Growth of permutational extensions

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Abstract We study the geometry of a class of group extensions, containing permutational wreath products, which we call “permutational extensions”. We construct for all $k \in \mathbb{N}$ a torsion group K_k with growth function

$$v_{K_k}(n) \sim \exp(n^{1-(1-\alpha)^k}), \quad 2^{3-3/\alpha} + 2^{2-2/\alpha} + 2^{1-1/\alpha} = 2,$$

and a torsion-free group H_k with growth function

$$v_{H_k}(n) \sim \exp(\log(n)n^{1-(1-\alpha)^k}).$$

These are the first examples of groups of intermediate growth for which the asymptotics of their growth function is known.

We construct a group of intermediate growth that contains the group of finitely supported permutations of a countable set as a subgroup. This gives the first example of a group of intermediate growth containing an infinite simple group as a subgroup.

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1 Introduction

Let G be a finitely generated group. One of its fundamental invariants is the group's *growth*, studied since the 1950s [26, 35]: choose a finite set S generating G as a monoid, and define its *growth function*

$$v_{G,S}(n) = \#\{g \in G \mid g = s_1 \dots s_m \text{ for some } m \leq n \text{ and } s_i \in S\}.$$

This function depends on S , but only mildly: given two functions $f, g: \mathbb{N} \rightarrow \mathbb{R}_+$, we say that g is asymptotically smaller than f , and write $g \leq f$, if there exist $C > 0$ such $g(n) \leq f(Cn)$ for all sufficiently large n . We say that f is asymptotically equivalent to g , and write $f \sim g$, if $f \leq g$ and $g \leq f$. The asymptotic equivalence class of $v_{G,S}$ is then independent of S , and is written v_G .

More information on growth of groups may be found in [22, Chaps. VI–VIII] and in the monograph [30].

By a famous theorem of Gromov [21], the growth function is at most polynomial if and only if G is virtually nilpotent. Furthermore, in that case, there exists an integer d such that $v_G \sim n^d$; this fact is usually attributed to Guivarc'h and to Bass [8], see also [22, VII.26].

On the other hand, if G is linear, or solvable, or word-hyperbolic, then, unless G is virtually nilpotent, $v_G \sim \exp(n)$, so all such groups have asymptotically equivalent growth.

Milnor asked in [31] whether v_G could take values strictly between polynomial and exponential functions; such groups would be called groups of *intermediate growth*. This question was answered positively by Grigorchuk in 1983 [16, 17, 19]. Grigorchuk constructed a large class of groups of intermediate growth, showing in particular that for any subexponential function $v(n)$ there exists a group of intermediate growth whose growth function is greater than $v(n)$ for infinitely many n . Essentially the same argument shows that for any subexponential $v(n)$ there exists a group which is a direct sum of two Grigorchuk groups such that $v_{G,S}(n) \geq v(n)$ for all sufficiently large n [13]. Another result of Grigorchuk is the existence of groups with incomparable growth functions.

There was, up to now, no group of intermediate growth for which the asymptotics of its growth function is known. On the other hand, it was known since the early 1970s that semigroups can have intermediate growth, and some growth functions were computed explicitly: Govorov shows in [14] that the semigroup

$$\langle x, y \mid x^i y^{i-1} x^i = y^i x^{i-1} y^i = 0 \ \forall i \geq 2 \rangle_+$$

has growth function $\sim \exp(\sqrt{n})$. The precise asymptotics of the growth of more semigroups (including 2×2 matrix semigroups, and relatively free

semigroups) were computed by Lavrik [27], Shneerson [34], Okninski [33], and Reznikov and Sushchanskii [7]. The following growth types occur among these examples: $n^{\log n}$, $\exp(\sqrt{n})$, and $\exp(\sqrt{n}/\log n)$.

The most studied example of group of intermediate growth is the *first Grigorchuk group* G_{012} . The best known estimates for this group's growth are as follows: let η be the real root of the polynomial $X^3 + X^2 + X - 2$, and set $\alpha = \log(2)/\log(2/\eta) \approx 0.7674$. Then

$$\exp(n^{0.5153}) \lesssim v_{G_{012}} \lesssim \exp(n^\alpha). \quad (1)$$

For the lower bound, see [2] and Leonov [28]; for the upper bound, see [1].

1.1 Main results

Up to now no growth function $\not\sim n^d, \exp(n)$ had been determined. In this paper, we construct the first examples of groups of intermediate growth for which the asymptotics of the growth function is known:

Theorem A

- (1) *For every $k \in \mathbb{N}$ there exists a finitely generated torsion group K_k with growth*

$$v_{K_k} \sim \exp(n^{1-(1-\alpha)^k}).$$

- (2) *For every $k \in \mathbb{N}$ there exists a finitely generated torsion-free group H_k with growth*

$$v_{H_k} \sim \exp(\log(n)n^{1-(1-\alpha)^k}).$$

In fact, the group H_1 that we consider had already been studied by Grigorchuk in [17], who showed that it is torsion-free and has subexponential growth, though he did not compute its growth function.

A classical result due to Higman, Neumann and Neumann states that any countable group can be embedded into a finitely generated group [23]. Given a countable group, it is natural to ask which extra properties that finitely generated group may possess; and, in the case that interests us, when that finitely generated group may be taken of subexponential growth. We show:

Theorem B (= Theorem 6.1) *The group of finitely supported permutations of $\mathbb{Z}$ embeds as a normal subgroup in a group of intermediate growth.*

The theorem shows, in particular, that some groups of intermediate growth contain as a subgroup an infinite simple group. In our examples such infinite simple subgroups are not finitely generated. An open question due to Grigorchuk asks whether any infinite finitely generated simple group has exponential growth.

Question *Does every countable group locally of subexponential growth embed in a finitely generated group of subexponential growth?*

1.2 Outline of the approach

Our strategy may be summarized as follows. We consider groups B, G with respective generating sets R, S , such that: S is finite; G acts on B ; the action of G on B permutes R ; and R has finitely many G -orbits. Typically, G will act on a set X , and B will be either: a direct sum $\sum_X A$ of a finitely generated group; a group of permutations of X ; or a free Abelian, nilpotent etc. group generated by a set in bijection with X . We then take the semidirect product W of B with G . It turns out that the growth of W is well controlled by the growth of “inverted orbits” of G on X . By definition, the inverted orbit of a word $w = g_1 \dots g_n$ over G is $\{x_0 g_1 \dots g_n, x_0 g_2 \dots g_n, \dots, x_0 g_n\}$ for a basepoint $x_0 \in X$.

The idea behind the fact that groups of Grigorchuk have intermediate growth is a certain contraction property. For some of Grigorchuk’s groups G , including the first one, this property states that there exist an injective homomorphism $\psi = (\psi_1, \dots, \psi_d)$ from a finite index subgroup H in G to the direct product G^d , such that for some finite generating set S in G , some $\eta < 1$, some $C > 0$, and some proper norm $|\cdot|$ on G , the following holds for all $h \in H$:

$$\sum_{i=1}^d |\psi_i(h)| \leq \eta |h| + C. \quad (2)$$

This implies that the growth of G is bounded from above by $\exp(n^\beta)$ for a constant $\beta < 1$ depending only on d and η .

In this paper we introduce and study another contraction property, related to the sublinear growth of the inverted orbits for a group action. We prove that this property holds for the action on the boundary of many Grigorchuk groups, including the first Grigorchuk group. The property implies that not only these groups, but a large family of extensions of these groups have subexponential growth. We mention, however, that for some Grigorchuk groups the inverted orbit growth is linear (see Example 5.4).

The contracting property (2), in the case $d = 1$, deserves special mention; the group G then admits a *dilatation*. This implies that the growth of G is polynomial. Many nilpotent groups admit a dilatation. However, no nilpotent group may satisfy the “sublinear inverted orbit growth” property studied in this paper (see Remark 5.6).

2 Permutational wreath products and extensions

We consider groups A, G and a set X , such that G acts on X from the right. The *wreath product* $W = A \wr_X G$ is the semidirect product of $\sum_X A$ with G .

The *support* $\text{sup } f$ of a function $f : X \rightarrow A$ consists of those $x \in X$ such that $f(x) \neq 1$. We describe elements of $\sum_X A$ as finitely supported functions $X \rightarrow A$. The (left) action of G on $\sum_X A$ is then defined by $(g \cdot f)(x) = f(xg)$; observe that for g_1, g_2 in G

$$(g_1 g_2 \cdot f)(y) = f(y(g_1 g_2)) = f((y g_1) g_2) = (g_2 \cdot f)(y g_1) = (g_1 \cdot g_2 \cdot f)(y).$$

We have in particular $\text{sup}(g^{-1} \cdot f) = \text{sup}(f)g$. If A act on a set Y from the right, then W naturally acts on $X \times Y$ from the right, by $(x, y)f = (x, yf(x))$ and $(x, y)g = (xg, y)$.

Suppose now that A and G are finitely generated, and that the action of G on X is transitive. Fix generating sets S_A and S_G of A and G respectively, and fix a basepoint $x_0 \in X$. The wreath product is generated by $S = S_A \cup S_G$, in which we identify G with its image under the embedding $g \rightarrow (1, g)$, and identify A with its image under the embedding $a \rightarrow (f_a, 1)$; here $f_a : X \rightarrow A$ is defined by $f_a(x_0) = a$ and $f_a(x) = 1$ for all $x \neq x_0$. We call S the *standard generating set* of W defined by S_A, S_G . Analogously, if the action of G on X has finitely many orbits, then W is finitely generated by $S_G \cup (S_A \times (X/G))$.

2.1 The Cayley graph of a permutational wreath product

The Cayley graph of $W = (\sum_X A) \rtimes G$ with respect to the generating set S may be described as follows. Elements of W are written fg with $f \in \sum_X A$ and $g \in G$; multiplication is given by $(f_1 g_1)(f_2 g_2) = f_1(g_1 \cdot f_2) g_1 g_2$.

Consider a word $v = s_1 s_2 \dots s_\ell$, with all $s_i \in S$, and write its value in W as $f_v g_v$ with $f \in \sum_X A$ and $g_v \in G$. Set $u = f_u g_u = s_1 s_2 \dots s_{\ell-1}$. Here g_u, g_v belong to G , and $f_u, f_v : X \rightarrow A$.

We consider two cases, depending on whether $s_\ell \in S_A$ or $s_\ell \in S_G$. If $s_\ell \in S_A$, we have an edge of ‘‘A’’ type from u to v . The multiplication formula gives $g_v = g_u$ and $f_v(x) = f_u(x)$ for all $x \neq x_0 g_u^{-1}$, while $f_v(x_0 g_u^{-1}) = f_u(x_0 g_u^{-1}) s_\ell$. If $s_\ell \in S_G$, we have an edge of ‘‘G’’ type from u to v . Then $f_v = f_u$, and $g_v = g_u s_\ell$.

There is an alternative description of the edges in the Cayley graph, which appears if we write elements of $(\sum_X A) \rtimes G$ in the form gf with $g \in G$ and $f \in \sum_X A$. Their product is then given by $(g_1 f_1)(g_2 f_2) = g_1 g_2 (g_2^{-1} \cdot f_1) f_2$.

In that notation, if there is an edge of ‘‘A’’ type from $u = g_u f_u$ to $v = g_v f_v$, then we have $g_u = g_v$, and $f_v = f_u$ except at x_0 where $f_v(x_0) = f_u(x_0) s_\ell$. On the other hand, if there is an edge of ‘‘G’’ type from u to v , then we have $g_v = g_u s_\ell$ and $f_v = s_\ell^{-1} \cdot f_u$.

For $i = 1, \dots, \ell$, set now $g_i = s_i, a_i = 1$ whenever $s_i \in S_G$, and $g_i = 1, a_i = s_i$ whenever $s_i \in S_A$. Still writing $v = g_v f_v$, we then have

$$v = a_1 g_1 \cdots a_\ell g_\ell = g_1 \cdots g_\ell a_1^{g_1 \cdots g_\ell} \cdots a_{\ell-1}^{g_{\ell-1} \cdots g_\ell} a_\ell^{g_\ell},$$

so $g_v = g_1 g_2 \cdots g_\ell$ and $f_v = ((g_1 \cdots g_\ell)^{-1} \cdot f_{a_1})((g_2 \cdots g_\ell)^{-1} \cdot f_{a_2}) \cdots (g_\ell^{-1} \cdot f_{a_\ell})$; we observe that the support of f_v is contained in $\{x_0 g_\ell, x_0 g_{\ell-1} g_\ell, \dots,$

$x_0 g_1 g_2 \cdots g_\ell\}$. In other words, in order to understand the support of the configuration, we have to study *inverted orbits* of the action of G on X and the number of distinct points visited by these orbits.

Remark 2.1 In case $G = X$, there is no difference between counting the number of points on the orbits or on the inverted orbits ($x_0 = 1$, G acts on X both from the right and from the left, and the inverted orbits for the right action are usual orbits for the left action). This is no longer the case if $X \neq G$.

Remark 2.2 We might wonder to which degree the geometry of the Cayley graphs of A and G , and of the Schreier graph of X (the graph with vertex set X , and an edge from x to xs for all $x \in X$ and generator s of G), determine the geometry of the wreath product.

In contrast with the case $X = G$ (“usual” wreath products), the Cayley graph of the permutational wreath product is in no way defined by the unmarked Cayley graphs of A and G and the Schreier graph of X . We will see in the sequel that the following may happen: a group G acts on X_1 and X_2 , the unmarked Schreier graphs of X_1 and X_2 are the same, but $A \wr_{X_1} G$ has exponential growth (for some finite group A), and $A \wr_{X_2} G$ has intermediate growth (see Example 5.5).

2.2 Inverted orbits

We formalize the discussion above as follows. Fix a group G acting on the right on a set X ; fix a set S generating G as a monoid; and fix a basepoint $x_0 \in X$. Denote by S^* the set of words over S . For a word $w = w_1 \dots w_\ell \in S^*$, its *inverted orbit* is

$$\mathcal{O}(w) = \{x_0, x_0 w_\ell, x_0 w_{\ell-1} w_\ell, \dots, x_0 w_1 w_2 \dots w_\ell\}.$$

Its *inverted orbit growth* is

$$\delta(w) = \#\mathcal{O}(w).$$

The *inverted orbit growth function* of G is the function

$$\Delta(n) = \max\{\delta(w) \mid n = |w|\}.$$

Clearly $\Delta(n) \leq n + 1$; and, if the orbit of x_0 is infinite, $\Delta(n(n-1)/2) \geq n$, so $\Delta(n) \gtrsim \sqrt{n}$. Indeed, consider a word $w = u_n \dots u_1$ in which u_i is a word of length $\leq i$, chosen such that $x_0 u_i \notin \{x_0, x_0 u_{i-1}^{-1}, x_0 u_{i-2}^{-1} u_{i-1}^{-1}, \dots, x_0 u_1^{-1} \dots u_{i-1}^{-1}\}$; then $\delta(w) \geq n + 1$.

The functions δ and Δ depend on the choice of x_0 and S . However, it is easy to see that their asymptotics do not depend on the choice of the basepoint x_0 and the generating set S :

Lemma 2.3 *If G is finitely generated, then the $\sim$ -equivalence class of Δ does not depend on the choice of S .*

If G acts transitively on X , then the $\sim$ -equivalence class of Δ does not depend on the choice of x_0 .

Proof Let S and S' be two finite generating sets for G ; write each element of S as a word over S' ; and let C be the maximum of such lengths. We temporarily write Δ_S and $\Delta_{x_0,S}$ to remember the dependence on the choices of $x_0 \in X$ and $S \subset G$.

Given $w \in S^*$, let w' be the corresponding rewritten word over S' . We have $|w'| \leq C|w|$ and $\delta_S(w) \leq \delta_{S'}(w')$, so $\Delta_S(n) \leq \Delta_{S'}(Cn)$ for all $n \in \mathbb{N}$, and $\Delta_S \lesssim \Delta_{S'}$. The reverse inequality gives $\Delta_S \sim \Delta_{S'}$, and proves the first part of the lemma.

Now consider two points $x_0, x_1 \in X$, and an element $g \in G$ with $x_0g = x_1$, of length k . Set $S' = \{s^g = g^{-1}sg \mid s \in S\}$. It is clear that S' is a generating set of G . Let $w = w_1w_2 \dots w_\ell$ be a given word, and consider the word $w^g = (g^{-1}w_1g)(g^{-1}w_2g) \dots (g^{-1}w_\ell g)$ over the alphabet S' . Given $i, j \in \{0, \dots, \ell\}$, observe that $x_0w_iw_{i+1} \dots w_\ell \neq x_0w_jw_{j+1} \dots w_\ell$ if and only if $x_0w_i \dots w_\ell g \neq x_0w_j \dots w_\ell g$, if and only if

$$x_1(g^{-1}w_i g) \dots g^{-1}w_\ell g \neq x_1(g^{-1}w_j g) \dots g^{-1}w_\ell g.$$

This implies $\delta_{x_0,S}(w) = \delta_{x_1,S'}(w^g)$. Therefore, we have $\Delta_{x_0,S}(n) = \Delta_{x_1,S'}(n)$; and the first part of the lemma gives $\Delta_{x_0,S} \sim \Delta_{x_1,S}$. $\square$

3 Self-similar groups

Below we recall the definition of some of Grigorchuk's groups. They are groups acting on a rooted tree. The first Grigorchuk group belongs to a smaller class of self-similar groups. We fix our notation for such groups; for more information on self-similar groups, see Nekrashevych's book [32]. Fix an integer $d \geq 2$ called the *degree*. Words $q = q_1 \dots q_n \in \{0, \dots, d-1\}^*$ form the vertex set of a rooted regular tree $\mathcal{T}$, with root the empty word; and $q_1 \dots q_n$ connected by an edge to $q_1 \dots q_{n-1}$.

A *self-similar group* is, by definition, a group *presented* by a map, called the *wreath recursion*, $\psi : G \rightarrow G \wr \mathfrak{S}_d$, from G to its permutational wreath product with the symmetric group $\mathfrak{S}_d$. We write images under ψ in the form

$$\psi(g) = \langle\langle g_0, \dots, g_{d-1} \rangle\rangle \pi \quad \text{with } (g_0, \dots, g_{d-1}) \in G^d \text{ and } \pi \in \mathfrak{S}_d.$$

The wreath recursion ψ defines an action of G by isometries on $\mathcal{T}$, as follows. Consider $g \in G$ and $q = q_1 \dots q_n \in \mathcal{T}$. If $n = 0$, then $qg = q$.

Otherwise, compute $\psi(g) = \langle\langle g_0, \dots, g_{d-1} \rangle\rangle \pi$, and set inductively $qg = (q_1^\pi)(q_2 \dots q_n)g_{q_1}$.

When a self-similar group is given by its wreath recursion, it is assumed that the action on $\mathcal{T}$ is faithful; namely, the group G defined by the wreath recursion $\psi : \Gamma \rightarrow \Gamma \wr \mathfrak{S}_d$ is the quotient G of Γ by the kernel of Γ 's action on $\mathcal{T}$. We then drop ' ψ ' from the notation, and write the wreath recursion on G in the form

$$g = \langle\langle g_0, \dots, g_{d-1} \rangle\rangle \pi \quad \text{or} \quad g = \pi \langle\langle g_0, \dots, g_{d-1} \rangle\rangle.$$

The boundary $\partial\mathcal{T}$ of $\mathcal{T}$ consists of infinite sequences; its elements are called *rays*. If G is a self-similar group acting on $\mathcal{T}$, then G also acts on $\partial\mathcal{T}$. Mainly, the action of G we will be interested in is that on a ray orbit ρG .

3.1 The first Grigorchuk group

An important example of self-similar group was extensively studied by Grigorchuk [20]. It may be defined by its wreath recursion as the 4-generated group G_{012} with generators $\{a, b, c, d\}$; if ε denote the transposition $(0, 1)$,

$$\psi : a \mapsto \langle\langle 1, 1 \rangle\rangle \varepsilon, \quad b \mapsto \langle\langle a, c \rangle\rangle, \quad c \mapsto \langle\langle a, d \rangle\rangle, \quad d \mapsto \langle\langle 1, b \rangle\rangle. \quad (3)$$

Grigorchuk proved in [15] that G_{012} is an infinite, finitely generated torsion group; and, in [16], that G_{012} is a group of intermediate growth.

A presentation of G_{012} by generators and relators was given by Lyssonok [29]: define the endomorphism σ of $\{a, b, c, d\}^*$ by

$$\sigma : a \mapsto aca, \quad b \mapsto d, \quad c \mapsto b, \quad d \mapsto c. \quad (4)$$

Then

$$G_{012} = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, \sigma^n([d, d^a]), \sigma^n([d, d^{(ac)^2a}]) \text{ for all } n \in \mathbb{N} \rangle. \quad (5)$$

The notation G_{012} arises as follows: Grigorchuk defined a continuum of groups G_ω , for $\omega \in \{0, 1, 2\}^\mathbb{N}$. The first Grigorchuk group is in fact the group $G_{(012)^\infty}$ defined by a periodic sequence ω .

3.2 Another Grigorchuk group

By G_{01} we denote in the sequel the following Grigorchuk group. It is the group generated by a, b, c, d and given by the recursion

$$\psi : a \mapsto \langle\langle 1, 1 \rangle\rangle \varepsilon, \quad b \mapsto \langle\langle 1, c \rangle\rangle, \quad c \mapsto \langle\langle a, b \rangle\rangle, \quad d \mapsto \langle\langle a, d \rangle\rangle. \quad (6)$$

In contrast with the first Grigorchuk group, G_{01} contains elements of infinite order. Indeed, the subgroup generated by $\{a, d\}$ is isomorphic to the infinite dihedral group. Furthermore, the infinite-order element ad acts freely on the boundary $\partial\mathcal{T}$ of the tree on which G_{01} acts [17, proof of Lemma 9.10]. If we denote by ρ the ray 1^∞ , we then have for all integers $m \neq n$

$$\rho(ad)^n \neq \rho(ad)^m. \quad (7)$$

The group G_{01} has intermediate growth [11]; the best known lower and upper bounds are, for all $\epsilon > 0$,

$$\exp(n/\log^{2+\epsilon}(n)) \lesssim v_{G_{01}}(n) \lesssim \exp(n/\log^{1-\epsilon}(n)).$$

4 Inverted orbit growth for Grigorchuk's first group

We fix $\rho = 1^\infty$ the ray in the binary tree $\mathcal{T}$. This ray is fixed by b, c, d . We write $\Omega = \{a, b, c, d\}^*$ the set of words over the standard generators. The length of $w \in \Omega$ is written $|w|$. The recursion (3) gives rise to a map $\Omega \rightarrow \mathfrak{S}_2 \times \Omega \times \Omega$, defined by the same formulas. We still write it in the form $w \mapsto \varepsilon^s \langle\langle u, v \rangle\rangle$.

We call a word $w \in \Omega$ *pre-reduced*, if it does not contain two consecutive occurrences of b, c, d ; the *pre-reduction* of w is the word obtained from w by deleting consecutive bb, cc, dd and replacing bc or cb by d , cd or dc by b , and db or bd by c . These operations do not change the image of the word in G_{012} . Recall that, for a word $w = w_1 \dots w_n \in \Omega$, we defined

$$\delta(w) = \#\{\rho w_{i+1} \dots w_n \mid i = 0, \dots, n\}.$$

Lemma 4.1 $\delta(w) = \delta(\text{its pre-reduction})$.

Proof Consider a subword $w_j w_{j+1}$ of w consisting only of b, c, d 's, and let u denote the shorter word obtained by replacing $w_j w_{j+1}$ by its value. In computing $\delta(w)$, either $i \leq j$ or $i > j + 1$, in which case $\rho w_i \dots w_n = \rho u_i \dots u_{n-1}$; or $i = j + 1$, in which case $\rho w_i \dots w_n = \rho w_{j+2} \dots w_n = \rho u_{i+1} \dots u_{n-1}$, because w_j and w_{j+1} fix ρ . $\square$

Let $\eta \approx 0.811$ be the real root of the polynomial $X^3 + X^2 + X - 2$, and consider on Ω the norm defined by

$$\|a\| = 1 - \eta^3, \quad \|b\| = \eta^3, \quad \|c\| = 1 - \eta^2, \quad \|d\| = 1 - \eta;$$

namely, for a word $w = w_1 \dots w_n \in \Omega$ set $\|w\| = \|w_1\| + \dots + \|w_n\|$. The norm induced on G_{012} by $\|\cdot\|$ was considered by the first author in [1]. As we

have already mentioned, the first Grigorchuk group satisfies the contraction property in (2). Here as a word metric in G_{012} one can consider the word metric with respect to the generating set a, b, c, d . The idea of [1] was that if instead of the word metric we consider the norm as defined above, this leads to a better contraction coefficient η in (2) and a better upper bound of the form $\exp(n^\alpha)$ for the growth of G_{012} . In this paper we use this norm in order to get upper bounds on the growth for extensions of G_{012} .

Note that the norm $\|\cdot\|$ and the word length $|\cdot|$ are equivalent. If w is pre-reduced of length n , then it contains at least $(n-1)/2$ times the letter 'a'. We may therefore apply the argument in [1, Proposition 4.2], which we reproduce here for completeness, with words *in lieu* of group elements:

Lemma 4.2 (See [1, Proposition 4.2]) *Consider $w \in \Omega$ pre-reduced, and write $w = \varepsilon^s \langle\langle u, v \rangle\rangle$ for $u, v \in \Omega$ and $s \in \{0, 1\}$. Define $C = \eta\|a\|$. We then have*

$$\|u\| + \|v\| \leq \eta\|w\| + C.$$

Proof Since w is pre-reduced, say of length n , it contains at most $(n+1)/2$ letters in $\{b, c, d\}$. For each of these letters, we consider the corresponding letter(s) in u and v given by the wreath recursion. We have

$$\eta(\|a\| + \|b\|) = \|a\| + \|c\|,$$

$$\eta(\|a\| + \|c\|) = \|a\| + \|d\|,$$

$$\eta(\|a\| + \|d\|) = 0 + \|b\|.$$

As $b = \langle\langle a, c \rangle\rangle$ and $aba = \langle\langle c, a \rangle\rangle$, each b in w contributes $\|a\| + \|c\|$ to the total weight of u and v ; the same argument applies to c and d . Now, grouping together each letters in $\{b, c, d\}$ with the 'a' after it (this is possible for all letters except possibly the last), we see that $\eta\|w\|$ is a sum of left-hand terms, possibly $-\eta\|a\|$; while $\|u\| + \|v\|$ is the sum of the corresponding right-hand terms. $\square$

Let us write $\Delta(n) = \max\{\delta(w) \mid n \geq \|w\|\}$; this function is equivalent to that defined in Sect. 2.2. We state the following general lemma:

Lemma 4.3 *Let $\Delta : \mathbb{N} \rightarrow \mathbb{N}$ be a function. Let $\eta \in (0, 1)$ and C be such that, for all $n \in \mathbb{N}$, there exists $\ell, m \in \mathbb{N}$ with $\ell + m \leq \eta n + C$ and $\Delta(n) \leq \Delta(\ell) + \Delta(m)$. Set*

$$\alpha = \log(2)/\log(2/\eta).$$

Then we have for all $n \in \mathbb{N}$

$$\Delta(n) \lesssim n^\alpha.$$

Proof Define $K = C/(2 - \eta)$ and $M = C/(1 - \eta)$. We will prove, in fact, $\Delta(n) \leq L(n - K)^\alpha$ for some constant L and all n large enough.

For that purpose, let L be large enough so that $\Delta(n) \leq L(n - K)^\alpha$ for all $n \leq M$. Set $N = K/(1 - \alpha)$, define Δ^* by

$$\Delta^*(n) = \begin{cases} L(n - K)^\alpha & \text{if } n \geq N, \\ 1 + (L(N - K)^\alpha - 1)n/N & \text{if } n \leq N, \end{cases} \quad (8)$$

and note that Δ^* is the convex hull of 1 and $L(n - K)^\alpha$; it is a monotone concave function satisfying $\Delta(n) \leq \Delta^*(n)$ for all $n \leq M$.

Consider now $n > M$. We then have $\ell, m < n$. By induction, we have $\Delta(\ell) \leq \Delta^*(\ell)$ and $\Delta(m) \leq \Delta^*(m)$; so, using concavity of Δ^* ,

$$\begin{aligned} \Delta(n) &\leq \Delta^*(\ell) + \Delta^*(m) \leq 2\Delta^*\left(\frac{1}{2}(\ell + m)\right) \leq 2\Delta^*\left(\frac{\eta}{2}\left(n + \frac{C}{\eta}\right)\right) \\ &= 2L\left(\frac{\eta}{2}\left(n + \frac{C}{\eta}\right) - K\right)^\alpha = 2L\left(\frac{\eta}{2}(n - K)\right)^\alpha \\ &= L(n - K)^\alpha = \Delta^*(n). \end{aligned}$$

Therefore, $\Delta(n) \leq \Delta^*(n) \sim (n - K)^\alpha$ for all $n \in \mathbb{N}$. $\square$

Proposition 4.4 *We have $\delta(w) \lesssim \|w\|^\alpha$ for all $w \in \Omega$. Equivalently, $\Delta(n) \lesssim n^\alpha$.*

Proof We show that, for some C and all $n \in \mathbb{N}$, there exists $\ell, m \in \mathbb{N}$ with $\ell + m \leq \eta n + C$ and $\Delta(n) \leq \Delta(\ell) + \Delta(m)$; the claimed upper bound on Δ will then follow by Lemma 4.3.

Consider a word $w = w_1 \dots w_n$ realizing the maximum in Δ , assumed without loss of generality to be pre-reduced. We will study the inverted orbit of w on X . Write as above $w = \varepsilon^s \langle\langle u, v \rangle\rangle$ with $u = u_1 \dots u_\ell$ and $v = v_1 \dots v_m$; then, as we will see, the inverted orbit of w is made of rays $0x$ for x in the inverted orbit of u , and of rays $1x$ for x in the inverted orbit of v . By Lemma 4.2, we have $\ell + m \leq \eta(n + \|a\|)$. A suffix $w' = w_{i+1} \dots w_n$ has the form $w' = \varepsilon^{s'} \langle\langle u', v' \rangle\rangle$, in which u', v' are respectively suffixes of u, v . We have $\rho w' = 1\rho v'$ if $s' = 0$, and $\rho w' = 0\rho u'$ if $s' = 1$. Therefore,

$$\begin{aligned} \Delta(n) &= \#\{\rho w_{i+1} \dots w_n \mid i = 0, \dots, n\} \\ &\leq \#\{0(\rho u_{j+1} \dots u_\ell) \mid j = 0, \dots, \ell\} + \#\{1(\rho v_j \dots v_m) \mid j = 0, \dots, m\} \\ &\leq \Delta(\ell) + \Delta(m). \end{aligned} \quad \square$$

We now explore the range of $\delta(w)$, for various words $w \in \{a, b, c, d\}^n$. Let us insist that different words $u, v \in \Omega$ which have the same value in G_{012} may

very well have widely different δ -values. The following result is not used in this text, but is included to stress the difference between direct and inverted orbit growth:

Remark 4.5 There exists for all n a word $w \in \Omega$ of length 2^n , whose direct orbit on X has length 2^n , and such that $\delta(w) \sim n$. In particular, w has length n in G_{012} .

Proof Consider again the substitution $\sigma : \Omega \rightarrow \Omega$ given by (4), and the word $w_n = \sigma^{n-1}(ad)$. Let v be a suffix of w_n .

Note that $w_n = \langle\langle u, w_{n-1} \rangle\rangle$ for some word u over $\{a, d\}$; therefore, the suffix v has the form $\varepsilon^s \langle\langle u', v' \rangle\rangle$ for some $s \in \{0, 1\}$ and suffixes u', v' of u, w_{n-1} respectively. Now either $i = 1$, so $\rho v = 0(\rho u') \in \{0\rho, 00\rho\}$, or $i = 0$, and $\rho v = 1(\rho v')$; by induction, ρv can take at most $2n$ values when v ranges over all suffixes of w_n .

On the other hand, $\rho w_n = 0^{n+1}\rho$ is at distance $2^n - 1$ from ρ , so the direct orbit of w_n traverses 2^n points. $\square$

Remark 4.6 For comparison, let $\delta'(w)$ denote the size of the direct orbit of w ; namely, $\delta'(w) = \#\{\rho, \rho w_1, \rho w_1 w_2, \dots, \rho w_1 \dots w_n\}$ if $w = w_1 \dots w_n$. Then, for $w = \langle\langle u, v \rangle\rangle \varepsilon^s$, we have the inequality $\delta'(w) \leq 2\delta'(v)$, from which nothing can be deduced, instead of $\delta(w) \leq \delta(u) + \delta(v)$. As soon as G is infinite, there is for all $n \in \mathbb{N}$ a word of length n whose direct orbit visits n points.

We now show that the estimate in Proposition 4.4 is optimal:

Proposition 4.7 *There exists a constant C such that, for all $n \in \mathbb{N}$, there exists a word $w_n \in \Omega$ of length at most $C(2/\eta)^n$ with $\delta(w) \geq 2^n$.*

Proof Write $\Omega' = \{ab, ac, ad\}^* \subset \Omega$, consider the substitution $\zeta : \Omega' \rightarrow \Omega'$ given by

$$\zeta : ab \mapsto abadac, \quad ac \mapsto abab, \quad ad \mapsto acac,$$

and consider the word $w_n = \zeta^n(ad)$. For example, we have $w_0 = ad$, $w_1 = acac$, $w_2 = abababab$, $w_3 = (abadac)^4, \dots$

Counting the number of occurrences of ab, ac, ad in the word w_n , we get

$$|w_n| = (2 \ 2 \ 2) \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}^n \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix};$$

the characteristic polynomial of the 3×3 matrix is $X^3 - X^2 - 2X - 4$. This polynomial has a positive real root $2/\eta$, and two conjugate complex roots of

a smaller absolute value. Therefore, there exists a constant C such that the length of w_n is at most $C(2/\eta)^n$ for all n . (In fact, it is also $\geq C'(2/\eta)^n$ for another constant C' , because the polynomial is irreducible over $\mathbb{Q}$.)

Consider $w = \varepsilon^s \langle\langle u, v \rangle\rangle$ and $w' = \varepsilon^{s'} \langle\langle u', v' \rangle\rangle$. We write $w \simeq w'$ to mean that $s = s'$, that u, u' have the same pre-reduction, and that v, v' have the same pre-reduction. Under the wreath recursion, we have

$$\begin{aligned}\zeta(ab) &= \varepsilon \langle\langle aba, c1d \rangle\rangle \simeq \varepsilon \langle\langle aba, b \rangle\rangle, \\ \zeta(ac) &= \langle\langle ca, ac \rangle\rangle, \quad \zeta(ad) = \langle\langle da, ad \rangle\rangle.\end{aligned}$$

It follows that, for any word $av \in \Omega'$, we have

$$\zeta(av) \simeq \begin{cases} \varepsilon \langle\langle ava, v \rangle\rangle & \text{if } \zeta(av) \text{ contains an odd number of } a\text{'s,} \\ \langle\langle va, av \rangle\rangle & \text{if } \zeta(av) \text{ contains an even number of } a\text{'s.} \end{cases}$$

In particular, let us denote by $w'_{n-1} := a^{-1}w_{n-1}a$ the word obtained from $w_{n-1}a$ by deleting its initial a ; then $w_n \simeq \langle\langle w'_{n-1}, w_{n-1} \rangle\rangle$.

Let now u be a suffix of w_{n-1} . There exists then a suffix v of w_n such that $v \simeq \langle\langle *, u \rangle\rangle$. Similarly, let u' be a suffix of w'_{n-1} . There exists then a suffix v' of w_n such that $v' \simeq \varepsilon \langle\langle u', * \rangle\rangle$. Now $\rho v = 1\rho u$, and $\rho v' = 0\rho u'$; so $\mathcal{O}(w_n) \supseteq 1\mathcal{O}(w_{n-1}) \sqcup 0\mathcal{O}(w'_{n-1})$, and

$$\delta(w_n) \geq \delta(w_{n-1}) + \delta(w'_{n-1}).$$

A similar reasoning shows $\delta(w'_n) \geq 2\delta(w'_{n-1})$. Since $\delta(w_0) = 2$ and $\delta(w'_0) = 1$, we get

$$\delta(w_n) \geq 2^n + 1, \quad \delta(w'_n) \geq 2^n.$$

(These inequalities are in fact equalities.) □

Corollary 4.8 *There exist constants $C_1, C_2 > 0$ such that, for any $\ell \in \mathbb{N}$,*

- (1) *for all words w of length ℓ , we have $\delta(w) \leq C_1 \ell^\alpha$;*
- (2) *there exists a word w of length ℓ with $\delta(w) \geq C_2 \ell^\alpha$.*

We will also need to control the total number of possibilities $\Sigma(n)$ for the inverted orbit of a word of length $\leq n$. Since the inverted orbit has cardinality at most $\sim n^\alpha$, and lies in the ball of radius n in the Schreier graph ρG , which contains at most $2n + 1$ points (see Fig. 1), we have $\Sigma(n) \lesssim \binom{2n+1}{n^\alpha}$; but this estimate is too crude for our purposes. We improve it as follows:

Lemma 4.9 *Set $\Sigma(n) = \#\{\mathcal{O}(w) \mid n \geq |w|\}$. Then $\Sigma(n) \lesssim \exp(n^\alpha)$.*

Proof Recalling that the norms $\|\cdot\|$ and $|\cdot|$ are equivalent, we consider w with $\|w\| \leq n$, assumed without loss of generality to be pre-reduced (because the inverted orbit is invariant under pre-reduction). We write $w = \varepsilon^s \langle\langle u, v \rangle\rangle$ with $\|u\| = \ell$, $\|v\| = m$, and recall that a suffix w' of w has the form $w' = \varepsilon^{s'} \langle\langle u', v' \rangle\rangle$ for suffixes u', v' of u, v respectively. As in Proposition 4.4, we then have $\mathcal{O}(w) = \mathcal{O}(u) \sqcup \mathcal{O}(v)\varepsilon$; we get

We reuse the notation of Proposition 4.4, and take as Ansatz

for a function Δ^* as in (8), with a large enough constant L that (9) holds whenever $n \leq M$. Then, because $\exp(\Delta^*(n))/n$ is log-concave, we get as before

and (9) holds by induction for all $n \in \mathbb{N}$. $\square$

Recall that G_{012} denotes the first Grigorchuk group, and that X denotes the orbit of $\rho = 1^\infty$ under the right G_{012} -action on the (boundary of the) binary tree $\mathcal{T} = \{0, 1\}^*$.

Lemma 5.1 *Let A be a non-trivial group having growth $\sim \exp(n/\phi(n))$, and assume that $n/\phi(n)$ is concave. Consider the wreath product $W = A \wr_X G_{012}$. Then the growth of W is $\sim \exp(n/\phi(n^{1-\alpha}))$.*

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for $k \sim n^\alpha$. Choose then k elements $a_1, \dots, a_k$ of length $\leq n^{1-\alpha}$ in A . Define $f \in \sum_X A$ by $f(x_i) = a_i$, all unspecified values being 1. Then $wf \in W$ may be expressed as a word of length $n + |a_1| + \dots + |a_k| \sim 2n$ in the standard generators of W .

Furthermore, different choices of a_i yield different elements of W ; and there are $\sim \exp(n^{1-\alpha}/\phi(n^{1-\alpha}))^{n^\alpha} = \exp(n/\phi(n^{1-\alpha}))$ choices for all the elements of A . This proves the lower bound.

For the upper bound, consider a word w of length n in W , and let $f \in \sum_X A$ denote its value in the base of the wreath product. The support of f has cardinality at most $\delta(w) \lesssim n^\alpha$ by Proposition 4.4, and may take at most $\sim \exp(n^\alpha)$ values by Lemma 4.9.

Write then $\text{supp}(f) = \{x_1, \dots, x_k\}$ for some $k \lesssim n^\alpha$, and let $a_1, \dots, a_k \in A$ be the values of w at its support; write $\ell_i = \|a_i\|$. We now consider two cases. If A is finite, then each of the a_i may be chosen among, at most, $\#A$ possibilities, so there are $\sim \exp(n^\alpha)$ possibilities in total for the element f .

Assume now that A is infinite, so that $v_A(n) \gtrsim n$. Since $\sum \ell_i \leq n$, the lengths of the different elements on the support of f define a composition of a number not greater than n into at most n^α summands; such a composition is determined by n^α “marked positions” among $n + n^\alpha$, so there are at most $\binom{n+n^\alpha}{n^\alpha} \sim \exp(\log(n)n^\alpha)$ such compositions. Each of the a_i is then chosen among $v_A(\ell_i)$ elements, and there are $\sim \exp(\ell_i/\phi(\ell_i))$ such choices for each i .

By our concavity assumption, there are at most $\sim \prod \exp(\ell_i/\phi(\ell_i)) \lesssim \exp(n/\phi(n^{1-\alpha}))$ choices for the elements in A .

We have now decomposed w into data that specify it uniquely, and we multiply the different possibilities for each of the pieces of data. First, there are $\lesssim \exp(n^\alpha)$ possibilities for the value of w in G_{012} , by the upper bound (1). There are $\lesssim \exp(n^\alpha)$ possibilities for the support of w . There are $\lesssim \exp(\log(n)n^\alpha) \exp(n/\phi(n^{1-\alpha}))$ possibilities for the values of w at its support, the first factor counting the number of compositions of n as a sum of n^α terms and the second factor counting the number of elements in A of these lengths. Altogether, we get

$$v_W(n) \lesssim \exp(n^\alpha + n^\alpha + n^\alpha \log(n) + n/\phi(n^{1-\alpha})) \sim \exp(n/\phi(n^{1-\alpha})),$$

and we have obtained the claimed upper bound. $\square$

We are ready to prove the first part of Theorem A.

Theorem 5.2 *Consider the following sequence of groups: $K_0 = \mathbb{Z}/2\mathbb{Z}$, and $K_{k+1} = K_k \wr_X G_{012}$. Then every K_k is a finitely generated infinite torsion group, with growth function*

$$v_{K_k}(n) \sim \exp(n^{1-(1-\alpha)^k}).$$

Proof We start by $\phi_0 = n$; then $\phi_{k+1}(n) = \phi_k(n^{1-\alpha})$, so $\phi_k(n) = n^{(1-\alpha)^k}$. $\square$

Theorem 5.3 *Consider the following sequence of groups: $L_0 = \mathbb{Z}$, and $L_{k+1} = L_k \wr_X G_{012}$. Then their growth functions satisfy*

$$v_{L_k}(n) \sim \exp(\log(n)n^{1-(1-\alpha)^k}).$$

Proof We start by $\phi_0 = n/\log(n)$; then $\phi_{k+1}(n) = \phi_k(n^{1-\alpha})$. Now $\log(n^{1-\alpha}) \sim \log(n)$, so we get $\phi_k(n) \sim n^{(1-\alpha)^k}/\log(n)$. $\square$

Example 5.4 The inverted orbits of G_{01} , for its action on the orbit X of the rightmost ray $\rho = 1^\infty$, have linear growth. Moreover, because G_{01} contains the infinite-order element ad acting freely on X , as we saw in (7), the permutational wreath product of any group A with (G_{01}, X) contains the wreath product $A \wr \mathbb{Z}$. Therefore, for any $A \neq \{1\}$ the growth of this permutational wreath product is exponential.

Example 5.5 Consider $G = G' = G_{012} \times G_{01}$, and let X and X' be respectively the orbits of the rightmost ray $\rho = 1^\infty$ under the action of G_{012} , respectively G_{01} , on the rooted tree. Extend the action of G_{012} on X to an action of G by making G_{01} act trivially, and extend the action of G_{01} on X' to an action of G' by making G_{012} act trivially.

Let A be a finite group containing at least two elements. The unmarked Schreier graph of (G, X) is the same as the unmarked Schreier graph of (G', X') .

However, the growth of the wreath product $A \wr_X G$ is subexponential, whereas the growth of $A \wr_{X'} G'$ is exponential.

Indeed, observe that $W = A \wr_X G = A \wr_X G_{012} \times G_{01}$. By Theorem 5.2 we know that $K_1 = A \wr_X G_{012}$ has intermediate growth. We see that W is a direct sum of two groups of intermediate growth, and hence the growth of this group is intermediate.

On the other hand $W' = A \wr_{X'} G' = A \wr_{X'} G_{01} \times G_{012}$, and already the first factor has exponential growth, see Example 5.4.

The unmarked Schreier graph of (G_{012}, X) , as well as the Schreier graph of (G_{01}, X') , are rays, in which every second edge has been duplicated, a loop has been added at each vertex, and three loops are drawn at the origin (see Fig. 1). The unmarked Schreier graphs of (G, X) and (G', X') are obtained from that graph by drawing four additional loops at each vertex.

Remark 5.6 Let N be a finitely generated nilpotent group, acting transitively on an infinite set X . Then the inverted orbits for this action have linear growth: that is, there exists $C > 0$ such that for any $n > 1$ there exist a word w_n of

length n such that its inverted orbit for the action on X visits at least Cn points.

Proof Take $A = \mathbb{Z}/2\mathbb{Z}$ and let G be the wreath product of A with (N, X) . Observe that G is an extension of an Abelian group by a nilpotent group, so G is solvable. Since N and A are finitely generated, so is G . We know that G contains as a subgroup $\sum_X A$. Since G contains an infinitely generated subgroup, we conclude that G is not virtually nilpotent. Therefore, G has exponential growth. However, N has subexponential growth, so if it also had sublinear inverted orbit growth then G would have subexponential growth. $\square$

5.1 Torsion-free examples

Grigorchuk constructed in [18, Sect. 5] a torsion-free group H of intermediate growth. We recall the basic steps: start by

$$H_0 = \langle a, b, c, d \mid [a^2, b], [a^2, c], [a^2, d], [b, c], [b, d], [c, d] \rangle,$$

and define a wreath recursion $\psi : H_0 \rightarrow H_0 \wr \mathfrak{S}_2$ by the same formula (3) as for Grigorchuk's first group, namely

$$\psi : a \mapsto \langle \langle 1, 1 \rangle \rangle \varepsilon, \quad b \mapsto \langle \langle a, c \rangle \rangle, \quad c \mapsto \langle \langle a, d \rangle \rangle, \quad d \mapsto \langle \langle 1, b \rangle \rangle.$$

Set $K_0 = 1$ and inductively $K_{n+1} = \psi^{-1}(K_n \times K_n)$. Define then $H = H_0 / \bigcup_{n \geq 0} K_n$, and use the same notation for the generators a, b, c, d of H and the induced homomorphism $\psi : H \rightarrow H \wr \mathfrak{S}_2$. Note that H is the largest quotient of H_0 such that the restriction of ψ to $\langle b, c, d, b^a, c^a, d^a \rangle$ is injective.

We view the group H in terms of permutational extensions, and compute its growth function by adapting Lemma 5.1.

Note first that the natural map $\xi : a \mapsto a, b \mapsto b, c \mapsto c, d \mapsto d$ defines a homomorphism from H to G_{012} . Consider the subgroup $C = \langle b^2, c^2, d^2, bcd \rangle$ of H ; then, for instance using the presentation (5) of G_{012} , the normal closure $\langle a^2, C \rangle^H$ equals $\ker \xi$.

Grigorchuk proves in [18, pp. 199–200] that $C \cong \mathbb{Z}^3$, and that $\langle a^2 \rangle \cong \mathbb{Z}$. Because a^2 is central in H , we have exact sequences

$$\begin{aligned} 1 &\longrightarrow \langle a^2 \rangle \longrightarrow H \longrightarrow H / \langle a^2 \rangle \longrightarrow 1, \\ 1 &\longrightarrow C^H \longrightarrow H / \langle a^2 \rangle \longrightarrow G_{012} \longrightarrow 1. \end{aligned}$$

Let ψ_0, ψ_1 be the coordinates of ψ , namely, the set-maps defined by the projections $H \rightarrow H \wr \mathfrak{S}_2 \rightarrow H \times H \rightarrow H$, as in $\psi(g) = \langle \langle \psi_0(g), \psi_1(g) \rangle \rangle \varepsilon^s$. Note that ψ_0, ψ_1 are not homomorphisms, but their restriction to $B := \langle b, c, d \rangle$ is a homomorphism; ψ_0 maps to $\langle a \rangle$, while ψ_1 permutes cyclically

b, c, d . Consider $\tau = \tau_1 \tau_2 \dots \in \{0, 1\}^\infty$ a ray in $\mathcal{T}$. Given $g \in H$, it is easy to see (see [18, p. 200]) that $\psi_{\tau_n} \psi_{\tau_{n-1}} \dots \psi_{\tau_1}(g)$ belongs to $B := \langle b, c, d \rangle$ for all n large enough. Recall the endomorphism σ from (4); it induces an automorphism of B permuting cyclically b, d, c , so we have $\sigma(\psi_1(g)) = g$ for all $g \in B$. The sequence $\sigma^n \psi_{\tau_n} \psi_{\tau_{n-1}} \dots \psi_{\tau_1}(g)$ eventually stabilizes, and we call its limit $g_\tau \in B$ the *germ* of g at τ . Note that $g_\tau = 1$ unless τ is in the H -orbit of $\rho = 1^\infty$. For an element $g \in B$, its germs are $g_\rho = g$ and $g_\tau = 1$ for all $\tau \neq \rho$. Similarly, for $g \in B$ and $x \in H$, the germs of the conjugate $x^{-1}gx = g^x \in B^x$ are $(g^x)_{\rho x} = g$ and $(g^x)_\tau = 1$ for all $\tau \neq \rho x$.

Lemma 5.7 *An element of H is determined by its projection to G_{012} , its a -exponent sum, and its germs; different choices give different elements of H :*

$$\ker \xi = \langle a^2 \rangle \times \sum_{x \in (G_{012})_\rho \backslash G_{012}} C^x \cong \mathbb{Z} \times \sum_X \mathbb{Z}^3.$$

Proof On the one hand, a^2 is central, and generates a split copy of $\mathbb{Z}$. As we noted above, $\ker \xi$ is generated by a^2 and conjugates of C .

Consider next $y, z \in C$, and $g \in H$; we show that y and z^g commute. Write $g = \langle\langle g', g'' \rangle\rangle \varepsilon^s$ for $g', g'' \in H$ and $s \in \{0, 1\}$. Write also $y = \langle\langle a^{2k}, y' \rangle\rangle$ and $z = \langle\langle a^{2\ell}, z' \rangle\rangle$. If $s = 1$, then $z^g = \langle\langle (z')^{g'}, (a^{2\ell})^{g''} \rangle\rangle = \langle\langle (z')^{g'}, a^{2\ell} \rangle\rangle$ using the relations in H_0 ; so $[y, z^g] = \langle\langle [(z')^{g'}, a^{2k}], [a^{2\ell}, y'] \rangle\rangle = 1$. If $s = 0$, then $z^g = \langle\langle (a^{2\ell})^{g'}, (z')^{g''} \rangle\rangle = \langle\langle a^{2\ell}, (z')^{g''} \rangle\rangle$; so $[y, z^g] = \langle\langle [a^{2\ell}, a^{2k}], [y', (z')^{g''}] \rangle\rangle = \langle\langle 1, [y', (z')^{g''}] \rangle\rangle$, and now, because g'' is shorter than g , we eventually have $g \in B$, so by induction $[y, z^g] = 1$.

Consider finally $y \in C$ and $h \in H$ whose image $\xi(h)$ in G_{012} fixes ρ ; we show that y and h commute. Write again $y = \langle\langle a^{2k}, y' \rangle\rangle$ and $h = \langle\langle h', h'' \rangle\rangle$; then $[y, h] = \langle\langle 1, [y', h''] \rangle\rangle$, and by induction eventually $h \in B$ so $[y, h] = 1$. It now follows that $\ker \xi$ is a quotient of $\mathbb{Z} \times \sum_X \mathbb{Z}^3$.

On the other hand, consider an element h of H of the form $y_1^{g_1} \dots y_\ell^{g_\ell}$ for some distinct $g_i \in (G_{012})_\rho \backslash G_{012}$ and $y_i \in C$. Consider some $i \in \{1, \dots, \ell\}$; then the germ $h_{\rho g_i}$ equals y_i , so no relations occur among the elements of $\mathbb{Z} \times \sum_X \mathbb{Z}^3$ when they are mapped to $\ker \xi$. $\square$

The difference between H and the wreath product $\mathbb{Z}^3 \wr_X G_{012}$ is twofold: H is not a split extension of $\sum_X \mathbb{Z}^3$; and the generator $a \in G_{012}$ lifts to an infinite-order element of H . We nevertheless show that H and $\mathbb{Z}^3 \wr_X G_{012}$ have the same asymptotic growth:

Proposition 5.8 *The group H has growth $\sim \exp(\log(n)n^\alpha)$.*

Proof We define a set-theoretic splitting ν of $\xi : H \rightarrow G_{012}$ by the condition that, for all $g \in G_{012}$, the germs $\nu(g)_\tau$ all belong to $\{1, b, c, d\}$, and that the total exponent sum $|\nu(g)|_a$ of a in $\nu(g)$ is 0 or 1.

By Lemma 5.7, elements $h \in H$ can, and will, be put in the form $a^{2\ell} f v(g)$, with $g \in G_{012}$, $f : X \rightarrow C$ finitely supported, and $\ell \in \mathbb{Z}$. We consider the effect of left-multiplying such an expression by a generator $t \in \{a, b, c, d\}^{\pm 1}$. First, consider $t = a^k$ for $k \in \{1, -1\}$. Write $|v(g)|_a + k = n + 2m$, with $n \in \{0, 1\}$. Then

$$th = a^{2\ell} t f v(g) = a^{2(\ell+m)} (t \cdot f) v(ag).$$

Consider next $t \in \{b, c, d\}^{\pm 1}$. Then

$$th = a^{2\ell} t f v(g) = a^{2\ell} (t \cdot f) t v(g);$$

now, in $\langle b, c, d \rangle$, write $t v(g)_\rho = z r$ for $z \in C$ and $r \in \{1, b, c, d\}$. Denote still by z the function $X \rightarrow C$ which takes value z at ρ and is trivial everywhere else. We then have

$$th = a^{2\ell} (t \cdot f) z v(tg).$$

It follows that the action of a generator on an element of H , written in the form $a^{2\ell} f v(g)$, is by translation of f (just as in the wreath product $\mathbb{Z}^3 \wr_X G_{012}$), possibly followed by a multiplication at ρ of f by a generator of C or its inverse.

More pedantically, the computation above shows that the cocycle $\eta(g, h) := v(gh)^{-1} v(g) v(h)$ associated with the extension $1 \rightarrow C^H \rightarrow H \rightarrow G_{012} \rightarrow 1$ is controlled in the following manner: if $|g|, |h| \leq n$, then $\eta(g, h) : X \rightarrow C$ is supported on a set of cardinality n^α , and takes values bounded in $\{-n, \dots, n\}$.

The remainder of the growth computation follows closely the argument in Lemma 5.1; we repeat it briefly.

Consider the representations as $a^{2\ell} f v(g)$ of elements $h \in H$ of norm at most n . The element $g = \xi(h)$ belongs to G_{012} and has norm at most n , so may take at most $\exp(Dn^\alpha)$ values, for a predefined constant D . The function f is supported on a set of cardinality at most Cn^α , for another predefined constant C , and takes values between $-n$ and n ; so there are at most $\exp(\log(2n+1)Cn^\alpha)$ possibilities for f . Finally $|\ell| \leq n$. In total, there are $\lesssim \exp(\log(n)n^\alpha)$ values for h .

For the lower bound, consider a word $w = w_1 \dots w_n$ of length n over $\{a, b, c, d\}$ with $\delta(w) \sim n^\alpha$, which exists by Lemma 4.7; write $\mathcal{O}(w) = \{x_1, \dots, x_k\}$ for $k \sim n^\alpha$, and let $i_1, \dots, i_k \in \{1, \dots, n\}$ be such that $\rho w_{i_j} \dots w_n = x_j$ for $j = 1, \dots, k$. Choose then k integers $a_1, \dots, a_k$ between 1 and $n^{1-\alpha}$. Insert $(bcd)^{a_j}$ before position i_j in w , and call the resulting word $v(a_1, \dots, a_k)$.

First, the length of $v(a_1, \dots, a_k)$ is at most $n + 3n^\alpha n^{1-\alpha} = 4n$. Then, the germ at x_j of $v(a_1, \dots, a_k)$ belongs to $\{1, b, c, d\}(bcd)^{a_j}$, so all $v(a_1, \dots, a_k)$ are distinct. It follows that there are at least $(n^{1-\alpha})^{n^\alpha}$ elements of length $4n$ in H . $\square$

Proof of Theorem A, second part For $k = 0$, consider the group $H_0 = \mathbb{Z}$; for $k = 1$, consider the group $H_1 = H$ from Proposition 5.8. For $k > 1$, consider inductively $H_k = H_{k-1} \wr_X H$. They are torsion-free, as extensions of a torsion-free group by a torsion-free group. $\square$

5.2 Orbits on pairs of rays

We gather here some results from [6]. Consider the ray $\rho = 1^\infty$ the ray in the binary tree $\mathcal{T}$, and its orbit $X := \rho G_{012}$. The group G_{012} acts on X , and therefore acts (diagonally) on $X \times X$.

Because G_{012} acts transitively on X , the G_{012} -orbits on $X \times X$ are in bijection with the orbits of the stabilizer $P = (G_{012})_\rho$ on X , and also in bijection with the double cosets PgP of P in G_{012} .

The set of orbits of G_{012} on $X \times X \setminus \{(x, x) \mid x \in X\}$ may be readily described. A pair of distinct points $(x, y) \in X \times X$ determines a bi-infinite path in $\mathcal{T}$, namely the path γ that starts from x , goes to the root of $\mathcal{T}$, and leaves towards y . Let $\bar{\gamma}$ denote the geodesic path (without backtracking) associated with γ , and let $(x|y) \in \mathbb{N}$ denote the minimal distance of this geodesic to the root of $\mathcal{T}$. The action of G_{012} on $\mathcal{T}$ and on $\partial\mathcal{T}$ induces an action on bi-infinite geodesics in $\mathcal{T}$, so $(x|y)$ is preserved by the G_{012} -action. We now show that pairs (x, y) and (x', y') belong to the same G_{012} -orbit if and only if $(x|y) = (x'|y')$.

We summarize the results:

Lemma 5.9 [6, Lemma 9.10] *The orbits of the stabilizer $P = (G_{012})_\rho$ of ρ on $X \setminus \{\rho\}$ are described as follows:*

$$\mathcal{O} = \{1^k 0\{0, 1\}^* \rho \mid k \in \mathbb{N}\}. \quad (10)$$

In particular, there are infinitely many orbits of G_{012} on $X \times X$. For $x \neq y \in X$, we denote by $(x|y) \in \mathbb{N}$ the length of the maximal common prefix of x, y ; it is the distance to the root of the geodesic in $\mathcal{T}$. We also set $(x|x) = \infty$. The orbit of (x, y) is then completely determined by $(x|y) \in \mathbb{N} \cup \{\infty\}$.

Recall the endomorphism σ from (4). The set $T = \{\sigma^n(a) \mid n \in \mathbb{N}\}$ is a set of non-trivial double coset representatives of P . $\square$

A generating set for P has also been computed:

Lemma 5.10 [6, Theorem 4.4] *The stabilizer $P = (G_{012})_\rho$ is generated by*

$$U = \bigcup_{n \in \mathbb{N}} \sigma^n \{b, c, d, d^a, (ac)^4, \sigma([a, b])^a, \sigma^2([a, b])^a\}.$$

5.3 Presentations for wreath products

We recall the notion of L -presentation, introduced in [3]. A group G is *finitely L -presented* if there exists a finitely generated free group $F = \langle S \rangle$, a finite set Φ of endomorphisms of F , and finite subsets Q, R of F , such that $G \cong F / \langle Q \cup \bigcup_{\phi \in \Phi} \phi(R) \rangle^F$. The expression $\langle S | Q | \Phi | R \rangle$ is the corresponding *finite L -presentation*.

In particular, the first Grigorchuk group is finitely L -presented as

$$G_{012} = \langle a, b, c, d | \sigma | a^2, bcd, [d, d^a], [d, d^{(ac)^2a}] \rangle.$$

Proposition 5.11 *Let A be a finitely L -presented group. Then $A \wr_X G_{012}$ is finitely L -presented.*

Proof Cornuier characterizes in [10] when permutational wreath products are finitely presented. A permutational wreath product $A \wr_X G$, for G acting transitively on a transitive G -set $X = P \backslash G$, is presented as follows: as generators, take those of A and G . As relations, take: those of A and G ; the relation $[a, u]$ for every generator a of A and every u in a generating set U of P ; and the relations $[a, b^t]$ for every generators a, b of A and q in a set of double coset representatives of P in G ; namely $t \in T$ with $G = P \sqcup \bigsqcup_{g \in T} PgP$.

In the case of the first Grigorchuk group, a generating set for $P = (G_{012})_\rho$, and a set of double coset representatives, have been computed in [6]; see Lemmata 5.9 and 5.10. Let $\langle S | Q | \Phi | R \rangle$ be a finite L -presentation of A . A finite L -presentation for $A \wr_X G_{012}$ is then $\langle S, a, b, c, d | Q | \Phi' \cup \{\sigma'\} | R \cup R' \rangle$, with

Φ' the endomorphisms in Φ , extended by fixing a, b, c, d ;

$\sigma' = \sigma$ on $\{a, b, c, d\}$, and fixing S ;

$$R' = \{[s, b], [s, d^a], [s, (ac)^4], [s, \sigma([a, b])^a], [s, \sigma^2([a, b])^a] \mid s \in S\} \\ \cup \{[s', s^a] \mid s, s' \in S\}. \quad \square$$

Corollary 5.12 *The groups K_k from Theorem 5.2 and L_k from Theorem 5.3 are finitely L -presented.* $\square$

Example 5.13 A recursive presentation for the group $K_1 = \mathbb{Z}/2\mathbb{Z} \wr_X G_{012}$ is

$$K_1 = \langle a, b, c, d, s \mid a^2, b^2, c^2, d^2, s^2, bcd, \sigma^n(r_1), \dots, \sigma^n(r_8) \text{ for all } n \in \mathbb{N} \rangle$$

for σ the same endomorphism as in (4), extended by $\sigma(s) = s$, and iterated relations

$$\begin{aligned} r_1 &= [d, d^a], & r_2 &= [d, d^{(ac)^2a}], \\ r_3 &= [s, s^a], & r_4 &= [s, b], \end{aligned}$$

$$\begin{aligned} r_5 &= [s, d^a], & r_6 &= [s, (ac)^4], \\ r_7 &= [s, \sigma([a, b])^a], & r_8 &= [s, \sigma^2([a, b])^a]. \end{aligned}$$

6 Embeddings of the group of finitely supported permutations

Theorem 6.1 *There exists a group H of intermediate growth that contains as a subgroup the group $\mathfrak{S}_\infty$ of finitely supported permutations of an infinite countable set.*

Moreover, the group H can be chosen in such a way that its growth function satisfies

$$\exp(n^\alpha) \lesssim v_H(n) \lesssim \exp(\log(n)n^\alpha).$$

Proof If G acts on a set X , then it acts on the group of finitely supported permutations of X : for $\sigma : X \rightarrow X$ with $\sigma(x) = x$ for all $x \in X$ except finitely many, $g^{-1}\sigma g$ is still finitely supported for all $g \in G$.

Let X denote the orbit of the ray $\rho = 1^\infty$ under the action of the first Grigorchuk group G_{012} , and let $\mathfrak{S}_\infty$ denote the group of finitely supported permutations of X . Set $H = \mathfrak{S}_\infty \rtimes G_{012}$. Take as generating set for H the union of the generating set $\{a, b, c, d\}$ of G with the involution s switching 0ρ and ρ .

Consider an element $g = s^{e_1}h_1 \dots s^{e_n} \dots h_n$ of length $\leq 2n$ in this group, for some $h_i \in \{a, b, c, d\}$ and $e_i \in \{0, 1\}$, and write $g_i = h_i h_{i+1} \dots h_n$. Observe that the finitely supported permutation corresponding to this element is obtained as a product of some of the involutions switching $(0\rho)g_i$ and ρg_i . By Proposition 4.4, there are $\lesssim n^\alpha$ possibilities for the first, and in view of Lemma 2.3 also for the second of these two points; so the support of the permutation has cardinality $\lesssim n^\alpha$. An element of H of length $\sim n$ may therefore be described by: an element of G_{012} of length $\lesssim n$; a subset of X of cardinality $\lesssim n^\alpha$; and a permutation of that subset. There are at most, respectively, $v_{G_{012}}(n)$, $\binom{n}{n^\alpha}$, and $(n^\alpha)!$ choices for each of these pieces of data. We get

$$v_H(n) \lesssim \exp(n^\alpha) \exp(\log(n)n^\alpha) (n^\alpha)! \sim \exp(\log(n)n^\alpha).$$

On the other hand, consider the word $w_n = g_1 \dots g_\ell$ given by Proposition 4.7. Set inductively $S_0 = \emptyset$ and $S_k = S_{k-1} \cup \{\rho g_k \dots g_\ell, 0\rho g_k \dots g_\ell\}$; and select positions $k_1, k_2, \dots, k_{2^n}$ such that $S_k \neq S_{k-1}$. Consider then the 2^{2^n} elements of H obtained by inserting, at each position k_i in w_n , the word s^{e_i} for all choices of $e_i \in \{0, 1\}$.

An easy induction shows that all these elements are distinct in H : given such an element, expressed as σg with $\sigma \in \mathfrak{S}_\infty$ and $g = w_n \in G_{012}$, we recover the e_i as follows. Let $x \in X$ be in the support of σ , and in $S_{k_i} \setminus S_{k_i-1}$

for maximal k_i . This determines $e_i = 1$. Right-divide then by $s^{e_i} g_{k_i+1} \dots g_\ell$, and proceed inductively. All other e_j 's are 0.

This gives $v_H(2|w_n|) \geq 2^{2^n}$, proving the lower bound. $\square$

The first examples of groups of intermediate growth that are not residually finite are constructed in [12]. Theorem 6.1 gives new examples of this kind:

Corollary 6.2 *There exist finitely generated groups of growth $\lesssim \exp(\log(n)n^\alpha)$ that contain an infinite simple group as a (normal) subgroup. In particular, such groups provide new examples of non residually finite groups of intermediate growth.*

Proof We recall that the group of even permutations is a characteristic subgroup of the group $\mathfrak{S}_\infty$ of finitely supported permutations, and is simple. $\square$

Remark 6.3 It is shown in [11] that there exist groups of intermediate growth which admit a non-degenerate measure with non-trivial Poisson-Furstenberg boundary. Kaimanovich shows in [24] that the group of finitely supported permutations of a countable set admits a symmetric measure with non-trivial boundary. This provides an example of a measure with non-trivial boundary on the group H from Theorem 6.1. However, the measures obtained in this way are degenerate.

It can be shown that the group H considered in the proof of Theorem 6.1, as well as other groups constructed in this paper, also admit non-degenerate measures with non-trivial Poisson-Furstenberg boundary. We will study random walks on permutational extensions, not restricted to those considered in this paper, in [4]. Some of the groups that will be treated in [4] lead to new phenomena in boundary behavior.

Remark 6.4 After this paper was completed, further developments on the growth of groups appeared in the preprints [5, 9, 25].

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