For brevity, let us call a finite étale surjective $f: X' \rightarrow X$ a p -cover if it is Galois under the action of a finite p -group. Let M be a finite-dimensional $\mathbf{F}_p$ -vector space and let $\rho: \pi_1(X, \bar{x}) \rightarrow GL(M)$ be a representation whose image is a p -group. Let $\mathcal{F}$ be the associated sheaf on $\mathbf{A}_k^\eta$. Consider the commutative triangle

$$\begin{array}{ccc} H^*(\pi_1(X, \bar{x})^{\text{pro-}p}, M) & & \\ \downarrow & \searrow & \\ H^*(\pi_1(X, \bar{x}), M) & \longrightarrow & H^*(X, \mathcal{F}). \end{array}$$

Since the bottom map is an isomorphism by Theorem 1.1, the vertical arrow is an isomorphism if and only if the diagonal one is. To this end, let ρ_p be the natural map of topoi

$$\rho_p: X_{\text{ét}} \rightarrow X_{p\text{-fét}} \simeq B\pi_1(X, \bar{x})^{\text{pro-}p}.$$

Here $X_{p\text{-fét}}$ denotes the topos of sheaves on the full subcategory of the étale site of X consisting of p -covers, with the induced topology. We must show that $R^q \rho_{p*} \mathcal{F} = 0$ for $q > 0$. As usual, this is automatic for $q = 1$. By the definition of the higher direct images, this amounts to showing that for every p -cover $X' \rightarrow X$, and every class $\zeta \in H^q(X', \mathcal{F})$, there exists a p -cover X'' of X and a map $f: X'' \rightarrow X'$ over X such that $f^* \zeta = 0 \in H^q(X'', \mathcal{F})$. Since $\mathcal{F}$ is trivialized by the p -cover corresponding to the image of ρ , we can assume $\mathcal{F}$ is constant, and hence $\mathcal{F} = \mathbf{F}_p$. In this case $H^q(X, \mathbf{F}_p) = 0$ for $q > 1$ because X is affine, by Artin–Schreier theory.

- (2) By [37, Theorem 7.7.4], to prove that $\pi_1(X, \bar{x})^{\text{pro-}p}$ is free, it suffices to show that $H^2(\pi_1(X, \bar{x})^{\text{pro-}p}, \mathbf{F}_p) = 0$. This follows from (1) and the fact that $H^q(X, \mathbf{F}_p) = 0$ for $q > 1$. The rank can be read off of $\text{Hom}(\pi_1(X, \bar{x})^{\text{pro-}p}, \mathbf{F}_p) \simeq H^1(X, \mathbf{F}_p) \simeq \Gamma(X, \mathcal{O}_X)/(1 - F)$.
- (3) Let $\Pi \subseteq \pi_1(X, \bar{x})$ be a p -Sylow subgroup. Let $\Pi_\alpha \subseteq \pi_1(X, \bar{x})$ be a projective system of open subgroups such that $\Pi = \bigcap_\alpha \Pi_\alpha$, and let $\{X_\alpha \rightarrow X\}$ be the corresponding projective system of finite étale coverings of X . Since the X_α are also affine $K(\pi, 1)$, we have

$$H^2(\Pi, \mathbf{F}_p) = \varinjlim H^2(\Pi_\alpha, \mathbf{F}_p) = \varinjlim H^2(X_\alpha, \mathbf{F}_p) = 0.$$

Again, we conclude by [37, Theorem 7.7.4]. $\square$

7.5 Relation to $K(\pi, 1)$ pro- ℓ

Let k be an infinite field of characteristic $p > 0$ and let $\ell \neq p$ a prime. In [17] (see also [18, Theorem 7.1.1]), Friedlander has constructed coverings of smooth schemes X over k by connected open subsets U which are ‘ $K(\pi, 1)$ pro- ℓ .’ This means that the pro- ℓ completion of the étale homotopy type of U is weakly equivalent to the classifying space of the pro- ℓ completion $\pi_1(U, u)^{\text{pro-}\ell}$. In simpler terms, for every locally constant constructible $\mathbf{F}_\ell$ -sheaf $\mathcal{F}$ on U which is ‘ ℓ -monodromic’ (i.e., the image of the associated $\pi_1(U, u)$ -representation factors through $\pi_1(U, u)^{\text{pro-}\ell}$), the natural maps

$$H^*(\pi_1(U, u)^{\text{pro-}\ell}, \mathcal{F}_u) \longrightarrow H^*(U, \mathcal{F})$$

are isomorphisms. Being a $K(\pi, 1)$ does not imply being a $K(\pi, 1)$ pro- ℓ [31, Mise en garde 1.4.5]. A variant of this construction due to Gabber is used in [31] in order to prove computability of $\mathbf{F}_\ell$ -cohomology of schemes of finite type over k . Friedlander’s construction is a variation of Artin’s, and in particular the neighborhoods U he obtains are Artin neighborhoods in the sense defined in the Introduction.

Since the opens constructed by Friedlander are affine, we deduce that a smooth scheme X over k can be covered by affine open subsets U which are simultaneously $K(\pi, 1)$ and $K(\pi, 1)$ pro- ℓ (for a single fixed ℓ). The two properties imply that for an ℓ -monodromic $\mathbf{F}_\ell$ -sheaf $\mathcal{F}$ on U , we have a commutative triangle of isomorphisms

$$\begin{array}{ccc} H^*(\pi_1(U, u)^{\text{pro-}\ell}, \mathcal{F}_u) & & \\ \downarrow & \searrow & \\ H^*(\pi_1(U, u), \mathcal{F}_u) & \longrightarrow & H^*(U, \mathcal{F}). \end{array}$$

This implies that $\pi_1(U, u)$ is ℓ -good (cf. Sect. 7.4).

7.6 Anabelian geometry speculations

Anabelian geometry started with a series of conjectures formulated by Grothendieck in his letter to Faltings [22] (cf. [47, §5] for a more recent but not completely up to date survey). Its recurring theme is the question whether some class C of connected schemes is ‘anabelian,’ that is, whether for $X, Y \in C$, the morphism

$$\text{Hom}(X, Y) \longrightarrow \text{Hom}_{\text{ext}}(\pi_1(X), \pi_1(Y))$$

is bijective. It only makes sense to consider such classes in characteristic zero, for if X is an $\mathbb{F}_p$ -scheme, then the absolute Frobenius $F_X: X \rightarrow X$ induces the identity on $\pi_1(X)$.

To fix this problem, we can introduce an auxiliary category Sch^F , whose objects are schemes of positive characteristic, and whose morphisms are morphisms of schemes modulo the relation $F_X = \text{id}_X$ (F_X being the absolute Frobenius). More precisely, let $\text{Sch}_{\text{pos.char.}}$ be the full subcategory of the category of schemes Sch consisting of schemes X such that $p \cdot \mathcal{O}_X$ for some prime p (depending on X). Then there is a functor

$$\varphi: \text{Sch}_{\text{pos.char.}} \rightarrow \text{Sch}^F$$

such that

- (1) $\varphi(F_X) = \text{id}_{\varphi(X)}$ for all objects X ,
- (2) whenever $\psi: \text{Sch}_{\text{pos.char.}} \rightarrow \mathcal{C}$ is a functor such that $\psi(F_X) = \text{id}_{\psi(X)}$ for all objects X , then there exists a unique functor $\bar{\psi}: \text{Sch}^F \rightarrow \mathcal{C}$ and an isomorphism $\psi \simeq \bar{\psi} \circ \varphi$.

The category Sch^F and the functor φ are uniquely characterized by those properties, up to equivalence. In simple terms, the objects of Sch^F are those of $\text{Sch}_{\text{pos.char.}}$, the morphisms $\text{Hom}_{\text{Sch}^F}(X, Y)$ are the quotient of $\text{Hom}(X, Y)$ modulo the relation $f \sim g$ if $F_Y^a \circ f = F_Y^b \circ g$ for some $a, b \geq 0$, and composition is well-defined thanks to the relations $F_Y \circ f = f \circ F_X$.

By the universal property, the functor

$$X \mapsto X_{\text{ét}}: \text{Sch}_{\text{pos.char.}} \rightarrow (\text{Topoi})$$

descends to Sch^F , and hence so does the Artin–Mazur homotopy type functor Π .

Definition 7.8 A class C of objects of $\text{Sch}_{\text{pos.char.}}$ is *anabelian* if for $X, Y \in C$, the morphism

$$\text{Hom}_{\text{Sch}^F}(X, Y) \rightarrow \text{Hom}_{\text{pro-H}}(\Pi(X), \Pi(Y)) \quad (7.1)$$

is a bijection.

We put $\Pi(X)$ here instead of $\pi_1(X)$ because it behaves better in the absence of basepoints (cf. [46, §2.2]), and because the schemes we are interested in are $K(\pi, 1)$ anyway.

The following question seems very natural in the light of our results.

Question 7.9 Let C be the class of integral normal affine schemes $X = \text{Spec } R$ with $p \cdot \mathcal{O}_X = 0$ for some prime p depending on X , such that if $R^{p^\infty} = \bigcap_{e \geq 0} F_R^e(R)$, then

- (1) the perfect ring R^{p^∞} is a field,
- (2) R is finitely generated over R^{p^∞} ,
- (3) $\dim X > 0$.

Is C an anabelian class?

Note the contrast with Grothendieck's conjectures, where the fields in question are finitely generated. In fact, it makes sense to ask the above question under the additional restriction that R^{p^∞} is an algebraically closed field.

It is not difficult to show, using the Artin–Schreier isomorphism $\mathrm{Hom}(\pi_1(X), \mathbf{F}_p) \simeq H^1(X, \mathbf{F}_p) \simeq R/(1 - F)$, that the map (7.1) is always injective. In other words, if X and Y are in the class C , and $f, g: X \rightarrow Y$ are two morphisms inducing the same map $H^1(Y, \mathbf{F}_p) \rightarrow H^1(X, \mathbf{F}_p)$, then $f = g$. The details will appear elsewhere.

Surjectivity of (7.1) seems beyond reach at the present moment. A seemingly more tractable special case is the following:

Question 7.10 *Let k be a perfect field of characteristic p . Does $\pi_1(\mathbf{A}_k^1)$ determine k ?*

Acknowledgements I would like to thank Ahmed Abbes, Bhargav Bhatt, Ofer Gabber, Kiran Kedlaya, Laurent Lafforgue, Martin Olsson, Arthur Ogus, Fabrice Orgogozo, Takeshi Saito, Vasudevan Srinivas, and Karol Szumilo for helpful conversations. I am especially grateful to Takeshi Saito for his help with the proof of Theorem 1.3, and to Ofer Gabber for pointing out that Theorem 1.1 follows from its special case Corollary 5.4. We thank an anonymous referee for pointing out a mistake in an earlier version of the paper and for many valuable comments. We would also like to thank Maciej Borodzick for providing Fig. 1. The author was supported by NCN OPUS grant number UMO-2015/17/B/ST1/02634.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

Appendix A: The rank one case of the Bertini theorem

We present a second proof of Theorem 1.3 in the case when the sheaf $\mathcal{F}$ has rank one. In this case, we are able to explicitly find the desired automorphism φ . Interestingly, we do not have to assume that k is infinite.

A.1: Ramification of Artin–Schreier–Witt sheaves

For $r \geq 0$, let $\mathbf{W}_{r+1}$ denote the group scheme of Witt vectors of length $r + 1$ over k , and let $F: \mathbf{W}_{r+1} \rightarrow \mathbf{W}_{r+1}$ be the Frobenius homomorphism. As a

variety, $\mathbf{W}_{r+1} \simeq \mathbf{A}_k^{r+1}$, and $F(x_0, \dots, x_r) = (x_0^p, \dots, x_r^p)$. We have a short exact (for the étale topology) sequence of k -group schemes:

$$0 \longrightarrow \mathbf{Z}/p^{r+1}\mathbf{Z} \longrightarrow \mathbf{W}_{r+1} \xrightarrow{1-F} \mathbf{W}_{r+1} \longrightarrow 0, \quad (\text{A.1.1})$$

yielding for any k -scheme X a boundary map

$$\delta: \mathbf{W}_{r+1}(X) \longrightarrow H^1(X, \mathbf{Z}/p^{r+1}\mathbf{Z}). \quad (\text{A.1.2})$$

If $H^1(X, \mathcal{O}_X) = 0$ (e.g. if X is affine), then δ induces an isomorphism

$$\mathbf{W}_{r+1}(X)/(1-F)\mathbf{W}_{r+1}(X) \xrightarrow{\sim} H^1(X, \mathbf{Z}/p^{r+1}\mathbf{Z}).$$

Definition A.1.1 If $\ell \neq p$ is a prime and $\psi: \mathbf{Z}/p^{r+1}\mathbf{Z} \longrightarrow \bar{\mathbf{F}}_\ell^\times$ is an injective character, we denote by $\mathcal{L}_\psi$ the rank one $\bar{\mathbf{F}}_\ell$ -sheaf on $\mathbf{W}_{r+1}$ associated to the $\mathbf{Z}/p^{r+1}\mathbf{Z}$ -torsor given by (A.1.1) and the character ψ .

If X is a k -scheme and $f: X \longrightarrow \mathbf{W}_{r+1}$ is a map, we denote by $\mathcal{L}_{\psi,f}$ the pull-back $f^*\mathcal{L}_\psi$; we call it the *Artin–Schreier–Witt sheaf* associated to f and ψ .

It is clear that if $f - h \in (1-F)\mathbf{W}_{r+1}(X)$, then $\mathcal{L}_{\psi,f} \simeq \mathcal{L}_{\psi,h}$. Our first goal in this section is to study the ramification of $\mathcal{L}_{\psi,f}$ along a divisor, in particular its Swan conductor.

To this end, suppose that $X = \eta = \text{Spec } K$ as in Sect. 2. In this context, and in the perfect residue field case ($\kappa = k$), the Swan conductor of an Artin–Schreier–Witt sheaf over X has been computed by Brylinski [13] and Kato [25]. To explain the result, we endow the groups $\mathbf{W}_{r+1}(K)$ and $H^1(K, \mathbf{Z}/p^{r+1}\mathbf{Z})$ with increasing filtrations as follows:

$$\text{fil}_n \mathbf{W}_{r+1}(K) = \left\{ f = (f_0, \dots, f_r) \in \mathbf{W}_{r+1}(K) : -p^{r-i}v(f_i) \leq n \right\}, \quad (\text{A.1.3})$$

$$\text{fil}_n H^1(X, \mathbf{Z}/p^{r+1}\mathbf{Z}) = \delta(\text{fil}_n \mathbf{W}_{r+1}(K)), \quad (\text{A.1.4})$$

where v is the valuation on K .

Theorem A.1.2 [13, Corollary to Theorem 1] *Suppose that the residue field of K is k . Let $f = (f_0, \dots, f_r) \in \mathbf{W}_{r+1}(K)$, $\psi: \mathbf{Z}/p^{r+1}\mathbf{Z} \longrightarrow \bar{\mathbf{F}}_\ell^\times$ an injective character, defining the $\bar{\mathbf{F}}_\ell$ -sheaf $\mathcal{L}_{\psi,f}$ on $\text{Spec } K$. Then*

$$\begin{aligned} \text{Sw}(\mathcal{L}_{\psi,f}) &= \min(\{0\} \cup \{n : \delta(f) \in \text{fil}_n H^1(K, \mathbf{Z}/p^{r+1}\mathbf{Z})\}) \\ &= \min_h \max\{0, -p^r v(h_0), -p^r v(h_1), \dots, -v(h_r)\}, \end{aligned}$$

where h ranges over all $h \in \mathbf{W}_{r+1}(K)$ such that $f - h \in (F - 1)\mathbf{W}_{r+1}(K)$.

The $\min_h$ in the above theorem might be difficult to compute in concrete situations. The following proposition provides a simple condition for that minimum to be attained at $h = f$.

Proposition A.1.3 *In the situation of Theorem A.1.2, suppose that*

- (1) *the function $i \mapsto -p^{r-i}v(f_i)$ has a unique maximum i_0 ,*
- (2) *the number $v(f_{i_0})$ is negative and prime to p .*

Then $\mathrm{Sw}(\mathcal{L}_{\psi, f}) = -p^{r-i_0}v(f_{i_0})$.

Following Abbes and Saito [8, §10], we endow $\Omega_{K/k}^1$ with the increasing filtration

$$\mathrm{fil}_n \Omega_{K/k}^1 = \mathrm{image}(\Omega_{\mathcal{O}_K/k}^1(\log) \otimes_{\mathcal{O}_K} \mathfrak{m}_{\mathcal{O}_K}^{-n} \longrightarrow \Omega_{K/k}^1). \quad (\text{A.1.5})$$

Here $\Omega_{\mathcal{O}_K/k}^1(\log)$ is the module of log differentials (cf. [8, §5.4]). The map

$$F^r d: \mathbf{W}_{r+1}(K) \longrightarrow \Omega_{K/k}^1, \quad f \mapsto \sum_{i=0}^r f_i^{p^{r-1}-1} df_i$$

is additive and compatible with the filtrations (the notation $F^r d$ comes from the de Rham–Witt complex). Moreover, upon passing to the associated graded quotients ($\mathrm{Gr}_n = \mathrm{fil}_n / \mathrm{fil}_{n-1}$), $F^r d$ factors uniquely through δ (A.1.2), that is, for every n there exists a unique map ψ_n making the following triangle commute

$$\begin{array}{ccc} \mathrm{Gr}_n \mathbf{W}_{r+1}(K) & \xrightarrow{F^r d} & \mathrm{Gr}_n \Omega_{K/k}^1 \\ \mathrm{Gr}_n \downarrow & \nearrow \psi_n & \\ \mathrm{Gr}_n H^1(K, \mathbf{Z}/p^{r+1}\mathbf{Z}) & & \end{array}$$

Furthermore, ψ_n is injective.

Lemma A.1.4 *Let $f = (f_0, \dots, f_r) \in \mathbf{W}_{r+1}(K)$ and*

$$a = \min\{n : f \in \mathrm{fil}_n \mathbf{W}_{r+1}(K)\}, \quad (\text{cf. (A.1.3)})$$

$$b = \min\{n : \delta(f) \in \mathrm{fil}_n H^1(K, \mathbf{Z}/p^{r+1}\mathbf{Z})\}, \quad (\text{cf. (A.1.4)})$$

$$c = \min\{n : F^r d(f) \in \mathrm{fil}_n \Omega_{K/k}^1\} \quad (\text{cf. (A.1.5)}).$$

Then $a = c$ if and only if $a = b$.

Proof This follows from the existence and injectivity of ψ_n . $\square$

Proof of Proposition A.1.3 Let a, b, c be as in Lemma A.1.4. Then $a = -p^{r-i_0}v(f_{i_0})$, and Theorem A.1.2 states that $b = \text{Sw}(\mathcal{L}_{\psi, f})$. Moreover,

$$F^r d(f) = \left(\alpha v(f_{i_0}) t^{p^{r-i_0}v(f_{i_0})} + \text{higher order terms} \right) d \log t$$

with $\alpha, v(f_{i_0}) \in k^*$. Thus $c = -p^{r-i_0}v(f_{i_0}) = a$, so $a = b$. $\square$

A.2: Artin–Schreier–Witt sheaves on A_k^η

Lemma A.2.1 *Let p be a prime number and let S be a non-empty finite subset of $\mathbb{N}^{n+1}$. Let $\leq$ be the order on S such that $(b_0, b_1, \dots, b_n) \leq (b'_0, b'_1, \dots, b'_n)$ if and only if $(b_0 + b_1 + \dots + b_n, b_0, b_1, \dots, b_n) \leq (b'_0 + b'_1 + \dots + b'_n, b'_0, b'_1, \dots, b'_n)$ in the lexicographical order. This is a linear order on S ; let $x = (b_0^*, b_1^*, \dots, b_n^*)$ be the largest element of S . There exist integers $d_1 > \dots > d_n > 0$ such that if we set $w(b_0, b_1, \dots, b_n) = b_0 + d_1 b_1 + \dots + d_n b_n$ then*

1. x is the unique maximum of w on S ,
2. $w(x)/\gcd(b_0^*, b_1^*, \dots, b_n^*)$ is not divisible by p .

Proof Let A be the diameter of S for the ℓ^∞ -norm, so if $(b_0, b_1, \dots, b_n), (b'_0, b'_1, \dots, b'_n) \in S$ then $A \geq |b_i - b'_i|$. It is easy to see that $w: S \rightarrow \mathbb{Z}$, $w(b_0, b_1, \dots, b_n) = b_0 + d_1 b_1 + \dots + d_n b_n$, will be injective as long as $d_{i+1} > A(1 + d_1 + \dots + d_i)$ for all $i = 1, \dots, n-1$. Moreover, the condition that $d_1 > \dots > d_n > 0$ ensures that $w: S \rightarrow \mathbb{Z}$ is non-decreasing. This implies that for every tuple of integers $e_1, \dots, e_n$, we can find a tuple $d_1 > \dots > d_n > 0$ satisfying (1) and such that $d_i \equiv e_i$ modulo p . We deduce that we can make condition (2) fulfilled as well. $\square$

Lemma A.2.2 *Let $f_0, \dots, f_r \in k[x_0, \dots, x_n]$ be polynomials without non-constant p -power monomials, such that not all f_i are constant. Then there exist $d_1 > \dots > d_n > 0$ such that if*

$$f'_i = f_i(x_0, x_1 + x_0^{d_1}, \dots, x_n + x_0^{d_n}),$$

then

- (1) $i \mapsto \deg(f'_i)p^{r-i}$ has a unique minimum $\deg(f'_{i_0})p^{r-i_0}$, and
- (2) p does not divide $\deg(f'_{i_0})$.

Proof Apply Lemma A.2.1 to $S = \bigcup \text{supp}(f_i) \cdot p^{r-i}$. By assumption, this union is disjoint. $\square$

Lemma A.2.3 *Let $A = k[x_0, \dots, x_n]$, $f \in \mathbf{W}_{r+1}(A)$. Then there exists a $g \in \mathbf{W}_{r+1}(A)$ such that if $f + (1 - F)g = (h_0, \dots, h_r)$ then the h_i have no nonzero p -th power monomials.*

Proof The proof is by induction on $r \geq 0$. Elementary combinatorics shows that for $u \in A$, there exists a $v(u) \in A$ (unique up to adding an element of $\mathbf{F}_p$) such that $w(u) = u + v(u) - v(u)^p$ has no non-zero p -th power monomials (note that we need k separably closed to deal with the constant term). This solves the base case $r = 0$. For the induction step, we take $f = (f_0, \dots, f_r)$ with $r > 0$, and set

$$f' = f + (1 - F)(v(f_0), 0, \dots, 0).$$

Clearly $f = f' \bmod (1 - F)\mathbf{W}_{r+1}(A)$. Moreover, $f'_0 = f_0 + v(f_0) - v(f_0)^p = w(f'_0)$ has no p -th power monomials. Set

$$f'' = (f'_1, \dots, f'_r) \in \mathbf{W}_r(A),$$

so that $f' = (f'_0, 0, \dots, 0) + Vf''$. By the induction assumption, there exists a g'' such that the entries of $h'' = f'' + (1 - F)g''$ have no p -th power monomials. Finally, let

$$g = (v(f_0), 0, \dots, 0) + (0, g''_0, \dots, g''_{r-1}) = (v(f_0), 0, \dots, 0) + Vg''.$$

Then

$$\begin{aligned} h &= f + (1 - F)g = f + (1 - F)(v(f_0), 0, \dots, 0) + (1 - F)Vg'' \\ &= f' + (1 - F)Vg'' = (f'_0, 0, \dots, 0) + Vf'' + V(1 - F)g'' \\ &= (w(f'_0), h''_0, \dots, h''_{r-1}), \end{aligned}$$

and we see that the entries of h have no p -th power monomials, as desired. $\square$

Proposition A.2.4 *The assertion of Theorem 1.3 holds if $\mathcal{F}$ is a rank one $\overline{\mathbf{F}}_\ell$ -sheaf.*

Proof As $\pi_1(A_k^{\eta+1})$ has no prime-to- p quotients, while $\overline{\mathbf{F}}_\ell^\times$ is cyclic, the representation $\pi_1(A_k^{\eta+1}) \rightarrow \overline{\mathbf{F}}_\ell^\times$ corresponding to $\mathcal{F}$ factors through an injective character $\psi: \mathbf{Z}/p^{r+1}\mathbf{Z} \rightarrow \overline{\mathbf{F}}_\ell^\times$ for some $r \geq 0$. Thus there exists an $f = (f_0, \dots, f_r) \in \mathbf{W}_{r+1}(k[z, x_1, \dots, x_n])$ such that $\mathcal{F} \simeq \mathcal{L}_{\psi, f}$. We can assume that not all of the f_i are constant, otherwise $\mathcal{F}$ is constant and there is nothing to prove. By Lemma A.2.3, we can moreover assume that the f_i have no nonzero p -th power monomials, as $\mathcal{L}_{\psi, f} \simeq \mathcal{L}_{\psi, f+(1-F)g}$ for any $g \in \mathbf{W}_{r+1}(k[z, x_1, \dots, x_n])$. By Lemma A.2.2, there exist $d_1 > \dots > d_n > 0$

such that if $f'_i = f_i(z, x_1 + z^{d_1}, \dots, x_n + z^{d_n})$, then $i \mapsto \deg(f'_i) p^{r-i}$ has a unique minimum $\deg(f'_{i_0}) p^{r-i_0}$, and p does not divide $\deg(f'_{i_0})$. But by Proposition A.1.3 this means that $\mathrm{Sw}_\infty(\mathcal{F}_{\pi^{-1}(y)}) = \deg(f'_{i_0}) p^{r-i_0}$ is independent of $y \in \mathbf{A}_k^n$, where $\pi = \mathrm{pr} \circ \varphi$,

$$\varphi(z, x_1, \dots, x_n) = (z, x_1 + z^{d_1}, \dots, x_n + z^{d_n})$$

and $\mathrm{pr}(z, x_1, \dots, x_n) = (x_1, \dots, x_n)$. $\square$

Remark A.2.5 We expect that the sheaf $\varphi^* \mathcal{F}$ in the proof above is non-fierce at infinity.

Remark A.2.6 Our calculations (especially Lemma A.2.1) resemble those of Barrientos [9, §5] in a similar context of restricting rank one sheaves to curves.

References

1. Abbes, A.: *Éléments de géométrie rigide. Volume I. Progress in Mathematics*, vol. 286, Birkhäuser/Springer Basel AG, Basel (2010). Construction et étude géométrique des espaces rigides. With a preface by Michel Raynaud
2. Abhyankar, S.: Coverings of algebraic curves. *Am. J. Math.* **79**, 825–856 (1957)
3. Achinger, P.: $K(\pi, 1)$ -neighborhoods and comparison theorems. *Compos. Math.* **151**(10), 1945–1964 (2015)
4. Abbes, A., Gros, M.: Covanishing topos and generalizations. In: *The p -Adic Simpson Correspondence*. *Ann. of Math. Stud.*, vol. 193, pp. 485–576. Princeton Univ. Press, Princeton (2016)
5. Artin, M., Mazur, B.: *Étale Homotopy. Lecture Notes in Mathematics*, vol. 100. Springer, Berlin (1969)
6. Artin, M.: Comparaison avec la cohomologie classique: cas d'un préschéma lisse, SGA 4 Exposé XI. *Théorie des Topos et Cohomologie Étale des Schémas. Lecture Notes in Mathematics*, vol. 305, pp. 64–78 (1973)
7. Artin, M.: Théorème de finitude pour un morphisme propre; dimension cohomologique des schémas algébriques affines, SGA 4 Exposé XIV. *Théorie des Topos et Cohomologie Étale des Schémas. Lecture Notes in Mathematics*, vol. 305, pp. 145–167 (1973)
8. Abbes, A., Saito, T.: Analyse micro-locale l -adique en caractéristique $p > 0$: le cas d'un trait. *Publ. Res. Inst. Math. Sci.* **45**(1), 25–74 (2009)
9. Barrientos, I.: Log ramification via curves in rank 1. *Int. Math. Res. Not.* (2016). doi:[10.1093/imrn/rnw177](https://doi.org/10.1093/imrn/rnw177)
10. Beilinson, A.: Constructible sheaves are holonomic. *Selecta Math. (N.S.)* **22**(4), 1797–1819 (2016)
11. Bessis, D.: Finite complex reflection arrangements are $K(\pi, 1)$. *Ann. Math. (2)* **181**(3), 809–904 (2015)
12. Brieskorn, E.: Sur les groupes de tresses, Séminaire Bourbaki, 24ème année (1971/1972). Exp. No. 401. *Lecture Notes in Math.*, vol. 317, pp. 21–44. Springer, Berlin (1973)
13. Brylinski, J.-L.: Théorie du corps de classes de Kato et revêtements abéliens de surfaces. *Ann. Inst. Fourier Grenoble* **33**(3), 23–38 (1983)
14. Deligne, P.: Les immeubles des groupes de tresses généralisés. *Invent. Math.* **17**, 273–302 (1972)
15. Deligne, P.: *Cohomologie étale. Lecture Notes in Mathematics*, vol. 569. Springer, Berlin (1977)

16. Fried, M.D., Jarden, M.: Field Arithmetic, 3rd eds., *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*, vol. 11. Springer, Berlin (2008). Revised by Jarden
17. Friedlander, E.M.: $K(\pi, 1)$'s in characteristic $p > 0$. *Topology* **12**, 9–18 (1973)
18. Friedlander, E.M.: Étale Homotopy of Simplicial Schemes. *Annals of Mathematics Studies*, vol. 104. Princeton University Press, Princeton (1982)
19. Fujiwara, K.: Theory of tubular neighborhood in étale topology. *Duke Math. J.* **80**(1), 15–57 (1995)
20. Gabber, O.: Affine analog of the proper base change theorem. *Israel J. Math.* **87**(1–3), 325–335 (1994)
21. Grothendieck, A.: Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II. *Inst. Hautes Études Sci. Publ. Math.* **231**(24), 231 (1965)
22. Grothendieck, A.: Brief an G. Faltings. *Geometric Galois Actions, 1*. London Math. Soc. Lecture Note Ser., vol. 242, pp. 49–58. Cambridge Univ. Press, Cambridge (1997). With an English translation on pp. 285–293
23. Hattori, A.: Topology of C^n minus a finite number of affine hyperplanes in general position. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **22**(2), 205–219 (1975)
24. Huber, R.: Étale Cohomology of Rigid Analytic Varieties and Adic Spaces. *Aspects of Mathematics*, E30, Friedr. Vieweg & Sohn, Braunschweig (1996)
25. Kato, K.: Swan conductors for characters of degree one in the imperfect residue field case. In: *Algebraic K-Theory and Algebraic Number Theory* (Honolulu, HI, 1987), *Contemp. Math.*, vol. 83, pp. 101–131. Am. Math. Soc., Providence (1989)
26. Katz, N.M.: L -functions and monodromy: four lectures on Weil II. *Adv. Math.* **160**(1), 81–132 (2001)
27. Kedlaya, K.S.: Étale covers of affine spaces in positive characteristic. *C. R. Math. Acad. Sci. Paris* **335**(11), 921–926 (2002)
28. Kashiwara, M., Schapira, P.: *Sheaves on Manifolds*. Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 292. Springer, Berlin (1994). With a chapter in French by Christian Houzel, Corrected reprint of the 1990 original
29. Laumon, G.: Semi-continuité du conducteur de Swan (d'après P. Deligne). *The Euler–Poincaré characteristic* (French), *Astérisque*, vol. 83, pp. 173–219. Soc. Math. France, Paris (1981)
30. Laumon, G.: Caractéristique d'Euler–Poincaré des faisceaux constructibles sur une surface. Analysis and topology on singular spaces, II, III (Luminy, 1981), *Astérisque*, vol. 101, pp. 193–207. Soc. Math. France, Paris (1983)
31. Madore, D.A., Orgogozo, F.: Calculabilité de la cohomologie étale modulo ℓ . *Algebra Number Theory* **9**(7), 1647–1739 (2015)
32. Mumford, D.: *The red book of varieties and schemes*, expanded ed., *Lecture Notes in Mathematics*, vol. 1358, Springer, Berlin (1999). Includes the Michigan lectures (1974) on curves and their Jacobians, With contributions by Enrico Arbarello
33. Olsson, M.C.: On Faltings' method of almost étale extensions. *Algebraic geometry—Seattle 2005. Part 2*, *Proc. Sympos. Pure Math.*, vol. 80, pp. 811–936. Amer. Math. Soc., Providence, RI (2009)
34. Orlik, P., Terao, H.: *Arrangements of Hyperplanes*. Grundlehren der Mathematischen Wissenschaften, vol. 300. Springer, Berlin (1992)
35. Raynaud, M.: Anneaux locaux henséliens. *Lecture Notes in Mathematics*, vol. 169. Springer, Berlin (1970)
36. Raynaud, M.: Revêtements de la droite affine en caractéristique $p > 0$ et conjecture d'Abhyankar. *Invent. Math.* **116**(1–3), 425–462 (1994)

37. Ribes, L., Zalesskii, P.: Profinite groups, second ed., *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*, vol. 40. Springer, Berlin (2010)
38. Saito, T.: Wild ramification and the cotangent bundle. *J. Algebr. Geom.* **26**(2017), 399–473 (2016)
39. Saito, T.: The characteristic cycle and the singular support of a constructible sheaf. *Invent. Math.* **207**(2), 597–695 (2017)
40. Scholze, P.: Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.* **116**, 245–313 (2012)
41. Scholze, P.: p -adic Hodge theory for rigid-analytic varieties. *Forum Math. Pi* **1**, e177 (2013)
42. Serre, J.-P.: Cohomologie galoisienne. Cours au Collège de France, vol. 1962. Springer, Berlin (1962/1963)
43. Serre, J.-P.: Linear Representations of Finite Groups. Springer, New York (1977). Translated from the second French edition by Leonard L. Scott, *Graduate Texts in Mathematics*, vol. 42
44. Serre, J.-P.: Local Fields. *Graduate Texts in Mathematics*, vol. 67. Springer, New York (1979). Translated from the French by Marvin Jay Greenberg
45. Grothendieck, A.: rédigé par I. Bucur, Formule d'Euler-Poincaré en cohomologie étale, Cohomologie l -adique et Fonction L , SGA 5 Exposé X, *Springer Lecture Notes in Math.* **589**(1977), 372–406
46. Schmidt, A., Stix, J.: Anabelian geometry with étale homotopy types. *Ann. Math. (2)* **184**(3), 817–868 (2016)
47. Szamuely, T.: Heidelberg lectures on fundamental groups. In: *The Arithmetic of Fundamental Groups—PIA 2010*, *Contrib. Math. Comput. Sci.*, vol. 2, pp. 53–74. Springer, Heidelberg (2012)