Uniformization of Sierpiński carpets in the plane

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Abstract Let S_i , $i \in I$, be a countable collection of Jordan curves in the extended complex plane $\widehat{\mathbb{C}}$ that bound pairwise disjoint closed Jordan regions. If the Jordan curves are uniform quasicircles and are uniformly relatively separated, then there exists a quasiconformal map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ such that $f(S_i)$ is a round circle for all $i \in I$. This implies that every Sierpiński carpet in $\widehat{\mathbb{C}}$ whose peripheral circles are uniformly relatively separated uniform quasicircles can be mapped to a round Sierpiński carpet by a quasiasymmetric map.

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1 Introduction

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the extended complex plane equipped with the chordal metric σ given by

$$\sigma(x, y) = \frac{2|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}} \quad \text{for } x, y \in \mathbb{C}, \quad (1)$$

and by a suitable limit of this expression if $x = \infty$ or $y = \infty$. As usual $\widehat{\mathbb{C}}$ can be identified with the unit sphere in $\mathbb{R}^3$ equipped with the restriction of the Euclidean metric by stereographic projection.

A Jordan curve $S \subseteq \widehat{\mathbb{C}}$ is called a *quasicircle* if the following condition holds: there exists a constant $k \geq 1$ such that for all points $x, y \in S$, $x \neq y$, we have the inequality

$$\text{diam}(\gamma) \leq k\sigma(x, y) \quad (2)$$

for the diameter of one of the subarcs γ of S with endpoints x and y . Essentially, this condition rules out cusps for S . Typical examples of quasicircles are von Koch snowflake-type curves. It is well-known that S is a quasicircle if and only if there exists a quasiconformal map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ such that $f(S)$ is a round circle. So the quasicircles are precisely the images of round circles under quasiconformal homeomorphisms on $\widehat{\mathbb{C}}$.

One can ask whether a similar statement is true for a collection $\mathcal{S} = \{S_i : i \in I\}$ of pairwise disjoint quasicircles S_i , where I is a countable index set. So we want to find a quasiconformal homeomorphism f on $\widehat{\mathbb{C}}$ that makes all the quasicircles in the collection simultaneously round.

It is clear that such a map f need not exist if we do not impose further restrictions on the collection $\mathcal{S}$. Indeed, as follows from standard distortion estimates for quasiconformal maps, a necessary condition for the existence of the map f is that $\mathcal{S}$ consists of *uniform quasicircles*: there exists a constant $k \geq 1$ such that each S_i for $i \in I$ is a k -quasicircle, i.e., it satisfies condition (2). Even if the Jordan curves S_i are uniform quasicircles, the desired map f need not exist. An example can be obtained from an infinite collection of disjoint squares that contains a sequence of pairs of squares with parallel sides of equal length such that the distance between the sides goes to zero faster than the sidelength (see Example 10.3).

A way to exclude such examples is to impose *uniform relative separation* on the collection $\mathcal{S}$: there exists a constant $s > 0$ such that

$$\frac{\text{dist}(S_i, S_j)}{\min\{\text{diam}(S_i), \text{diam}(S_j)\}} \geq s, \quad (3)$$

whenever $i, j \in I, i \neq j$. This requirement stipulates that the relative distance of two distinct quasicircles in $\mathcal{S}$ (the distance rescaled by the smaller diameter of the sets) is uniformly bounded from below. The condition of uniform relative separation still allows rather tight collections of quasicircles. For example, the peripheral circles of the standard Sierpiński carpet T (given by the boundaries of the squares used in the construction of T ; see Sect. 12) form a collection of uniformly relatively separated uniform quasicircles.

Even if the collection $\mathcal{S}$ consists of uniformly relatively separated uniform quasicircles, a map f as desired need not exist due to possible nesting of the quasicircles S_i (see Example 10.4). This problem is ruled out if we require that the curves S_i bound pairwise disjoint closed Jordan regions.

If we impose all the conditions on $\mathcal{S}$ as discussed, then we actually get a positive statement as our first main result shows.

Theorem 1.1 *Suppose that $\mathcal{S} = \{S_i : i \in I\}$ is a family of Jordan curves in $\widehat{\mathbb{C}}$ that bound pairwise disjoint closed Jordan regions. If $\mathcal{S}$ consists of uniformly relatively separated uniform quasicircles, then there exists a quasiconformal map $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ such that $f(S_i)$ is a round circle for all $i \in I$.*

The proof will show that this statement is quantitative in the following sense: if $\mathcal{S}$ consists of s -relatively separated k -quasicircles, then the map f can be chosen to be H -quasiconformal with H only depending on s and k .

One can ask to what extent the map f is uniquely determined. Suppose that $\{D_i : i \in I\}$ is a collection of pairwise disjoint closed Jordan regions such that $\partial D_i = S_i$, where the collection $\mathcal{S} = \{S_i : i \in I\}$ is as in Theorem 1.1. Obviously, it is easy to perturb f on the interior $\text{int}(D_i)$ of any of the sets D_i while retaining the roundness of the circles $f(S_i)$. So it is only meaningful to ask for uniqueness of f on the complementary set

$$T = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I} \text{int}(D_i) \quad (4)$$

of the regions $D_i, i \in I$. Again if T has non-empty interior, then f is not unique on T , but it turns out that if T has measure zero, then f is uniquely determined on T up to post-composition with a Möbius transformation. This follows from rigidity statements for *Schottky sets* in $\widehat{\mathbb{C}}$, i.e., compact subsets of $\widehat{\mathbb{C}}$ whose complementary components consist of pairwise disjoint open disks [8, Theorem 1.1].

If one uses this rigidity result in combination with Theorem 1.1, one obtains the following existence and uniqueness statement for the uniformization of Sierpiński carpets by round Sierpiński carpets, i.e., Sierpiński carpets in $\widehat{\mathbb{C}}$ whose complementary components are round disks (see Sect. 12 for terminology).

Corollary 1.2 *Suppose that $T \subseteq \widehat{\mathbb{C}}$ is a Sierpiński carpet whose peripheral circles are uniformly relatively separated uniform quasicircles. Then T can be mapped to a round Sierpiński carpet T' by a quasimetric homeomorphism $f: T \rightarrow T'$.*

If T has spherical measure zero, then the quasimetric map f is unique up to post-composition with a Möbius transformation on $\widehat{\mathbb{C}}$.

In particular, this corollary applies to the standard Sierpiński carpet T (see Sect. 12). For this special case the existence part of the statement was proved earlier by methods different from the ones used in this paper in unpublished joint work by B. Kleiner and the author.

Corollary 1.2 is an analog of a classical uniformization theorem due to Koebe. It states that every finitely connected region in $\widehat{\mathbb{C}}$ can be mapped to a *circle domain* (a region whose complementary components are closed, possibly degenerate disks) by a conformal map. Moreover, this map is unique up to post-composition with an orientation-preserving Möbius transformation. Actually, we will use Koebe's theorem in the proof of Theorem 1.1.

Our investigations were partly motivated by a problem in Geometric Group Theory, the Kapovich-Kleiner conjecture. This conjecture predicts that if a Gromov hyperbolic group G has a boundary at infinity $\partial_\infty G$ that is a Sierpiński carpet, then G should arise from a standard situation in hyperbolic geometry. More precisely, G should admit an action on a convex subset of hyperbolic 3-space $\mathbb{H}^3$ with non-empty totally geodesic boundary where the action is isometric, properly discontinuous, and cocompact [22]. If G admits such an action on $\mathbb{H}^3$, then $\partial_\infty G$ can be identified with a round Sierpiński carpet. The Kapovich-Kleiner conjecture is equivalent to the following uniformization conjecture for metric Sierpiński carpets arising as boundaries of hyperbolic groups.

Conjecture 1.3 *Suppose that G is a Gromov hyperbolic group such that $\partial_\infty G$ is a Sierpiński carpet. Then $\partial_\infty G$ is quasimetrically equivalent to a round Sierpiński carpet $T \subseteq \widehat{\mathbb{C}}$.*

Here the set $\partial_\infty G$ can be considered as a metric space in a natural way by equipping it with a “visual” metric. Though in general there is no unique choice of such a metric, these metrics are quasimetrically equivalent by the identity map.

For Gromov hyperbolic groups G with Sierpiński carpet boundary $\partial_\infty G$ one can show the following properties of the collection of peripheral circles of $\partial_\infty G$.

Proposition 1.4 *Let G be a Gromov hyperbolic group such that $\partial_\infty G$ is a Sierpiński carpet, and let $\mathcal{S}$ be the collection of peripheral circles of $\partial_\infty G$. Then $\mathcal{S}$ consists of uniform quasicircles that are uniformly relatively separated and occur on all locations and scales.*

This proposition will not be a surprise to experts, but it cannot be found in the literature. We will record a proof in Sect. 13 where we also explain the terminology used in the statement. If one combines this proposition with Corollary 1.2, then Conjecture 1.3 is reduced to showing that every Sierpiński carpet $\partial_\infty G$ arising as the boundary at infinity of a Gromov hyperbolic group G can be mapped to a Sierpiński carpet $T \subseteq \widehat{\mathbb{C}}$ by a quasimorphism.

In view of Proposition 1.4, one can ask the more general question whether any metric Sierpiński carpet T whose peripheral circles are uniformly relatively separated uniform quasicircles is quasimetrically equivalent to a round Sierpiński carpet in $\widehat{\mathbb{C}}$. This is not true in general, but in [7] Corollary 1.2 is used to show that this holds under the additional assumption that T has Ahlfors regular conformal dimension less than 2.

Corollary 1.2 is an instance of a new phenomenon that can be formulated as a heuristic principle: *Quasimetric maps on Sierpiński carpets of measure zero behave similarly as conformal maps on regions in $\widehat{\mathbb{C}}$.*

The main point here is that we have analogies of quasimetric maps on Sierpiński carpets with conformal maps, and not as expected, and less surprising, with quasiconformal maps on regions.

The following fact supports our heuristic principle. If Γ is a path family in $\widehat{\mathbb{C}}$ and T is a Sierpiński carpet, then one can assign a type of conformal modulus, the *carpet modulus* $\mathcal{M}_T(\Gamma)$, to Γ that is preserved under quasiconformal maps (and not only quasi-preserved as expected). See Sect. 12 for the details. This notion corresponds to the classical modulus of path families that is preserved under conformal maps. Applications of the carpet modulus to proving rigidity statements for Sierpiński carpets are studied in [9].

Corollary 1.2 in combination with the main result in [8] leads to surprising uniqueness results. Here is an example.

Theorem 1.5 (Three-circle theorem) *Suppose that $T \subseteq \widehat{\mathbb{C}}$ is a Sierpiński carpet of spherical measure zero whose peripheral circles are uniformly relatively separated uniform quasicircles. Let $f: T \rightarrow T$ be an orientation-preserving quasimetric homeomorphism of T onto itself.*

If there exist three distinct peripheral circles S_1, S_2, S_3 of T with $f(S_i) = S_i$ for $i = 1, 2, 3$, or if there exist three distinct fixed points of f in T , then f is the identity map on T .

This theorem is in complete contrast to the topological flexibility of Sierpiński carpets: if T is a carpet, $\{S_i : i = 1, \dots, n\}$ a family of distinct peripheral circles of T , and $\{S'_i : i = 1, \dots, n\}$ another such family, then there exists an homeomorphism $f : T \rightarrow T$ such that $f(S_i) = S'_i$ for $i = 1, \dots, n$ (the author was unable to locate this result in the literature, but it can easily be established by using the methods in [36]).

We will prove another uniformization theorem for Sierpiński carpets that has an application to an extremal problem for carpet modulus.

Theorem 1.6 *Suppose that*

$$T = \widehat{\mathbb{C}} \setminus \bigcup_{i \in \mathbb{N}_0} \text{int}(D_i),$$

is a Sierpiński carpet, where the sets D_i , $i \in \mathbb{N}_0$, are pairwise disjoint closed Jordan regions, and that the collection ∂D_i , $i \in \mathbb{N}_0$, of peripheral circles of T consists of uniformly relatively separated uniform quasicircles.

Then there exists a finite $\mathbb{C}^$ -cylinder A , pairwise disjoint $\mathbb{C}^*$ -squares $Q_i \subseteq A$ for $i \geq 2$, and a quasymmetric map*

$$f : T \rightarrow T' := \overline{A} \setminus \bigcup_{i \geq 2} \text{int}(Q_i)$$

such that $f(\partial D_0) = \partial_o A$, $f(\partial D_1) = \partial_i A$, and $f(\partial D_i) = \partial Q_i$ for $i \geq 2$.

See the discussion after Theorem 9.5 for the terminology employed here. One can show that if T has spherical measure zero, then f is unique up to a Euclidean similarity fixing the origin in $\mathbb{C}$ (this follows from [9, Theorem 1.5]).

To formulate the mentioned application of this theorem to an extremal problem, let Γ be the family of all open paths γ in $\widehat{\mathbb{C}} \setminus (D_0 \cup D_1)$ connecting ∂D_0 and ∂D_1 (see the end of Sect. 6 for the precise definition of such paths). If $A = \{z \in \mathbb{C} : r < |z| < R\}$ we denote by $h_A = \log(R/r)$ the height of A . Then the carpet modulus of Γ with respect to T is given by

$$\mathcal{M}_T(\Gamma) = \frac{2\pi}{h_A}.$$

See Corollary 12.2, where we will also identify the unique extremal weight sequence for $\mathcal{M}_T(\Gamma)$.

Theorem 1.6 and Corollary 12.2 can be considered as limiting cases of statements in classical uniformization (see Theorem 9.12, Corollary 9.13, and Proposition 11.2) or of combinatorial facts related to square tilings (see

[12, 28]). Our proof of Theorem 1.6 relies on the corresponding uniformization statement Corollary 9.13 and a limiting argument. It would be very interesting to find a different proof that proceeds from results on square tilings as a starting point.

We will now give an outline of the proof of Theorem 1.1. The main point is to find a quasisymmetric map g of the set T in (4) onto a set $T' \subseteq \widehat{\mathbb{C}}$ whose complementary components are round disks. The desired quasiconformal map f is then found by “filling the holes” (see Proposition 5.1). Finding this extension f of g involves some subtleties, but can be derived from the classical Beurling-Ahlfors extension theorem (see Theorem 5.2) without too much trouble.

The construction of g is based on the obvious idea to obtain this map as a sublimit of conformal maps that map finite approximations of T to circle domains; more precisely, assuming $I = \mathbb{N}$ we let

$$T_n = \widehat{\mathbb{C}} \setminus \bigcup_{i=1}^n \text{int}(D_i)$$

and invoke Koebe’s Uniformization Theorem to find a map g_n for each $n \in \mathbb{N}$ that is suitably normalized and conformally maps the interior of T_n to a circle domain. We then show that these maps g_n are *uniformly quasisymmetric* (see Theorem 10.2) and hence have a sublimit g with the desired properties (see the proof of Theorem 1.1 in Sect. 10).

The proof of the uniform quasisymmetry of the maps g_n is the main difficulty. The standard method for establishing distortion estimates as required for the quasisymmetry of a map are modulus estimates. In our situation one cannot expect that this method gives the required uniform bounds. The reason is that by removing more and more of the sets $\text{int}(D_i)$ from $\widehat{\mathbb{C}}$, the remaining sets T_n may carry smaller and smaller path families. In particular, if T has measure zero, then every path family in T has vanishing modulus and it is unlikely that classical modulus will lead to the desired bounds.

To overcome these obstacles we use *transboundary modulus* (see Sect. 6). This concept (under the different name “transboundary extremal length”) was introduced by O. Schramm [29] and is a variant of classical conformal modulus. Since in its definition transboundary modulus uses the “holes” (i.e., the complementary components) of a region, we can hope to get uniform positive lower bounds for the transboundary modulus of path families that are relevant for desired distortion estimates of the maps g_n (see Proposition 8.1 for a general statement in this direction).

Unfortunately, while classical modulus is too small for our purpose, transboundary modulus will be too large in general. Essentially, one wants a quantity that is not too small in the source domain, but not too large in the target. Subtle considerations are necessary to navigate around this problem: one only

considers the transboundary modulus of path families in the complement of a controlled number of carefully selected holes of the target domain (see Proposition 8.7 and the further discussion following the statement of this proposition). This will lead to the right balance of modulus estimates for source and target. Carrying out the details involves substantial technicalities. The key steps in the proof are Propositions 7.5, 8.1, and 8.7. They enter the proof of the uniform quasisymmetry statement Theorem 10.2 from which Theorem 1.1 can easily be derived.

The paper is organized as follows. We fix notation and some terminology in Sect. 2. In Sect. 3 we review quasiconformal and related maps, and in Sect. 4 relevant facts about quasicircles. Most of this material is standard, but we have included many details in order to make the paper as self-contained as possible. The extension result Proposition 5.1, already mentioned in the outline of the proof of Theorem 1.1, is proved in the next Sect. 5. Classical and transboundary modulus appear in Sect. 6. In Sect. 7 we discuss Loewner domains and establish Proposition 7.5 which is used in the proof of our main result.

Section 8 is devoted to estimates for transboundary modulus. The main results are the rather technical Propositions 8.1 and 8.7, the former giving a lower and the latter an upper bound for transboundary modulus. They are crucial in the proof of Theorem 1.1. Proposition 8.4 is later applied in Sect. 11.

Results on classical uniformization are discussed in Sect. 9. Apart from Koebe's Uniformization Theorem and some rather standard results on boundary extension of conformal maps, none of this material is used in the proof of Theorem 1.1. The main results in this section are Theorem 9.12 and Corollary 9.13. This corollary is later invoked in the proof of Theorem 1.6. Theorem 9.12 can be derived from results by Schramm [30], but we decided to present the details for the convenience of the reader.

All the preparation is wrapped up in Sect. 10, where a proof of Theorem 1.1 is finally given. It is based on Theorem 10.2 which is of independent interest. Examples 10.3 and 10.4 in this section show that one can neither drop the assumption of uniform relative separation in our main theorem, nor the assumption that the quasicircles S_i bound pairwise disjoint closed Jordan regions.

In Sect. 11 we solve an extremal problem for transboundary modulus (Proposition 11.2). As an application we prove a uniqueness statement for conformal maps (Corollary 11.3). We also prepare and give the proof of Theorem 11.7 which is a slightly more general version of Theorem 1.6. In Sect. 12 we recall the definition of the standard Sierpiński carpet and some related topological facts. In this section we prove Corollary 1.2, Theorem 1.5, and Theorem 1.6, and define the concept of carpet modulus of a curve family. In the final Sect. 13 we establish Proposition 1.4.

2 Notation and terminology

For $a, b \in \mathbb{R}$ we set $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. We let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathbb{C}^* = \{z \in \mathbb{C} : z \neq 0\}$. The symbol i stands for the imaginary unit in the complex plane $\mathbb{C}$.

The chordal metric on $\widehat{\mathbb{C}}$ is denoted by σ (see (1)). We will use the letter Σ to denote the spherical measure on $\widehat{\mathbb{C}}$. So if $M \subseteq \mathbb{C}$ is measurable, then

$$\Sigma(M) = \int_M \frac{4 dm_2(z)}{(1 + |z|^2)^2},$$

where integration is with respect to Lebesgue measure m_2 on $\mathbb{C} \cong \mathbb{R}^2$. Integrals will be extended over $\widehat{\mathbb{C}}$ unless otherwise indicated. We say that a measurable set $M \subseteq \widehat{\mathbb{C}}$ has (spherical) measure zero if $\Sigma(M) = 0$.

Let (X, d) be a metric space. If $a \in X$ and $r > 0$, we denote by

$$B(a, r) = \{x \in X : d(x, a) < r\}$$

the open and by

$$\overline{B}(a, r) = \{x \in X : d(x, a) \leq r\}$$

the closed ball of radius r centered at a . If $A \subseteq X$, then we write $\overline{A}$ for the closure, $\text{int}(A)$ for the interior, ∂A for the topological boundary, and $\text{diam}(A)$ for the diameter of the set A .

For sets $A, B \subseteq X$ we write

$$\text{dist}(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$$

for their distance, and

$$\Delta(A, B) = \frac{\text{dist}(A, B)}{\text{diam}(A) \wedge \text{diam}(B)}$$

for their relative distance if in addition $\text{diam}(A) > 0$ and $\text{diam}(B) > 0$. For $x \in X$ and $A \subseteq X$, we set $\text{dist}(x, A) = \text{dist}(\{x\}, A)$. If $\epsilon > 0$ we denote by

$$N_\epsilon(A) = \{x \in X : \text{dist}(x, A) < \epsilon\}$$

the open ϵ -neighborhood of A in X .

Mostly, it will be clear from the context what metric d we are using. If necessary we put the symbol for the metric as subscript on metric notation. For example, $B_d(x, r)$ denotes the open ball with respect to the metric d , etc. By default, all sets in $\widehat{\mathbb{C}}$ carry the restriction of the chordal metric σ . We

sometimes use the *spherical metric* on $\widehat{\mathbb{C}}$, i.e., the Riemannian metric with length element

$$ds = \frac{2|dz|}{1 + |z|^2}.$$

For sets in $\mathbb{C}$ we also use the Euclidean metric $d_{\mathbb{C}}$ defined by $d_{\mathbb{C}}(x, y) = |x - y|$ for $x, y \in \mathbb{C}$, and for sets in $\mathbb{C}^*$ the flat metric $d_{\mathbb{C}^*}$ (see Sect. 9 for its definition). To distinguish metric notions that refer to $d_{\mathbb{C}}$ or $d_{\mathbb{C}^*}$ from their counterparts with respect to the metric σ , we use the subscript $\mathbb{C}$ or $\mathbb{C}^*$. For example, we denote by $\text{diam}_{\mathbb{C}}(A)$ the Euclidean diameter of a set $A \subseteq \mathbb{C}$, by $\text{length}_{\mathbb{C}^*}(\gamma)$ the length of a path γ (see below) in $\mathbb{C}^*$ with respect to the metric $d_{\mathbb{C}^*}$, etc.

A *circle* in $\widehat{\mathbb{C}}$ is a set of the form

$$S(x, r) := \{y \in \widehat{\mathbb{C}} : \sigma(y, x) = r\},$$

where $x \in \widehat{\mathbb{C}}$ and $0 < r < \text{diam}(\widehat{\mathbb{C}}) = 2$. Sometimes we call these sets also *round circles* to emphasize their distinction from quasicircles or metric circles (see Sect. 4). Similarly, a *round disk* is a (closed or open) metric ball in $\widehat{\mathbb{C}}$ with respect to the metric σ .

If $f: X \rightarrow Y$ is a map between two sets X and Y , and $M \subseteq X$, then $f|_M$ denotes the restriction of f to M .

A *path* in a metric space (X, d) is a continuous map $\gamma: I \rightarrow X$ of an interval $I \subseteq \mathbb{R}$ (i.e., a non-empty connected subset of $\mathbb{R}$) into X . If now confusion can arise, we will also denote by γ the image set $\gamma(I) \subseteq X$ of a path γ . We denote by $\text{length}(\gamma) \in [0, \infty]$ the length of γ . The path $\gamma: I \rightarrow X$ is *rectifiable* if $\text{length}(\gamma) < \infty$, and *locally rectifiable* if $\text{length}(\gamma|_J) < \infty$ for each compact subinterval $J \subseteq I$.

A *region* Ω in $\widehat{\mathbb{C}}$ is an open and connected set. A *Jordan curve* S in $\widehat{\mathbb{C}}$ is a subset of $\widehat{\mathbb{C}}$ homeomorphic to the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. A *closed Jordan region* in $\widehat{\mathbb{C}}$ is a set homeomorphic to the closed unit disc $\overline{\mathbb{D}}$, and an *open Jordan region* a set in $\widehat{\mathbb{C}}$ that is the interior of a closed Jordan region. According to the Schönflies theorem for each Jordan curve $S \subseteq \widehat{\mathbb{C}}$ there exists a homeomorphism $F: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ such that $F(\partial\mathbb{D}) = S$. In particular, every Jordan curve $S \subseteq \widehat{\mathbb{C}}$ has two complementary components in $\widehat{\mathbb{C}}$ both of which are open Jordan regions.

In this paper it is very important to keep track of the dependence of constants and functions on parameters (i.e., other constants and functions). We will write $C = C(a, b, \dots)$ if the constant C can be chosen only depending on the parameters $a, b, \dots$, and $A \leq C(a, b, \dots)$ if the quantity A is bounded by a constant only depending on $a, b, \dots$. For the dependence of functions on parameters we use subscripts to distinguish this dependence from the evaluation of the function on elements of its domain of definition; so $\phi = \phi_{a,b,\dots}$.

means that ϕ is a function that can be chosen only depending on the parameters $a, b, \dots$

Sometimes a property of a space, function, etc. depending on some parameters $a, b, \dots$ implies another property depending on other parameters $a', b', \dots$. If we can choose the parameters $a', b', \dots$ as fixed functions of $a, b, \dots$, that is, only depending on $a, b, \dots$, then we say that the first property implies the second one *quantitatively*. If we have implications of this type in both directions, we call the properties *quantitatively equivalent*. See the remark after Proposition 3.1 for the discussion of a specific example.

We always assume that the parameters are in their natural range or of appropriate type. So, for example, in the phrase “the family $\{S_i : i \in I\}$ is s -relatively separated” it is understood that $s > 0$, and in “ f is an η -quasisymmetry” (see Sect. 3) that η is a distortion function with the right properties, i.e., a homeomorphism $\eta : [0, \infty) \rightarrow [0, \infty)$.

We also often omit quantifying statements (“there exists” or “for all”) related to these parameters for ease of formulation if the intended meaning is clear. For example, we say “ J is a k -quasicircle” instead of “there exists $k \geq 1$ such that J is a k -quasicircle” (cf. statement (i) in Proposition 4.1) or “the maps f_n are η -quasisymmetries for $n \in \mathbb{N}$ ” instead of “there exists a homeomorphism $\eta : [0, \infty) \rightarrow [0, \infty)$ such that the maps f_n are η -quasisymmetries for all $n \in \mathbb{N}$ ” (cf. Lemma 3.3).

3 Quasiconformal and related maps

In this section we summarize basic facts on quasiconformal and related maps (see [20, 23, 33] for general background). Let $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a homeomorphism, and for $x \in \widehat{\mathbb{C}}$ and small $r > 0$ define

$$L_f(r, x) = \sup\{\sigma(f(y), f(x)) : y \in \widehat{\mathbb{C}} \text{ and } \sigma(y, x) = r\}, \quad (5)$$

$$l_f(r, x) = \inf\{\sigma(f(y), f(x)) : y \in \widehat{\mathbb{C}} \text{ and } \sigma(y, x) = r\}, \quad (6)$$

and

$$H_f(x) = \limsup_{r \rightarrow 0} \frac{L_f(x, r)}{l_f(x, r)}. \quad (7)$$

The map f is called *quasiconformal* if

$$\sup_{x \in \widehat{\mathbb{C}}} H_f(x) < \infty.$$

A quasiconformal map f is called *H -quasiconformal*, $H \geq 1$, if

$$H_f(x) \leq H \quad \text{for almost every } x \in \widehat{\mathbb{C}}.$$

We refer to H as the *dilatation* of the map f .

Quasiconformality can be defined similarly in other settings, for example for homeomorphisms between regions in $\widehat{\mathbb{C}}$ or $\mathbb{R}^n$, or between Riemannian manifolds.

The composition of an H -quasiconformal and an H' -quasiconformal map is an (HH') -quasiconformal map. If a homeomorphism f on $\widehat{\mathbb{C}}$ is 1-quasiconformal, then f is a *Möbius transformation*, i.e., a conformal or anti-conformal map on $\widehat{\mathbb{C}}$, and so a fractional linear transformation, or the complex conjugate of such a map. Note that our definition of a Möbius transformation is slightly non-standard in complex analysis as we allow anti-conformal maps.

Let $f: X \rightarrow Y$ be a homeomorphism between metric spaces (X, d_X) and (Y, d_Y) . The map f is called η -*quasisymmetric* or an η -*quasisymmetry*, where $\eta: [0, \infty) \rightarrow [0, \infty)$ is a homeomorphism, if

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta \left(\frac{d_X(x, y)}{d_X(x, z)} \right) \quad (8)$$

for all $x, y, z \in X$ with $x \neq z$. The map f is called *quasisymmetric* or a *quasisymmetry* if it is η -*quasisymmetric* for some distortion function η . If $f: X \rightarrow Y$ is a homeomorphism of X onto a subset of Y and satisfies the distortion condition (8), then f is called an η -*quasisymmetric embedding*. Two metric spaces X and Y are called *quasisymmetrically equivalent* if there exists a quasisymmetry $f: X \rightarrow Y$.

If x_1, x_2, x_3, x_4 are four distinct points in a metric space (X, d) , then their *cross-ratio* is the quantity

$$[x_1, x_2, x_3, x_4] = \frac{d(x_1, x_3)d(x_2, x_4)}{d(x_1, x_4)d(x_2, x_3)}.$$

Let $\eta: [0, \infty) \rightarrow [0, \infty)$ be a homeomorphism, and $f: X \rightarrow Y$ a homeomorphism between metric spaces (X, d_X) and (Y, d_Y) . The map f is (an) η -*quasi-Möbius (map)* if

$$[f(x_1), f(x_2), f(x_3), f(x_4)] \leq \eta([x_1, x_2, x_3, x_4])$$

for every 4-tuple (x_1, x_2, x_3, x_4) of distinct points in X . For these maps we use terminology very similar as for quasisymmetry maps. For example, a *quasi-Möbius embedding* $f: X \rightarrow Y$ is a quasi-Möbius map of X onto a subset of Y .

Note that a Möbius transformation on $\widehat{\mathbb{C}}$ preserves cross-ratios of points. As a consequence every pre- or post-composition of an η -quasi-Möbius map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ with a Möbius transformation is η -quasi-Möbius.

The following proposition records interrelations between the classes of maps we discussed (see [34] for the proof of the statements).

Proposition 3.1

- (i) Every H -quasiconformal map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is η -quasi-Möbius with η depending only on H . Conversely, every η -quasi-Möbius map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is H -quasiconformal with H depending only on η .
- (ii) An η -quasisymmetric map between metric spaces is $\tilde{\eta}$ -quasi-Möbius with $\tilde{\eta}$ depending only on η .
- (iii) Let (X, d_X) and (Y, d_Y) be bounded metric spaces, $f: X \rightarrow Y$ an η -quasi-Möbius map, $\lambda \geq 1$, and $x_1, x_2, x_3 \in X$. Set $y_i = f(x_i)$, and suppose that

$$d_X(x_i, x_j) \geq \text{diam}(X)/\lambda \quad (9)$$

and

$$d_Y(y_i, y_j) \geq \text{diam}(Y)/\lambda \quad (10)$$

for $i, j = 1, 2, 3, i \neq j$. Then f is $\tilde{\eta}$ -quasisymmetric with $\tilde{\eta}$ depending only on η and λ .

The first statement (i) says that for a homeomorphism $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ the properties of being a quasiconformal map and of being a quasi-Möbius map are quantitatively equivalent, or, more informally, that f is a quasiconformal map if and only if f is a quasi-Möbius map, quantitatively.

Statements (ii) and (iii) of the previous proposition imply that a homeomorphism $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a quasisymmetric map if and only if f is a quasi-Möbius map. This statement is not quantitative. For if f is η -quasi-Möbius, then f is $\tilde{\eta}$ -quasisymmetric, but we cannot choose $\tilde{\eta}$ just to depend on η . If one wants a quantitative implication for this direction, one has to introduce additional parameters (such as the parameter λ in (iii)).

A metric space (X, d) is called N -doubling, where $N \in \mathbb{N}$, if every ball of radius $r > 0$ in X can be covered by N or less balls in X of radius $r/2$. Every subset of a doubling metric space is also doubling, quantitatively.

A homeomorphism $f: X \rightarrow Y$ between metric spaces (X, d_X) and (Y, d_Y) is called H -weakly quasisymmetric, $H \geq 1$, if for all $x, y, z \in X$ the following implication holds:

$$d_X(x, y) \leq d_X(x, z) \quad \Rightarrow \quad d_Y(f(x), f(y)) \leq H d_Y(f(x), f(z)).$$

Under mild extra assumptions on the spaces weak quasisymmetry of a map implies its quasisymmetry [20, Theorem 10.19].

Proposition 3.2 *Let (X, d_X) and (Y, d_Y) be metric spaces, and $f: X \rightarrow Y$ be weakly H -quasisymmetric. If X and Y are connected and N -doubling, then f is η -quasisymmetric with η only depending on N and H .*

A metric space is called *proper* if every closed ball in the space is compact. The following lemma will allow us to extract sublimits of a sequence of quasimetric embeddings into a proper metric space.

Lemma 3.3 (Subconvergence lemma) *Let (X, d_X) and (Y, d_Y) be metric spaces such that X is compact and Y is proper, and let $f_n: X \rightarrow Y$ be η -quasisymmetric embeddings for $n \in \mathbb{N}$. Suppose that there exists a constant $c > 0$, a set $A \subseteq X$, and a compact set $K \subseteq Y$ such that*

$$\text{diam}(f_n(A)) \geq c \quad \text{and} \quad f_n(A) \subseteq K$$

for all $n \in \mathbb{N}$. Then the sequence (f_n) subconverges uniformly to an η -quasisymmetric embedding $g: X \rightarrow Y$, i.e., there exists an increasing sequence (n_l) in $\mathbb{N}$ such that

$$\lim_{l \rightarrow \infty} \sup_{x \in X} d_Y(f_{n_l}(x), g(x)) = 0.$$

As discussed at the end of Sect. 2, the intended meaning of the phrase “let $f_n: X \rightarrow Y$ be η -quasisymmetric embeddings for $n \in \mathbb{N}$ ” in this lemma is that the maps f_n are η -quasisymmetric embeddings with the same distortion function η for all n .

Proof We may assume that $K = \overline{B}(y_0, R)$ for some $y_0 \in Y$ and $R > 0$.

We claim that the family (f_n) is uniformly bounded (i.e., there exists $R' > 0$ such that $f_n(X) \subseteq \overline{B}(y_0, R')$ for all n) and that it is equicontinuous. Let $u, v \in X$ be arbitrary. We have to show $d_Y(y_0, f_n(u))$ is uniformly bounded, and that if $d_X(u, v)$ is small, then $d_Y(f_n(u), f_n(v))$ is uniformly small. To see this we consider a fixed map $f = f_n$. For ease of notation we drop the subscript n .

Obviously, A must contain more than one point; so $\text{diam}(A) = a > 0$. There exist points $x_1, x_2 \in A$ such that $d_X(x_1, x_2) \geq a/2$. We can pick $x \in \{x_1, x_2\}$ such that $d_X(x, u) \geq a/4$. Let x' be the other point in $\{x_1, x_2\}$. Note that $f(x), f(x') \in K = \overline{B}(y_0, R)$ and so $d_Y(f(x'), f(x)) \leq 2R$.

Since f is an η -quasisymmetric embedding, this implies

$$\begin{aligned} d_Y(f(u), f(x)) &\leq d_Y(f(x'), f(x)) \eta \left(\frac{d_X(u, x)}{d_X(x', x)} \right) \\ &\leq 2R \eta(2 \text{diam}(X)/a), \end{aligned}$$

and so $f(u) \in \overline{B}(y_0, R')$, where $R' = R(1 + 2\eta(2 \text{diam}(X)/a))$. The uniform boundedness of the sequence (f_n) follows.

Moreover,

$$\begin{aligned} d_Y(f(u), f(v)) &\leq d_Y(f(u), f(x))\eta\left(\frac{d_X(u, v)}{d_X(u, x)}\right) \\ &\leq 2R'\eta(4d_X(u, v)/a). \end{aligned}$$

Since $\eta(t) \rightarrow 0$ as $t \rightarrow 0$ this gives the desired bound for $d_Y(f(u), f(v))$ that is uniformly small if $d_X(u, v)$ is small. The equicontinuity of the sequence (f_n) follows.

By the compactness theorem of Arzelà-Ascoli the sequence (f_n) subconverges to a continuous map $g: X \rightarrow Y$ uniformly on X . Since all the maps f_n are η -quasisymmetric embeddings, the map g satisfies the inequality

$$d_Y(g(u), g(v)) \leq d_Y(g(u), g(w))\eta\left(\frac{d_X(u, v)}{d_X(u, w)}\right),$$

whenever $u, v, w \in X$, $u \neq w$. This inequality implies that g is injective and hence a quasisymmetric embedding, or a constant map; but the latter possibility is ruled out, because a limiting argument shows that $\text{diam}(g(\bar{A})) \geq c > 0$. The proof is complete. $\square$

Lemma 3.4 *Let $a, b > 0$, and (X, d_X) and (Y, d_Y) be metric spaces. Suppose that $x_1, x_2, x_3 \in X$ and $y_1, y_2, y_3 \in Y$ are points such that*

$$d_X(x_i, x_j) \geq a \quad \text{and} \quad d_Y(y_i, y_j) \geq b \quad \text{for } i, j = 1, 2, 3, i \neq j.$$

Then for all $x \in X$ and $y \in Y$ there exists an index $l \in \{1, 2, 3\}$ such that $d_X(x, x_l) \geq a/2$ and $d_Y(y, y_l) \geq b/2$.

Proof At most one of the points x_i can lie in the ball $B(x, a/2)$. So there are at least two of the points x_i , say x_1 and x_2 , that have distance $\geq a/2$ to x . At most one of the points y_1 and y_2 can lie in $B(y, b/2)$; so one, say y_1 , has to lie outside this ball. Then $l = 1$ is an index as desired. $\square$

4 Quasicircles

A Jordan curve $J \subseteq \widehat{\mathbb{C}}$ is called a k -quasicircle for $k \geq 1$ if it satisfies condition (2), that is, whenever $x, y \in J$, $x \neq y$, are arbitrary, then

$$\text{diam}(\gamma) \leq k\sigma(x, y)$$

for one of the subarcs γ of J with endpoints x and y . The curve J is called a *quasicircle* if it is a k -quasicircle for some $k \geq 1$. A family $\mathcal{S} = \{S_i : i \in I\}$ of

Jordan curves S_i in $\widehat{\mathbb{C}}$ is said to consist of *uniform quasicircles* if there exists $k \geq 1$ such that S_i is a k -quasicircle for each $i \in I$.

Various equivalent characterizations of quasicircles and *quasidisks* (Jordan domains bounded by quasicircles) are known (see, for example, [15, 16]). Up to bi-Lipschitz equivalence all quasicircles can be constructed by a procedure similar to the one used in the definition of the von Koch snowflake curve [26].

The following proposition is essentially due to Ahlfors [1]. See [23, Chap. II, §8] for a discussion of related facts.

Proposition 4.1 *Suppose that $J \subseteq \widehat{\mathbb{C}}$ is a Jordan curve. Then the following conditions are quantitatively equivalent:*

- (i) J is a k -quasicircle.
- (ii) J is the image of a round circle under an H -quasiconformal map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$.
- (iii) J is the image of a round circle under an η -quasi-Möbius map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$.

As discussed before, “quantitative” equivalence here means that if condition (i) is true, then H in condition (ii) can be chosen only depending on k in (i), etc. The equivalence (i) $\Leftrightarrow$ (ii) is contained in [1], while (ii) $\Leftrightarrow$ (iii) follows from Proposition 3.1(i). An immediate consequence of Proposition 4.1 is the following fact: if $D \subseteq \widehat{\mathbb{C}}$ is a closed Jordan region whose boundary ∂D is a k -quasicircle, then there exists an η -quasi-Möbius map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ with $\eta = \eta_k$ such that $f(\overline{\mathbb{D}}) = D$.

The following lemma shows that the diameter of every Jordan curve in $\widehat{\mathbb{C}}$ is equal to the diameter of one of the closed Jordan regions whose boundary it is.

Lemma 4.2 *Let $D \subseteq \widehat{\mathbb{C}}$ be a closed Jordan region. Then $\text{diam}(D) = \text{diam}(\partial D)$, or $\text{diam}(D) = 2$ and $\text{diam}(\widehat{\mathbb{C}} \setminus \text{int}(D)) = \text{diam}(\partial D)$.*

Proof We first prove an elementary geometric fact. To state it we identify $\widehat{\mathbb{C}}$ with the unit sphere in $\mathbb{R}^3$ by stereographic projection, and denote by $A: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ the involution that assigns to each point $p \in \widehat{\mathbb{C}}$ its antipodal point (so A is the conjugate of the map $p \mapsto -p$ on the unit sphere by stereographic projection). We say that $p, q \in \widehat{\mathbb{C}}$ form a *pair of antipodal points* in $\widehat{\mathbb{C}}$ if $q = A(p)$ (then also $p = A(q)$). Now suppose $J \subseteq \widehat{\mathbb{C}}$ is a Jordan curve, and U and V are the closures of the two components of $\widehat{\mathbb{C}} \setminus J$. We claim that if each of the sets U and V contains a pair of antipodal points, then J also contains such a pair.

To see this we argue by contradiction and assume that J contains no such pair. In this case $J \cap A(J) = \emptyset$, and so the Jordan curve $A(J)$ must be contained in one of the closed Jordan regions U and V , say $A(J) \subseteq U$. Then

U must contain one of the closed Jordan regions $A(U)$ and $A(V)$ bounded by $A(J)$.

Suppose that $A(U) \subseteq U$. Since A is an involution, this implies $U \subseteq A(U)$, and so $A(U) = U$. Then we have $A(J) = A(\partial U) = \partial U = J$, which contradicts our assumption $J \cap A(J) = \emptyset$.

So we must have $A(V) \subseteq U$. By our hypotheses there exists an antipodal pair $\{v, A(v)\} \subseteq V$. Then $\{v, A(v)\} \subseteq A(V) \subseteq U$, and so $\{v, A(v)\} \subseteq U \cap V = J$. This contradicts our assumption that J contains no antipodal pair. The claim follows.

Now let $D \subseteq \widehat{\mathbb{C}}$ be an arbitrary closed Jordan region. Then there exist points $x, y \in D$ with $\sigma(x, y) = \text{diam}(D)$. If $x, y \in \partial D$, then $\text{diam}(D) \leq \text{diam}(\partial D)$, and so $\text{diam}(D) = \text{diam}(\partial D)$.

In the other case when x, y do not both belong to ∂D , one of these points must be an interior point of D , say $y \in \text{int}(D)$. Consider a minimizing spherical geodesic segment joining x and y . If we were able to slightly extend this geodesic segment beyond y to a minimizing geodesic segment, then we would obtain a point $y' \in D$ near y whose spherical distance to x is strictly larger than the distance of y to x . Since there is a strictly monotonic relation between spherical and chordal distance (spherical distance $s \in [0, \pi]$ corresponds to chordal distance $2 \sin(s/2)$), we would also have $\sigma(x, y') > \sigma(x, y)$. This is impossible, since $x, y' \in D$ and $\sigma(y, x) = \text{diam}(D)$. So the geodesic segment between x and y is not extendible as a minimizing geodesic segment and it must have length π . Then x and y form a pair of antipodal points which implies $\text{diam}(D) = \sigma(x, y) = 2$.

It now follows that $\text{diam}(D') = \text{diam}(\partial D)$ if $D' = \widehat{\mathbb{C}} \setminus \text{int}(D)$ denotes the other Jordan region bounded by ∂D . Indeed, by applying the first part of the argument also to D' , we see that the only case where this may possibly fail is if both D and D' contain a pair of antipodal points. By our claim established in the beginning of the proof, ∂D then contains such a pair as well and we get the desired relation $\text{diam}(D') = 2 = \text{diam}(\partial D)$ anyway. $\square$

The following proposition is a standard fact. We record a proof for the sake of completeness.

Proposition 4.3 *Suppose $D \subseteq \widehat{\mathbb{C}}$ is a closed Jordan region whose boundary ∂D is a k -quasicircle. Then there exists $\lambda = \lambda(k) \geq 1$, $x_0 \in D$, and $r \in (0, 2] = (0, \text{diam}(\widehat{\mathbb{C}})]$ such that*

$$\overline{B}(x_0, r/\lambda) \subseteq D \subseteq \overline{B}(x_0, r). \quad (11)$$

Proof Let $d = \text{diam}(\partial D)$. We first consider the case where

$$\text{diam}(D) > 2d. \quad (12)$$

Since $\text{diam}(D) \leq \text{diam}(\widehat{\mathbb{C}}) = 2$, this implies $d < 1$. Pick a point $p \in \partial D$, and let x_0 be the antipodal point of p on $\widehat{\mathbb{C}}$ (considered as the unit sphere in $\mathbb{R}^3$). Then $\partial D \subseteq \overline{B}(p, d)$. Therefore, the connected set $\widehat{\mathbb{C}} \setminus \overline{B}(p, d) \supseteq B(x_0, 2-d)$ does not meet ∂D and must hence be contained in one of the two closed Jordan regions bounded by ∂D . The other Jordan region must be contained in $\overline{B}(p, d)$, and so has diameter $\leq 2d$. By our assumption (12) this cannot be D . Hence $B(x_0, 2-d) \subseteq D$. Note that $2-d > 1$. Picking $r = 2$ and $\lambda = 2$ we see that we get the desired inclusion

$$\overline{B}(x_0, r/2) = \overline{B}(x_0, 1) \subseteq B(x_0, 2-d) \subseteq D \subseteq \widehat{\mathbb{C}} = \overline{B}(x_0, r).$$

In the remaining case we have

$$\text{diam}(D) \leq 2d. \quad (13)$$

The set D is the image of the closed unit disk

$$\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$$

under an η -quasi-Möbius map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$, where η only depends on k (see the remark after Proposition 4.1). We use a prime to denote image points under f , i.e., $x' = f(x)$ for $x \in \widehat{\mathbb{C}}$.

We can pick points $x_1, x_2, x_3 \in \partial \mathbb{D}$ such that for their image points we have

$$\sigma(x'_i, x'_j) \geq d/2 \quad \text{for } i \neq j. \quad (14)$$

By pre-composing f with a Möbius transformation if necessary, we may assume the points x_i are the third roots of unity. Then

$$\sigma(x_i, x_j) = \sqrt{3} \quad \text{for } i \neq j.$$

Define $z = 0$, and let $u \in \overline{\mathbb{D}}$ and $v \in \partial \mathbb{D}$ be arbitrary. By Lemma 3.4 there exists $w \in \{x_1, x_2, x_3\}$ such that

$$\sigma(u, w) \geq \sqrt{3}/2 \geq 1/2 \quad \text{and} \quad \sigma(v', w') \geq d/4.$$

We also have the inequalities $\sigma(u, z) \leq \sigma(v, z)$,

$$\sigma(v, w) \leq \text{diam}(\overline{\mathbb{D}}) = 2,$$

and

$$\sigma(u', w') \leq \text{diam}(D) \leq 2d.$$

Since f is η -quasi-Möbius, we obtain

$$\frac{\sigma(u', z')}{\sigma(v', z')} \leq \eta \left(\frac{\sigma(u, z)\sigma(v, w)}{\sigma(v, z)\sigma(u, w)} \right) \frac{\sigma(u', w')}{\sigma(v', w')}$$

$$\begin{aligned} &\leq \eta \left(\frac{\sigma(v, w)}{\sigma(u, w)} \right) \frac{\sigma(u', w')}{\sigma(v', w')} \\ &\leq 8\eta(4) =: \lambda. \end{aligned} \quad (15)$$

Since η only depends on k , the same is true for λ defined in the last line.

Since $u \in \mathbb{D}$ and $v \in \partial\mathbb{D}$ in (15) were arbitrary, we conclude that

$$\sup_{x \in D} \sigma(x, z') \leq \lambda \inf_{x \in \partial D} \sigma(x, z'). \quad (16)$$

Now define $x_0 = z' = f(0) \in \text{int}(D)$ and $r = \sup_{x \in D} \sigma(x, z') \in (0, 2]$. Then $D \subseteq \overline{B}(x_0, r)$ by definition of x_0 and r . Moreover, by (16) the set $B(x_0, r/\lambda)$ is disjoint from ∂D . So this disk must be contained in one of the open Jordan regions bounded by ∂D . Since its center is contained in $\text{int}(D)$, it follows that $B(x_0, r/\lambda) \subseteq \text{int}(D)$. Passing to closures we get the desired inclusion (11). $\square$

A metric circle S (that is, a metric space homeomorphic to a circle) is called a (metric) *quasicircle* if there exists a quasisymmetry $f: \partial\mathbb{D} \rightarrow S$ of the unit circle $\partial\mathbb{D} \subseteq \widehat{\mathbb{C}}$ onto S . Four distinct points x_1, x_2, x_3, x_4 on a metric circle S are in *cyclic order* if x_2 and x_4 lie in different components of $S \setminus \{x_1, x_3\}$.

Similarly as for quasicircles in $\widehat{\mathbb{C}}$, metric quasicircles admit various characterizations. We will record some of them in the next proposition.

Proposition 4.4 *Suppose (S, d) is a metric space homeomorphic to a circle. Then the following conditions are quantitatively equivalent:*

- (i) *There exists an η -quasisymmetric map $f: \partial\mathbb{D} \rightarrow S$.*
- (ii) *There exists a round circle $S' \subseteq \widehat{\mathbb{C}}$ and an $\tilde{\eta}$ -quasi-Möbius map $g: S' \rightarrow S$.*
- (iii) *S is N -doubling and there exists $k \geq 1$ such that*

$$\text{diam}(\gamma) \leq kd(x, y)$$

for one of the subarcs γ of S with endpoints x and y , whenever $x, y \in S$, $x \neq y$.

- (iv) *S is $\tilde{N}$ -doubling and there exists $\delta > 0$ such for all points $x_1, x_2, x_3, x_4 \in S$ in cyclic order on S we have $[x_1, x_2, x_3, x_4] \geq \delta > 0$.*

Note that every subset of $\widehat{\mathbb{C}}$ is N -doubling for a universal constant N . So condition (iii) in the proposition implies that a Jordan curve $S \subseteq \widehat{\mathbb{C}}$ equipped with the chordal metric is a metric quasicircle if and only if it is a quasicircle as defined in the beginning of this section. So for Jordan curves $S \subseteq \widehat{\mathbb{C}}$ the notions of metric quasicircle and quasicircle agree, quantitatively.

We will prove Proposition 4.4 below. It goes back to Tukia and Väisälä [32] whose work implies the equivalence of the first three conditions. For the fourth equivalence it is convenient to introduce a quantity that is quantitatively equivalent to the cross-ratio and is somewhat more manageable (see [6, Sect. 2]).

If (x_1, x_2, x_3, x_4) is a 4-tuple of distinct points in a metric space (X, d) define

$$\langle x_1, x_2, x_3, x_4 \rangle := \frac{d(x_1, x_3) \wedge d(x_2, x_4)}{d(x_1, x_4) \wedge d(x_2, x_3)}. \quad (17)$$

Then the following is true (this is essentially [6, Lemma 2.2]; we include a proof for the convenience of the reader).

Lemma 4.5 *Let (X, d) be a metric space, and define $\eta_1(t) = \frac{1}{3}(t \wedge \sqrt{t})$ and $\eta_2(t) = 3(t \vee \sqrt{t})$ for $t > 0$. Then whenever x_1, x_2, x_3, x_4 are distinct points in X we have*

$$\eta_1([x_1, x_2, x_3, x_4]) \leq \langle x_1, x_2, x_3, x_4 \rangle \leq \eta_2([x_1, x_2, x_3, x_4]). \quad (18)$$

The point of the lemma is that it shows that the cross-ratio $[x_1, x_2, x_3, x_4]$ is small if and only if the “modified” cross-ratio $\langle x_1, x_2, x_3, x_4 \rangle$ is small, quantitatively.

Proof We first prove the second inequality. Suppose that there exist distinct points x_1, x_2, x_3, x_4 in X for which

$$\langle x_1, x_2, x_3, x_4 \rangle > \eta_2([x_1, x_2, x_3, x_4]).$$

Let $t_0 = [x_1, x_2, x_3, x_4]$. We may assume $d(x_1, x_3) \leq d(x_2, x_4)$. Then our assumption implies

$$d(x_1, x_4) \wedge d(x_2, x_3) < \frac{1}{\eta_2(t_0)} d(x_2, x_4).$$

Moreover, we have

$$\begin{aligned} d(x_1, x_4) &\leq d(x_1, x_3) + d(x_3, x_2) + d(x_2, x_4) \\ &\leq 2d(x_2, x_4) + d(x_2, x_3). \end{aligned}$$

Similarly, $d(x_2, x_3) \leq 2d(x_2, x_4) + d(x_1, x_4)$, and so

$$|d(x_1, x_4) - d(x_2, x_3)| \leq 2d(x_2, x_4),$$

which implies

$$d(x_1, x_4) \vee d(x_2, x_3) \leq 2d(x_2, x_4) + d(x_1, x_4) \wedge d(x_2, x_3).$$

Hence

$$\begin{aligned} d(x_1, x_4) \vee d(x_2, x_3) &\leq 2d(x_2, x_4) + d(x_1, x_4) \wedge d(x_2, x_3) \\ &\leq \left(2 + \frac{1}{\eta_2(t_0)}\right) d(x_2, x_4), \end{aligned}$$

and so

$$\begin{aligned} t_0 &= [x_1, x_2, x_3, x_4] \\ &= \frac{d(x_1, x_3)d(x_2, x_4)}{(d(x_1, x_4) \wedge d(x_2, x_3))(d(x_1, x_4) \vee d(x_2, x_3))} \\ &\geq \frac{d(x_1, x_3)\eta_2(t_0)}{(d(x_1, x_4) \wedge d(x_2, x_3))(1 + 2\eta_2(t_0))} \geq \frac{\eta_2(t_0)^2}{1 + 2\eta_2(t_0)} > t_0. \end{aligned}$$

Here the last inequality follows from a simple computation based on the cases $0 < t_0 \leq 1$ and $t_0 > 1$ which is left to the reader. In conclusion, we obtain a contradiction showing the second inequality in (18).

The first inequality in (18) follows from the second, if one uses the symmetry relations

$$\begin{aligned} [x_2, x_1, x_3, x_4] &= 1/[x_1, x_2, x_3, x_4] \quad \text{and} \\ \langle x_2, x_1, x_3, x_4 \rangle &= 1/\langle x_1, x_2, x_3, x_4 \rangle, \end{aligned}$$

and the fact that $\eta_1(t) = 1/\eta_2(1/t)$ for $t > 0$. □

Proof of Proposition 4.4 The quantitative equivalence of the first three conditions is contained in [32].

To finish the proof it is enough to show that (iii) and (iv) are quantitatively equivalent.

(iii) $\Rightarrow$ (iv): Let x_1, x_2, x_3, x_4 be four distinct points in cyclic order on S . We may assume $d(x_1, x_3) \leq d(x_2, x_4)$. Denote by γ_1 and γ_2 the subarcs of S with endpoints x_1 and x_3 that contain the points x_2 and x_4 , respectively. Condition (iii) gives us the inequality

$$d(x_2, x_3) \wedge d(x_1, x_4) \leq \text{diam}(\gamma_1) \wedge \text{diam}(\gamma_2) \leq kd(x_1, x_3).$$

Hence

$$\langle x_1, x_2, x_3, x_4 \rangle = \frac{d(x_1, x_3)}{d(x_2, x_3) \wedge d(x_1, x_4)} \geq \frac{1}{k}.$$

By Lemma 4.5 this implies that $[x_1, x_2, x_3, x_4] \geq \delta$, where $\delta = \delta(k) > 0$ only depends on k .

(iv) $\Rightarrow$ (iii): Let $x, y \in S$ with $x \neq y$ be arbitrary, and denote by γ_1 and γ_2 the two subarcs of S with endpoints x and y . Define $x_1 := x$ and $x_3 := y$. There exists a point $x_2 \in \gamma_1 \setminus \{x_1, x_3\}$ such that

$$d(x_2, x_3) \geq \frac{1}{3} \operatorname{diam}(\gamma_1).$$

For otherwise, γ_1 would be contained in the closed ball of radius $\frac{1}{3} \operatorname{diam}(\gamma_1)$ centered at x_3 which is impossible.

Similarly, there exists a point $x_4 \in \gamma_2 \setminus \{x_1, x_3\}$ such that

$$d(x_1, x_4) \geq \frac{1}{3} \operatorname{diam}(\gamma_2).$$

The points x_1, x_2, x_3, x_4 are in cyclic order on S . Hence $[x_1, x_2, x_3, x_4] \geq \delta > 0$ by our hypothesis (iv), and so Lemma 4.5 implies that $\langle x_1, x_2, x_3, x_4 \rangle \geq \epsilon_0$, where $\epsilon_0 = \epsilon_0(\delta) > 0$ only depends on δ . It follows that

$$\begin{aligned} \operatorname{diam}(\gamma_1) \wedge \operatorname{diam}(\gamma_2) &\leq 3d(x_2, x_3) \wedge d(x_1, x_4) \\ &\leq \frac{3}{\epsilon_0} d(x_1, x_3) \wedge d(x_2, x_4) \leq kd(x_1, x_3), \end{aligned}$$

where $k = k(\delta) = 3/\epsilon_0$. This inequality shows that (iii) is true. $\square$

We will give another application of the modified cross-ratio defined in (17). We require the following fact.

Lemma 4.6 *Let (X, d) be a metric space, and E and F disjoint continua in X . Define*

$$D(E, F) = \inf_{x_1, x_4 \in E, x_2, x_3 \in F} \langle x_1, x_2, x_3, x_4 \rangle.$$

Then

$$\Delta(E, F) \leq D(E, F) \leq 2\Delta(E, F). \quad (19)$$

Recall that a *continuum* (in a metric space) is a compact connected set consisting of more than one point. The inequality in the lemma shows that the relative separation $\Delta(E, F)$ of two continua E and F is small if and only if $D(E, F)$ is small, quantitatively.

Proof It follows from the definitions that

$$\begin{aligned} \langle x_1, x_2, x_3, x_4 \rangle &= \frac{d(x_1, x_3) \wedge d(x_2, x_4)}{d(x_1, x_4) \wedge d(x_2, x_3)} \\ &\geq \frac{\operatorname{dist}(E, F)}{\operatorname{diam}(E) \wedge \operatorname{diam}(F)} \end{aligned}$$

whenever $x_1, x_4 \in E$ and $x_2, x_3 \in F$. The first inequality in (19) follows.

For the second inequality choose $x_1 \in E$ and $x_3 \in F$ such that $d(x_1, x_3) = \text{dist}(E, F)$. Then we can select points $x_4 \in E$ and $x_2 \in F$ such that $d(x_1, x_4) \geq \frac{1}{2} \text{diam}(E)$ and $d(x_2, x_3) \geq \frac{1}{2} \text{diam}(F)$. Hence

$$\begin{aligned} D(E, F) &\leq \langle x_1, x_2, x_3, x_4 \rangle = \frac{d(x_1, x_3) \wedge d(x_2, x_4)}{d(x_1, x_4) \wedge d(x_2, x_3)} \\ &\leq 2 \frac{\text{dist}(E, F)}{\text{diam}(E) \wedge \text{diam}(F)} = 2\Delta(E, F). \end{aligned}$$

The second inequality follows. $\square$

Let (X, d) be a metric space, and $\mathcal{S} = \{S_i : i \in I\}$ be a collection of pairwise disjoint continua in X . We say that the sets in $\mathcal{S}$ are *s-relatively separated* for $s > 0$ if

$$\Delta(S_i, S_j) \geq s$$

whenever $i, j \in I$, $i \neq j$. The sets in $\mathcal{S}$ are said to be *uniformly relatively separated* if they are *s-relatively separated* for some $s > 0$.

Corollary 4.7 *Let $\mathcal{S} = \{S_i : i \in I\}$ be a family of *s-relatively separated* k -quasicircles in $\widehat{\mathbb{C}}$, and $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be an η -quasi-Möbius map. Then the image family $\mathcal{S}' = \{f(S_i) : i \in I\}$ consists of *s'-relatively separated* k' -quasicircles, where $s' = s'(\eta, s) > 0$ and $k' = k'(\eta, k) \geq 1$.*

Proof It follows from Lemmas 4.5 and 4.6 that there exists a constant $s_1 = s_1(s) > 0$ such that $[x_1, x_2, x_3, x_4] \geq s_1$ whenever $i, j \in I$, $i \neq j$, $x_1, x_4 \in S_i$, and $x_2, x_3 \in S_j$. Since the quasi-Möbius map f distorts cross-ratios of points quantitatively controlled by η , this implies that there exists $s_2 = s_2(\eta, s_1) = s_2(\eta, s) > 0$ such that $[y_1, y_2, y_3, y_4] \geq s_2$, whenever $i, j \in I$, $i \neq j$, $y_1, y_4 \in f(S_i)$, and $y_2, y_3 \in f(S_j)$. Again invoking Lemmas 4.5 and 4.6 we see that the sets in $\mathcal{S}'$ are *s'-relatively separated*, where $s' = s'(s_2) = s_2(\eta, s) > 0$.

By Proposition 4.4 there exists $\eta' = \eta'_k$ such that each set in $\mathcal{S}$ is the image of a round circle under an η' -quasi-Möbius map on $\widehat{\mathbb{C}}$. Hence each set in $\mathcal{S}'$ is the image of a round circle under an η'' -quasi-Möbius map on $\widehat{\mathbb{C}}$, where $\eta'' = \eta \circ \eta' = \eta'_k$. Another application of Proposition 4.4 shows that the sets in $\mathcal{S}'$ are k' -quasicircles, where $k' = k'(\eta'') = k'(\eta, k)$. $\square$

We conclude this section with a lemma that implies that it does not matter in Theorem 1.1 whether we assume uniform relative separation for the curves in $\mathcal{S}$, or for the pairwise disjoint Jordan regions that the curves in $\mathcal{S}$ bound.

Lemma 4.8 *Let D and D' be disjoint Jordan regions in $\widehat{\mathbb{C}}$. Then $\Delta(D, D') = \Delta(\partial D, \partial D')$.*

An immediate consequence is that if $\{D_i : i \in I\}$ is a family of pairwise disjoint closed Jordan regions in $\widehat{\mathbb{C}}$, then this family is s -relatively separated if and only if the family $\{\partial D_i : i \in I\}$ of boundary curves is s -relatively separated.

Proof We can pick points $x \in D$ and $y \in D'$ such that $\text{dist}(D, D') = \sigma(x, y)$. If we run on a minimizing spherical geodesic segment from x to y , then we must meet ∂D and $\partial D'$. This implies that the spherical distance between the sets ∂D and $\partial D'$ is no larger than the spherical distance between D and D' . Since spherical distances and chordal distances are monotonically related, it follows that $\text{dist}(\partial D, \partial D') \leq \text{dist}(D, D')$, and so

$$\text{dist}(\partial D, \partial D') = \text{dist}(D, D'). \quad (20)$$

Moreover, by Lemma 4.2 we have

$$\text{diam}(\partial D) \geq \text{diam}(D) \wedge \text{diam}(\widehat{\mathbb{C}} \setminus \text{int}(D)) \geq \text{diam}(D) \wedge \text{diam}(D').$$

We also get the same lower bound for $\text{diam}(\partial D')$, and so

$$\text{diam}(\partial D) \wedge \text{diam}(\partial D') \geq \text{diam}(D) \wedge \text{diam}(D').$$

The reverse inequality is trivially true, which gives

$$\text{diam}(\partial D) \wedge \text{diam}(\partial D') = \text{diam}(D) \wedge \text{diam}(D').$$

If we combine this with (20), the claim follows. $\square$

5 Extending quasiconformal maps

In this section we will prove the following proposition that will be used in the proof of Theorem 1.1. Its proof is very similar to the considerations in [8, Sect. 4].

Proposition 5.1 *Suppose that $\{D_i : i \in I\}$ is a non-empty family of pairwise disjoint closed Jordan regions in $\widehat{\mathbb{C}}$, and let $f : T = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I} \text{int}(D_i) \rightarrow \widehat{\mathbb{C}}$ be an η -quasi-Möbius embedding. If the Jordan curves $S_i = \partial D_i$ are k -quasicircles for $i \in I$, then there exists an H -quasiconformal map $F : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ such that $F|_T = f$ where $H = H(\eta, k)$.*

We need the classical Beurling-Ahlfors [3] extension theorem that can be formulated as follows.

Theorem 5.2 (Beurling-Ahlfors 1956) *Every η -quasisymmetric map $f: \mathbb{R} \rightarrow \mathbb{R}$ has an H -quasiconformal extension $F: \mathbb{C} \rightarrow \mathbb{C}$, where H only depends on η .*

Here $\mathbb{R}$ and $\mathbb{C}$ are equipped with the Euclidean metric. See [23, p. 83, Theorem 6.3] for a streamlined proof of an equivalent version of this theorem.

The next proposition is a consequence of this result.

Proposition 5.3 *Let D and D' be closed Jordan regions in $\widehat{\mathbb{C}}$, and $f: \partial D \rightarrow \partial D'$ be a homeomorphism. Suppose that the Jordan curve ∂D is a k -quasicircle.*

- (i) *If f is an η -quasi Möbius map, then it can be extended to an η' -quasi-Möbius map $F: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ with $F(D) = D'$, where η' only depends on η and k .*
- (ii) *If f is η -quasisymmetric and*

$$\text{diam}(\widehat{\mathbb{C}} \setminus D) \wedge \text{diam}(\widehat{\mathbb{C}} \setminus D') \geq \delta > 0, \quad (21)$$

then f can be extended to an η' -quasisymmetric map $F: D \rightarrow D'$, where η' only depends on δ , k and η .

For a related result with a similar proof see [8, Proposition 4.3].

Proof We first prove (i). In this case $\partial D'$ is the image of a k -quasicircle under an η -quasi-Möbius map. Hence by the quantitative equivalence of conditions (ii) and (iii) in Proposition 4.4, the curve $\partial D'$ is a k' -quasicircle with $k' = k'(\eta, k)$. It follows from Proposition 4.1 that there exist $\tilde{\eta}$ -quasi-Möbius maps on $\widehat{\mathbb{C}}$ with $\tilde{\eta} = \tilde{\eta}_{k,\eta}$ that map ∂D and $\partial D'$ to $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \subseteq \widehat{\mathbb{C}}$ and the sets D and D' to the closed upper half-plane $U = \{z \in \mathbb{C} : \text{Im } z \geq 0\} \cup \{\infty\}$ in $\widehat{\mathbb{C}}$. So we are reduced to the case where $D = D' = U$, and f is an η -quasi-Möbius homeomorphism on $\widehat{\mathbb{R}}$. By post-composing f with a suitable Möbius transformation, which does not change the distortion function η of the map, we may further assume that $f(\infty) = \infty$.

Note that here $\widehat{\mathbb{R}}$ has to be considered as equipped with the chordal metric. Cross-ratios for points in $\mathbb{R}$ are the same if we take the chordal metric or the Euclidean metric. It follows that $f|_{\mathbb{R}}: \mathbb{R} \rightarrow \mathbb{R}$ is η -quasi-Möbius if $\mathbb{R}$ is equipped with the Euclidean metric. Since $f(\infty) = \infty$ a limiting argument shows that $f|_{\mathbb{R}}: \mathbb{R} \rightarrow \mathbb{R}$ is also η -quasisymmetric when $\mathbb{R}$ carries this metric. By the Beurling-Ahlfors Theorem 5.2 the map $f|_{\mathbb{R}}$ has an H -quasiconformal extension $F: \mathbb{C} \rightarrow \mathbb{C}$ where $H = H(\eta)$. Post-composing F with the reflection of $\widehat{\mathbb{C}}$ in $\widehat{\mathbb{R}}$ if necessary, we may assume that this H -quasiconformal extension F of f satisfies $F(\text{int}(U)) = \text{int}(U)$.

Letting $F(\infty) = \infty$ we get an H -quasiconformal mapping $F: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ that extends f . Note that points are “removable singularities” for quasiconformal maps [33, Theorem 17.3]. Moreover, the dilatation of F does not change by the passage from the Euclidean metric on $\mathbb{C}$ to the chordal metric on $\mathbb{C} \subseteq \widehat{\mathbb{C}}$, because these metrics are “asymptotically” conformal, i.e., the identity map from $\mathbb{C}$ equipped with the Euclidean metric to $\mathbb{C}$ equipped with the chordal metric is 1-quasiconformal. Then F will be η' -quasi-Möbius with η' only depending on H and hence only on η . So the map F is an extension of f with the desired properties.

To prove part (ii) we first show that our assumption $\text{diam}(\widehat{\mathbb{C}} \setminus D) \geq \delta$ implies that

$$\text{diam}(D) \leq \frac{2}{\delta} \text{diam}(\partial D).$$

Indeed, note that $\delta \leq \text{diam}(\widehat{\mathbb{C}} \setminus D) \leq \text{diam}(\widehat{\mathbb{C}}) = 2$. Hence by Lemma 4.2 we have that

$$\frac{2}{\delta} \text{diam}(\partial D) \geq \frac{2}{\delta} (\text{diam}(D) \wedge \text{diam}(\widehat{\mathbb{C}} \setminus D)) \geq \text{diam}(D) \wedge 2 = \text{diam}(D)$$

as desired. Similarly,

$$\text{diam}(D') \leq \frac{2}{\delta} \text{diam}(\partial D').$$

Now suppose that $f: \partial D \rightarrow \partial D'$ is η -quasisymmetric. Since quasisymmetric maps are quasi-Möbius maps, quantitatively (Proposition 3.1(ii)), it follows from the first part of the proof that there exists an $\tilde{\eta}$ -quasi-Möbius extension $F: D \rightarrow D'$, where $\tilde{\eta}$ only depends on k and η . We can pick points $x_1, x_2, x_3 \in \partial D$ such that

$$\sigma(x_i, x_j) \geq \text{diam}(\partial D)/2 \geq \frac{\delta}{4} \text{diam}(D) \quad \text{for } i \neq j,$$

and define $y_i = F(x_i) = f(x_i) \in \partial D'$. Now, since f is an η -quasisymmetry, we have

$$\sigma(f(z), f(x_i)) \leq \eta(2)\sigma(f(x_i), f(x_j))$$

for arbitrary $i \neq j$ and $z \in \partial D$. It follows that

$$\text{diam}(D') \leq \frac{2}{\delta} \text{diam}(\partial D') \leq \frac{4\eta(2)}{\delta} \sigma(y_i, y_j) \quad \text{for } i \neq j.$$

This shows that F satisfies the conditions (9) and (10) with $X = D$, $Y = D'$ and $\lambda = \frac{4}{\delta}(1 \vee \eta(2))$. Since λ only depends on δ and η , and F is $\tilde{\eta}$ -quasi-Möbius with $\tilde{\eta}$ only depending on k and η , it follows from Proposition 3.1(iii) that F is η' -quasisymmetric with η' only depending δ , k , and η . $\square$

Remark 5.4 If D and D' are closed Jordan regions in $\widehat{\mathbb{C}}$, and $f: \partial D \rightarrow \partial D'$ is a homeomorphism, then f can be extended to homeomorphism $F: D \rightarrow D'$.

Indeed, by the Schönflies theorem this statement can be reduced to the special case $D = D' = \overline{\mathbb{D}}$. Then F is obtained from $f: \partial \mathbb{D} \rightarrow \partial \mathbb{D}$ by “radial” extension, i.e.,

$$F(re^{it}) = rf(e^{it}) \quad \text{for } r \in [0, 1], t \in [0, 2\pi].$$

Lemma 5.5 Suppose that $\{D_i : i \in I\}$ is a family of pairwise disjoint closed Jordan regions in $\widehat{\mathbb{C}}$, where $I = \{1, \dots, n\}$ with $n \in \mathbb{N}$, or $I = \mathbb{N}$. If $I = \mathbb{N}$ assume in addition that $\text{diam}(D_i) \rightarrow 0$ as $i \rightarrow \infty$. Let $T = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I} \text{int}(D_i)$.

- (i) Suppose that we have a set T' with $T \subseteq T' \subseteq \widehat{\mathbb{C}}$, and a map $F: T' \rightarrow \widehat{\mathbb{C}}$ such that the restrictions $F|_T$ and $F|_{T' \cap D_i}$, $i \in I$, are continuous. If $I = \mathbb{N}$ assume in addition that $\text{diam}(F(T' \cap D_i)) \rightarrow 0$ as $i \rightarrow \infty$. Then F is continuous.
- (ii) The sets T and $T \setminus \partial D_i$, $i \in I$, are path-connected.
- (iii) If $f: T \rightarrow \widehat{\mathbb{C}}$ is an embedding, then the image of T under f can be written as $f(T) = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I} \text{int}(D'_i)$, where $\{D'_i : i \in I\}$ is a family of pairwise disjoint closed Jordan regions in $\widehat{\mathbb{C}}$ with $f(\partial D_i) = \partial D'_i$ for $i \in I$. Moreover, if $I = \mathbb{N}$, then we have $\text{diam}(D'_i) \rightarrow 0$ as $i \rightarrow \infty$.
- (iv) The set $\widetilde{T} = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I} D_i = T \setminus \bigcup_{i \in I} \partial D_i$ is non-empty and contains uncountably many elements.

Proof (i) We claim that F is continuous at each point $x \in T'$. This is clear if $x \in \text{int}(D_i) \cap T'$ for some $i \in I$. Otherwise, $x \in T$. Let $\epsilon > 0$ be arbitrary. Since the Jordan curves $\partial D_i \subseteq T$ are pairwise disjoint, the point x can lie on at most one of them.

Assume that $x \in \partial D_{i_0}$, where $i_0 \in I$. Then $F|_{T \cup (D_{i_0} \cap T')}$ is continuous and so we can choose $\delta > 0$ so that $\sigma(F(y), F(x)) < \epsilon/2$ for all $y \in B(x, \delta)$ that lie in $T \cup (D_{i_0} \cap T')$.

We have $x \notin D_i$ for $i \neq i_0$. Since there are only finitely many $i \in I$ with $\text{diam}(F(T' \cap D_i)) \geq \epsilon/2$ by our hypothesis, we can assume that $\delta > 0$ is so small that $\text{diam}(F(T' \cap D_i)) < \epsilon/2$ whenever $i \in I \setminus \{i_0\}$ and $D_i \cap B(x, \delta) \neq \emptyset$.

If $D_i \cap B(x, \delta) \neq \emptyset$ for $i \neq i_0$, then also $\partial D_i \cap B(x, \delta) \neq \emptyset$, and so there exists a point $y \in \partial D_i \cap B(x, \delta) \subseteq T$. It follows that $\sigma(F(x), F(y)) < \epsilon/2$ and $F(D_i \cap T') \subseteq B(F(y), \epsilon/2)$ by choice of δ . This implies $F(D_i \cap T') \subseteq B(F(x), \epsilon)$. We conclude that $F(B(x, \delta) \cap T') \subseteq B(F(x), \epsilon)$, and the continuity of F at x follows.

A similar argument shows that F is continuous at x if $x \in T \setminus \bigcup_{i \in I} \partial D_i$.

(ii) Pick a point $p_i \in \text{int}(D_i)$ for each $i \in I$. Let $P = \{p_i : i \in I\}$ and $T' = \widehat{\mathbb{C}} \setminus P \supset T$.

For each $i \in I$ there is a retraction of $D_i \setminus \{p_i\}$ onto ∂D_i , i.e., a continuous map $D_i \setminus \{p_i\} \rightarrow \partial D_i$ that is the identity on ∂D_i . These maps and the identity on T paste together to a map $R: T' \rightarrow T$. By (i) the map R is continuous, and so it is a continuous retraction of T' onto T .

Since P is countable, the set $T' = \widehat{\mathbb{C}} \setminus P$ is path-connected. Indeed, to find a path between any two points $x, y \in T'$ pick an uncountable family of arcs in $\widehat{\mathbb{C}}$ with endpoints x and y that have no common interior points. One of these arc will lie in T' .

Since T' is path-connected and R is a retraction, the image $T = R(T')$ is also path-connected.

For each $i \in I$ the set $T' \setminus D_i$ is path-connected as it is homeomorphic to the open unit disk $\mathbb{D}$ with at most countably many points removed. Hence $T \setminus \partial D_i = R(T' \setminus D_i)$ is path-connected.

(iii) By (ii) the set $T \setminus \partial D_i$ is connected for each $i \in I$. Hence $f(T \setminus \partial D_i)$ is also connected. Since this set is non-empty and does not meet the Jordan curve $f(\partial D_i)$, it must be contained in exactly one of the two components of $\widehat{\mathbb{C}} \setminus f(\partial D_i)$. Let D'_i be the closure of the other complementary component of $\widehat{\mathbb{C}} \setminus f(\partial D_i)$. Then D'_i is a closed Jordan region with $\partial D'_i = f(\partial D_i)$ for each $i \in I$, and we have

$$f(T) \subseteq \widehat{\mathbb{C}} \setminus \bigcup_{i \in I} \text{int}(D'_i). \quad (22)$$

Since the family $\{\partial D'_i = f(\partial D_i) : i \in I\}$ of Jordan curves consists of pairwise disjoint sets, the last inclusion implies that the family $\{D'_i : i \in I\}$ also consists of pairwise disjoint sets.

In order to prove that we have equality in (22), we first show that if $I = \mathbb{N}$, then $\text{diam}(D'_i) \rightarrow 0$ as $i \rightarrow \infty$. By our hypotheses we have $\text{diam}(\partial D_i) \rightarrow 0$. Since T is compact, the map f is uniformly continuous on T , and so $\text{diam}(\partial D'_i) = \text{diam}(f(\partial D_i)) \rightarrow 0$ as $i \rightarrow \infty$. The argument in the proof of Proposition 4.3 shows that for each $i \in I$ we have

$$\text{diam}(D'_i) \leq 2 \text{diam}(\partial D'_i), \quad (23)$$

or else D'_i contains a disk of radius 1. Since the Jordan regions D'_i are pairwise disjoint, it follows that inequality (23) holds for all $i \in I$ with at most finitely many exceptions. This implies the desired statement $\text{diam}(D'_i) \rightarrow 0$ as $i \rightarrow \infty$.

By Remark 5.4 the map $f|_{\partial D_i} : \partial D_i \rightarrow \partial D'_i$ extends to a homeomorphism of D_i onto D'_i for each $i \in I$. These extensions and the map f paste together to an injective map $F: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ so that $F|_T = f$ and $F|_{D_i}$ is continuous for each $i \in I$. Moreover, $\text{diam}(F(D_i)) = \text{diam}(D'_i) \rightarrow 0$ as $i \rightarrow \infty$ if $I = \mathbb{N}$. Hence by (i) the map F is continuous on $\widehat{\mathbb{C}}$.

Since $F: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is injective and continuous, this map is a homeomorphism onto its image. By “invariance of domain” this image is open. Since it is also compact, and hence closed, it follows that $F(\widehat{\mathbb{C}}) = \widehat{\mathbb{C}}$, and so F is a homeomorphism of $\widehat{\mathbb{C}}$ onto itself. Hence

$$f(T) = F(T) = F\left(\widehat{\mathbb{C}} \setminus \bigcup_{i \in I} \text{int}(D_i)\right) = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I} F(\text{int}(D_i)) = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I} \text{int}(D'_i)$$

as desired.

(iv) The statement is clear if I is a finite set, because then $\widetilde{T}$ has interior points (for example, points in $\widehat{\mathbb{C}} \setminus D_i$ sufficiently close to ∂D_i are interior points of $\widetilde{T}$); if I is an infinite set, then by (ii) we can find a path $\alpha: [0, 1] \rightarrow T$ with endpoints on different Jordan curves ∂D_{i_0} and ∂D_{i_1} , $i_0, i_1 \in I$, $i_0 \neq i_1$. We claim that $\alpha \cap \widetilde{T}$ is an uncountable set. Otherwise, this set consists of a countable (possibly empty) collection of distinct points x_λ , $\lambda \in \Lambda \subseteq \mathbb{N}$. Then $[0, 1]$ is the disjoint union of the countably many closed sets $\alpha^{-1}(\partial D_i)$, $i \in I$, and $\alpha^{-1}(\{x_\lambda\})$, $\lambda \in \Lambda$. Hence $[0, 1]$ must be contained in one of these sets (one cannot represent $[0, 1]$ as a countable union of pairwise disjoint closed sets in a non-trivial way; see [31, p. 219]). This implies that α is contained in one of the Jordan curves ∂D_i or is a constant path. Both alternatives are impossible, since α has one endpoint on ∂D_{i_0} , and one on the disjoint set ∂D_{i_1} . $\square$

Proof of Proposition 5.1 By pre- and post-composing f with suitable Möbius transformations, we may assume that there exists an index $i_0 \in I$ such that $0, 1, \infty \in \partial D_{i_0}$, and $f(0) = 0$, $f(1) = 1$, $f(\infty) = \infty$. In this reduction we used the fact that both the hypotheses of the proposition and the desired conclusion remain essentially unaffected by applying such auxiliary Möbius transformations; indeed, the image of a family of uniform quasicircles under a Möbius transformation consists of uniform quasicircles, quantitatively (this was shown in the proof of Corollary 4.7), and pre- and post-composition with Möbius transformations changes neither the distortion function of a quasi-Möbius map nor the dilatation of a quasiconformal map on $\widehat{\mathbb{C}}$.

The map f then satisfies conditions (9) and (10) in Proposition 3.1 with $X = T$, $Y = f(T)$, $x_1 = y_1 = 0$, $x_2 = y_2 = 1$, $x_3 = y_3 = \infty$, and $\lambda = \sqrt{2}$. Hence f is $\tilde{\eta}$ -quasisymmetric with $\tilde{\eta} = \tilde{\eta}_\eta$.

The set I is finite, or countably infinite in which case we may assume that $I = \mathbb{N}$. We show that if $I = \mathbb{N}$, then $\text{diam}(D_i) \rightarrow 0$ as $i \rightarrow \infty$. Indeed, by our hypotheses and Proposition 4.3, there exists $\lambda \geq 1$, $r_i > 0$, and points $x_i \in \widehat{\mathbb{C}}$ such that

$$\overline{B}(x_i, r_i/\lambda) \subseteq D_i \subseteq \overline{B}(x_i, r_i) \quad \text{for all } i \in I.$$

Since the regions D_i , $i \in I$, are pairwise disjoint, the first inclusion shows that $r_i \rightarrow 0$ as $i \rightarrow \infty$. Hence $\text{diam}(D_i) \rightarrow 0$ as $i \rightarrow \infty$ by the second inclusion, as desired.

By Lemma 5.5(iii) there exist pairwise disjoint closed Jordan regions D'_i for $i \in I$ such that $\partial D'_i = f(\partial D_i)$ and

$$T' = f(T) = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I} \text{int}(D'_i).$$

Moreover, if $I = \mathbb{N}$ we have $\text{diam}(D'_i) \rightarrow 0$ as $i \rightarrow \infty$.

By the normalization imposed in the beginning of the proof, the complement of each open Jordan region $\text{int}(D_i)$ and $\text{int}(D'_i)$, $i \in I$, contains the points $0, 1, \infty$. This implies that condition (21) in Proposition 5.3 for $D = D_i$ and $D' = D'_i$ is true with $\delta = \text{diam}\{0, 1, \infty\} = 2$. It follows that for each $i \in I$ we can extend the map $f|_{\partial D_i}: \partial D_i \rightarrow \partial D'_i$ to an η' -quasisymmetric map from D_i onto D'_i , where $\eta' = \eta'_{\eta, k} = \eta'_{\eta, k}$.

These maps paste together to a bijection $F: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ whose restriction to T agrees with the $\tilde{\eta}$ -quasisymmetric map $f: T \rightarrow T' = f(T)$ and whose restriction to each set D_i , $i \in I$, is an η' -quasisymmetric map onto D'_i .

By Lemma 5.5(i) the map F is continuous and hence a homeomorphism. We claim that F is H -quasiconformal with $H = H(\eta, k)$. We need to show that there exists a constant $H = H(\eta, k) \geq 1$ such that for every $x \in \widehat{\mathbb{C}}$,

$$\limsup_{r \rightarrow 0} \frac{L_F(x, r)}{l_F(x, r)} \leq H, \quad (24)$$

where L_F and l_F are defined as in (5) and (6). Below we will write $a \lesssim b$ for two quantities a and b , if there exists a constant C such that $a \leq Cb$ that depends only on the functions $\tilde{\eta}$ and η' , and hence only on η and k .

If x is in one of complementary components $\text{int}(D_i)$ of T , then (24) with $H = \eta'(1)$ follows from the definition of F . Thus it is enough to only consider the case $x \in T$.

Since T is connected (Lemma 5.5(ii)), there exists small $r_0 > 0$ such that the circle

$$S(x, r) := \{y \in \widehat{\mathbb{C}}: \sigma(y, x) = r\}$$

has non-empty intersection with T for each $0 < r \leq r_0$. Suppose that r is in this range. Since $F|_T = f$ is $\tilde{\eta}$ -quasisymmetric, it suffices to show that for each $y \in S(x, r)$, there exists a point $v \in T \cap S(x, r)$ such that

$$\sigma(F(v), F(x)) \lesssim \sigma(F(y), F(x)) \lesssim \sigma(F(v), F(x)). \quad (25)$$

For then $L_F(x, r)/l_F(x, r)$ will be bounded by a quantity comparable to $\tilde{\eta}(1)$.

This is trivial if y itself is in T . Thus we assume that y is not in T . Then y lies in one of the complementary components $\text{int}(D_i)$ of T . For simplicity we drop the index i and write $D = D_i$.

Since $S(x, r)$ contains $y \in D$ and points in T , and hence in the complement of $\text{int}(D)$, we have $S(x, r) \cap \partial D \neq \emptyset$. For v we pick an arbitrary point in $S(x, r) \cap \partial D$, and let u be a point in the intersection of ∂D and a minimizing spherical geodesic segment joining x and y . Since $\sigma(y, u) \leq \sigma(v, u)$, $\sigma(u, x) \leq \sigma(v, x)$, and $\sigma(v, u) \leq 2r = 2\sigma(v, x)$, and since $\{x, v, u\} \subseteq T$ and $\{y, v, u\} \subseteq D$, we have

$$\begin{aligned}\sigma(F(y), F(x)) &\leq \sigma(F(y), F(u)) + \sigma(F(u), F(x)) \\ &\lesssim \sigma(F(v), F(u)) + \sigma(F(v), F(x)) \lesssim \sigma(F(v), F(x)).\end{aligned}$$

This shows the right-hand side of (25). To prove the inequality on the left-hand side, we choose a point u' as the a preimage under F of a point in the intersection of $F(\partial D)$ with a minimizing spherical geodesic joining $F(x)$ and $F(y)$. Again, we have $\{x, v, u'\} \subseteq T$ and $\{y, v, u'\} \subseteq D$. We need to consider two cases:

Case 1. $\sigma(u', x) \geq \frac{1}{2}r$. In this case we have $r = \sigma(v, x) \leq 2\sigma(u', x)$, and therefore

$$\sigma(F(v), F(x)) \lesssim \sigma(F(u'), F(x)) \lesssim \sigma(F(y), F(x)).$$

Case 2. $\sigma(u', x) \leq \frac{1}{2}r$. Then we have $\sigma(v, u') \leq 2r \leq 4\sigma(y, u')$. Spherical distances are additive along minimizing spherical geodesic segments. By choice of u' this gives the inequality

$$\sigma(F(y), F(u')) + \sigma(F(u'), F(x)) \leq 2\sigma(F(y), F(x))$$

for chordal distances, and so

$$\begin{aligned}\sigma(F(v), F(x)) &\leq \sigma(F(v), F(u')) + \sigma(F(u'), F(x)) \\ &\lesssim \sigma(F(y), F(u')) + \sigma(F(u'), F(x)) \lesssim \sigma(F(y), F(x)).\end{aligned}$$

This completes the proof of (25), and thus of (24) and the proposition. $\square$

6 Classical and transboundary modulus

A *density* is a non-negative Borel function $\rho: M \rightarrow [0, \infty]$ defined on some Borel set $M \subseteq \widehat{\mathbb{C}}$. Let Γ be a family of paths in $\widehat{\mathbb{C}}$, and ρ a density on $\widehat{\mathbb{C}}$. Then ρ is called *admissible* (for Γ) if

$$\int_{\gamma} \rho ds \geq 1$$

for all locally rectifiable paths γ in Γ . Here integration is with respect to spherical arclength. The *modulus* of the family Γ is defined as

$$\text{mod}(\Gamma) = \inf_{\rho} \int \rho^2 d\Sigma,$$

where the infimum is taken over all densities ρ that are admissible and integration is with respect to spherical measure Σ on $\widehat{\mathbb{C}}$. We refer to the densities ρ over which the infimum is taken here also as the densities that are *admissible for* $\text{mod}(\Gamma)$. Note that if Γ is a family of paths in a region $\Omega \subseteq \widehat{\mathbb{C}}$, then we can restrict ourselves to considering densities ρ that vanish on $\widehat{\mathbb{C}} \setminus \Omega$. A density for which the infimum is attained is called *extremal* for $\text{mod}(\Gamma)$.

Remark 6.1 The modulus of a path family Γ in a region Ω does not change if the spherical base metric that was used to compute $\int_{\gamma} \rho ds$ and $\int \rho^2 d\Sigma$ is changed to a conformally equivalent metric.

More precisely, suppose that Ω is a region in $\widehat{\mathbb{C}}$, Γ is a path family in Ω , and $\lambda: \Omega \rightarrow (0, \infty)$ is a continuous and positive “conformal factor”. Consider the conformal metric on Ω with length element $ds_{\lambda} := \lambda ds$ and associated area element $dA_{\lambda} := \lambda^2 d\Sigma$.

Call a Borel function $\tilde{\rho}: \Omega \rightarrow [0, \infty]$ admissible if

$$\int_{\gamma} \tilde{\rho} ds_{\lambda} \geq 1$$

for all locally rectifiable paths γ in Γ , and define

$$\text{mod}_{\lambda}(\Gamma) = \inf_{\tilde{\rho}} \int_{\Omega} \tilde{\rho} dA_{\lambda},$$

where the infimum is taken over all admissible $\tilde{\rho}$. Then $\text{mod}_{\lambda}(\Gamma) = \text{mod}(\Gamma)$. This follows from the fact that the class of locally rectifiable paths is the same for the spherical metric and the conformal metric with length element ds_{λ} and that $\rho \leftrightarrow \tilde{\rho} = \rho/\lambda$ gives a bijection between admissible densities for $\text{mod}(\Gamma)$ and $\text{mod}_{\lambda}(\Gamma)$, respectively, that is mass preserving in the sense that

$$\int_{\Omega} \rho^2 d\Sigma = \int_{\Omega} \tilde{\rho}^2 dA_{\lambda}.$$

If $f: \Omega \rightarrow \Omega'$ is a continuous map between sets Ω and Ω' in $\widehat{\mathbb{C}}$ and Γ is a family of paths in Ω , then we denote by $f(\Gamma) = \{f \circ \gamma: \gamma \in \Gamma\}$ the family of image paths.

Conformal maps do not change the modulus of a path family: if $f: \Omega \rightarrow \Omega'$ is a conformal map between regions $\Omega, \Omega' \subseteq \widehat{\mathbb{C}}$ and Γ is a path family in Ω ,

then $\text{mod}(\Gamma) = \text{mod}(f(\Gamma))$. This is the fundamental property of modulus and easily follows from the previous remark on conformal change of the base metric.

Quasiconformal maps distort the moduli of path families in a controlled way (in [33, Chap. 2] this is the basis of the definition of a quasiconformal map; it is well-known that it is quantitatively equivalent to our definition [33, Theorem 34.1 and Remark 34.2]).

Proposition 6.2 *Let Ω and Ω' be regions in $\widehat{\mathbb{C}}$, Γ be a path family in Ω , and $f: \Omega \rightarrow \Omega'$ be an H -quasiconformal map. Then*

$$\frac{1}{K} \text{mod}(\Gamma) \leq \text{mod}(f(\Gamma)) \leq K \text{mod}(\Gamma), \quad (26)$$

where $K = K(H) \geq 1$.

Let Ω be a region in $\widehat{\mathbb{C}}$ and $\mathcal{K} = \{K_i : i \in I\}$ be a finite collection of pairwise disjoint compact subsets of Ω . Here I is a finite index set. Define $K := \bigcup_{i \in I} K_i$.

Let $\gamma: J \rightarrow \widehat{\mathbb{C}}$ be a path defined on an interval $J \subseteq \mathbb{R}$. Since $\Omega \setminus K$ is open, the set $\gamma^{-1}(\Omega \setminus K)$ is relatively open in J and so can it be written as

$$\gamma^{-1}(\Omega \setminus K) = \bigcup_{l \in \Lambda} J_l,$$

where Λ is a countable (possibly empty) index set, and the sets $J_l, l \in \Lambda$, are pairwise disjoint intervals in J . We call γ *locally rectifiable in $\Omega \setminus K$* , if the path $\gamma|_{J_l}$ is locally rectifiable for each $l \in \Lambda$.

In this case the path integral $\int_{\gamma|_{J_l}} \rho ds \in [0, \infty]$ is defined whenever $\rho: \Omega \setminus K \rightarrow [0, \infty]$ is a Borel function. We set

$$\int_{\gamma \cap (\Omega \setminus K)} \rho ds := \sum_{l \in \Lambda} \int_{\gamma|_{J_l}} \rho ds.$$

A *transboundary mass distribution* on Ω consists of a density on $\Omega \setminus K$, i.e., a Borel function $\rho: \Omega \setminus K \rightarrow [0, \infty]$, and non-negative weights $\rho_i \geq 0$ for $i \in I$ (so each of the sets K_i has a corresponding weight ρ_i). We call

$$\int_{\Omega \setminus K} \rho^2 d\Sigma + \sum_{i \in I} \rho_i^2 \in [0, \infty]$$

its *total mass*. The transboundary mass distribution is called *admissible* with respect to a path family Γ in $\widehat{\mathbb{C}}$ if

$$\int_{\gamma \cap (\Omega \setminus K)} \rho \, ds + \sum_{\gamma \cap K_i \neq \emptyset} \rho_i \geq 1, \quad (27)$$

whenever γ is a path in Γ that is locally rectifiable in $\Omega \setminus K$. Note that we do not require that Γ consists of paths in Ω .

The *transboundary modulus* of Γ with respect to Ω and $\mathcal{K}$ is defined as

$$M_{\Omega, \mathcal{K}}(\Gamma) = \inf_{\rho} \left\{ \int_{\Omega \setminus K} \rho^2 \, d\Sigma + \sum_{i \in I} \rho_i^2 \right\}, \quad (28)$$

where the infimum is taken over all transboundary mass distributions on Ω that are admissible for Γ . A transboundary mass distribution realizing the infimum is called *extremal* for $M_{\Omega, \mathcal{K}}(\Gamma)$.

The concept of transboundary modulus is due to Schramm. He introduced it in equivalent form as *transboundary extremal length* (the reciprocal of transboundary modulus) in [29].

As the next lemma shows, the transboundary modulus of a path family in Ω is invariant under homeomorphisms that are conformal on $\Omega \setminus K$ (see [29, Lemma 1.1] for a similar statement).

Lemma 6.3 (Invariance of transboundary modulus) *Let Ω and Ω' be regions in $\widehat{\mathbb{C}}$, and $\mathcal{K} = \{K_i : i \in I\}$ be a finite collection of pairwise disjoint compact sets in Ω . Suppose that Γ is a path family in Ω and that $f : \Omega \rightarrow \Omega'$ is a homeomorphism that is conformal on $\Omega \setminus K$. Set $\mathcal{K}' = \{f(K_i) : i \in I\}$ and $\Gamma' = f(\Gamma)$. Then*

$$M_{\Omega, \mathcal{K}}(\Gamma) = M_{\Omega', \mathcal{K}'}(\Gamma').$$

Proof Note that the sets $f(K_i)$, $i \in I$, are pairwise disjoint compact subsets of Ω' ; so $M_{\Omega', \mathcal{K}'}(\Gamma')$ is defined.

We denote by $Df(p) : T_p \widehat{\mathbb{C}} \rightarrow T_{f(p)} \widehat{\mathbb{C}}$ the differential of f at $p \in \Omega \setminus K$. This is a linear map between the tangent spaces $T_p \widehat{\mathbb{C}}$ and $T_{f(p)} \widehat{\mathbb{C}}$ of $\widehat{\mathbb{C}}$ (considered as a smooth manifold) at the points p and $f(p)$, respectively. Using the Riemannian structure on $\widehat{\mathbb{C}}$ induced by the spherical metric on $\widehat{\mathbb{C}}$, we can assign an operator norm $\|Df(p)\|$ to this map. If $p, f(p) \in \mathbb{C}$, then

$$\|Df(p)\| = \frac{(1 + |p|^2)|f'(p)|}{1 + |f(p)|^2}.$$

Let $\|Df\|$ be the map $p \mapsto \|Df(p)\|$.

A transboundary mass distribution on Ω' consisting of the Borel function $\rho: \Omega' \setminus K' \rightarrow [0, \infty]$ and the discrete weights $\rho_i \geq 0, i \in I$, is admissible for Γ' if and only if the transboundary mass distribution on Ω consisting of the density $(\rho \circ f)\|Df\|$ on $\Omega \setminus K$ and the discrete weights $\rho_i, i \in I$, is admissible for Γ . Indeed, in the admissibility conditions for a path $\gamma \in \Gamma$ and its image $f \circ \gamma \in \Gamma'$ the total contributions from the discrete weights are obviously equal; this is also true for the contributions from the densities, since we have the equation

$$\int_{\gamma \cap (\Omega \setminus K)} (\rho \circ f)\|Df\| ds = \int_{(f \circ \gamma) \cap (\Omega' \setminus K')} \rho ds$$

valid for all paths in Γ that are locally rectifiable in $\Omega \setminus K$. Note that a path $\gamma \in \Gamma$ is locally rectifiable in $\Omega \setminus K$ if and only if the path $f \circ \gamma$ is locally rectifiable in $\Omega' \setminus K'$.

Moreover, by the conformality of f on $\Omega \setminus K$ we have

$$\int_{\Omega \setminus K} (\rho \circ f)^2 \|Df\|^2 d\Sigma = \int_{\Omega' \setminus K'} \rho^2 d\Sigma.$$

This shows that every transboundary mass distribution that is admissible for $M_{\Omega', K'}(\Gamma')$ gives rise to a mass distribution that is admissible for $M_{\Omega, K}(\Gamma)$ of the same total mass. This implies that $M_{\Omega, K}(\Gamma) \leq M_{\Omega', K'}(\Gamma')$. The reverse inequality follows by applying the same argument to f^{-1} . $\square$

Let $\Omega \subseteq \widehat{\mathbb{C}}$ be a region and $E, F \subseteq \overline{\Omega}$. We say that γ is a (*closed*) *path in Ω connecting E and F* if the path is a continuous map $\gamma: [a, b] \rightarrow \widehat{\mathbb{C}}$ defined on a closed interval $[a, b] \subseteq \mathbb{R}$ such that $\gamma(a) \in E$, $\gamma(b) \in F$, and $\gamma((a, b)) \subseteq \Omega$. So γ lies in Ω with the possible exception of its endpoints. We denote by $\Gamma(E, F; \Omega)$ the family of all closed paths γ in Ω that connect E and F . In Sect. 12 it will be more convenient to consider *open paths in Ω connecting E and F* . By definition these are paths α for which there exists a path $\gamma: [a, b] \rightarrow \widehat{\mathbb{C}}$ in $\Gamma(E, F; \Omega)$ such that $\alpha = \gamma|(a, b)$. The family of these paths α is denoted by $\Gamma_o(E, F; \Omega)$ (so the subscript “*o*” indicates “open” paths).

Let $\mathcal{K} = \{K_i : i \in I\}$ be a finite collection of pairwise disjoint compact subsets of Ω . Set $K = \bigcup_{i \in I} K_i$. Note that $M_{\Omega, \mathcal{K}}(\Gamma(E, F; \Omega))$ can be different from $M_{\Omega, \mathcal{K}}(\Gamma_o(E, F; \Omega))$. One can easily obtain an example by assuming that E or F is contained in one of the sets in $\mathcal{K}$. Setting the discrete weight equal to 1 on this set and all the other discrete weights and the density equal to 0 produces an admissible mass distribution for $M_{\Omega, \mathcal{K}}(\Gamma(E, F; \Omega))$, but not necessarily for $M_{\Omega, \mathcal{K}}(\Gamma_o(E, F; \Omega))$. Hence $M_{\Omega, \mathcal{K}}(\Gamma(E, F; \Omega)) \leq 1$, but it is not hard to find a situation where $M_{\Omega, \mathcal{K}}(\Gamma_o(E, F; \Omega)) > 1$.

If $\Omega' \subseteq \widehat{\mathbb{C}}$ is another region, $f: \overline{\Omega} \rightarrow \overline{\Omega}'$ is a homeomorphism with $f(\Omega) = \Omega'$, and if we define $E' = f(E)$ and $F' = f(F)$, then

$$f(\Gamma(E, F; \Omega)) = \Gamma(E', F'; \Omega').$$

Moreover, if $\mathcal{K}' := \{f(K_i) : i \in I\}$ and $f|_{\Omega \setminus K}$ is conformal, then the same argument as in the proof of Lemma 6.3 shows that

$$M_{\Omega, \mathcal{K}}(\Gamma(E, F; \Omega)) = M_{\Omega', \mathcal{K}'}(\Gamma(E', F'; \Omega')).$$

Remark 6.4 Similarly as for classical modulus (see Remark 6.1), transboundary modulus does not change if we replace the integrals $\int_{\gamma \cap (\Omega \setminus K)} \rho \, ds$ and $\int_{\Omega \setminus K} \rho^2 \, d\Sigma$ in its definition by similar integrals with respect to a different base metric that is conformally equivalent to the spherical metric. This will be important in Sect. 11 where it is convenient to use the flat metric with length element $|dz|/|z|$ as a base metric on $\mathbb{C}^*$.

7 Loewner regions

Let $\Omega \subseteq \widehat{\mathbb{C}}$ be a region in $\widehat{\mathbb{C}}$. If there exists a non-increasing function $\phi: (0, \infty) \rightarrow (0, \infty)$ such that

$$\text{mod}(\Gamma(E, F; \Omega)) \geq \phi(\Delta(E, F)),$$

whenever E and F are disjoint continua in $\overline{\Omega}$, then we call Ω a *Loewner region* (or a ϕ -*Loewner region* if we want to emphasize ϕ). A region Ω is Loewner if and only if the following statement is true: for each $t > 0$ there exists $m = m(t) > 0$ such that if E and F are disjoint continua in $\overline{\Omega}$ with $\Delta(E, F) \leq t$ and ρ a density on $\widehat{\mathbb{C}}$ with $\int \rho^2 \, d\Sigma < m$, then there exists a rectifiable path in Ω connecting E and F such that $\int_{\gamma} \rho \, ds < 1$. Indeed, if Ω is ϕ -Loewner, then we can take $m = m(t) := \phi(t)$ for $t > 0$ in this condition. Conversely, if the condition is satisfied, then Ω is ϕ -Loewner with $\phi(s) := \sup\{m(t) : t \geq s\}$ for $s > 0$.

Loewner regions are examples for *Loewner spaces* as introduced by Heinonen and Koskela [21].

Let Ω be a proper subregion of $\widehat{\mathbb{C}}$. Then Ω is called *A-uniform*, where $A \geq 1$, if the following condition holds: for any points x, y in Ω there exists a parametrized arc $\gamma: [0, 1] \rightarrow \Omega$ such that $\gamma(0) = x$, $\gamma(1) = y$,

$$\text{length}(\gamma) \leq A\sigma(x, y),$$

and

$$\text{dist}(\gamma(t), \partial\Omega) \geq \frac{1}{A}(\text{length}(\gamma|[0, t]) \wedge \text{length}(\gamma|[t, 1]))$$

for all $t \in [0, 1]$.

The unit disk $\mathbb{D}$ is an example of a uniform region. If $\delta > 0$ and $\Omega = N_\delta(\mathbb{D}) \setminus \mathbb{D}$, then Ω is an annulus for $\delta \in (0, \sqrt{2})$ and so this region is A -uniform with $A = A(\delta)$ (recall from Sect. 2 that $N_\delta(A)$ denotes the open δ -neighborhood of a set A). We will use these facts below. They are essentially well-known and so we omit the easy (and tedious) proof.

Uniform regions are Loewner regions, quantitatively.

Proposition 7.1 *Every A -uniform region $\Omega \subseteq \widehat{\mathbb{C}}$ is ϕ -Loewner with $\phi = \phi_A$ only depending on A .*

Again this statement is essentially well-known and goes back to [14]. See [5, Chap. 6], and in particular [5, Remark 6.6], for more background. The statement can be derived from the fact that $\widehat{\mathbb{C}}$ is Loewner and from [5, Remark 6.38 and Theorem 6.47].

Images of Loewner regions under quasi-Möbius maps on $\widehat{\mathbb{C}}$ are Loewner, quantitatively.

Proposition 7.2 *Let $\Omega \subseteq \widehat{\mathbb{C}}$ be a ϕ -Loewner region and $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be an η -quasi Möbius map. Then $\Omega' = f(\Omega)$ is a ψ -Loewner region with ψ only depending on η and ϕ .*

Proof Let E' and F' be disjoint continua in $\overline{\Omega}'$. Then $E = f^{-1}(E')$ and $F = f^{-1}(F')$ are disjoint continua in $\overline{\Omega}$. Since f is η -quasi-Möbius, it follows from Lemmas 4.5 and 4.6 that there exists a homeomorphism $\theta: [0, \infty) \rightarrow [0, \infty)$ that can be chosen only depending on η such that

$$\Delta(E, F) \geq \theta(\Delta(E', F')).$$

Moreover, we have $f(\Gamma(E, F; \Omega)) = \Gamma(E', F'; \Omega')$, which by Proposition 3.1(i) and Proposition 6.2 implies that

$$\text{mod}(\Gamma(E', F'; \Omega')) \geq \frac{1}{K} \text{mod}(\Gamma(E, F; \Omega)),$$

where $K = K(\eta) \geq 1$. Since Ω is ϕ -Loewner we conclude that

$$\text{mod}(\Gamma(E', F'; \Omega')) \geq \frac{1}{K} \phi(\Delta(E, F)) \geq \psi(\Delta(E', F')),$$

where $\psi(t) = \frac{1}{K} \phi(\theta(t)) > 0$ for $t > 0$. Since ψ can be chosen only depending on η and ϕ , the statement follows. $\square$

Open Jordan regions in $\widehat{\mathbb{C}}$ bounded by quasicircles are Loewner regions, quantitatively.

Proposition 7.3 *Let $\Omega \subseteq \widehat{\mathbb{C}}$ be an open Jordan region whose boundary $\partial\Omega$ is a k -quasicircle. Then Ω is ϕ -Loewner with ϕ only depending on k .*

Proof By the remark following Proposition 4.1 the region Ω is the image of the unit disk $\mathbb{D}$ under an η -quasi Möbius map, where $\eta = \eta_k$. Since $\mathbb{D}$ is a uniform region and hence Loewner by Proposition 7.1, it follows from Proposition 7.2 that Ω is ϕ -Loewner, where $\phi = \phi_k$. $\square$

The goal of this section is to prove a similar statement for regions with finitely many complementary components (see Proposition 7.5). We first prove the following lemma for preparation.

Lemma 7.4 (Collar lemma) *Let $n \geq 2$, and let Ω be a region in $\widehat{\mathbb{C}}$ such that*

$$\Omega = \widehat{\mathbb{C}} \setminus \bigcup_{i=1}^n D_i,$$

where the sets D_i are pairwise disjoint closed Jordan regions. Suppose that the boundaries ∂D_i are k -quasicircles and the regions D_i are s -relatively separated for $i = 1, \dots, n$, and that $d = \text{diam}(D_n) \leq \text{diam}(D_i)$ for $i = 1, \dots, n$.

Then there exists an open Jordan region $V \supseteq D_n$ in $\widehat{\mathbb{C}}$ with the following properties:

- (i) $U := V \setminus D_n \subseteq \Omega$.
- (ii) $N_{cd}(D_n) \subseteq V$ where $c = c(s, k) > 0$ is a constant only depending on s and k .
- (iii) U is a ϕ -Loewner region with $\phi = \phi_{s,k}$ only depending on s and k .

This lemma says that under the given hypothesis one can put a “Loewner collar” U around the smallest complementary component D_n of Ω that lies in Ω , has a definite thickness proportional to the diameter d of D_n with a proportionality constant depending on s and k , and is ϕ -Loewner with ϕ controlled by s and k .

Proof Let $D = D_n$. Since ∂D is a k -quasicircle, there exists an η -quasi-Möbius map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ with $f(\overline{\mathbb{D}}) = D$, where η only depends on k (see the remark after Proposition 4.1). We denote by $u' = f(u)$ the image of an arbitrary point $u \in \widehat{\mathbb{C}}$.

Since $n \geq 2$ and $D = D_n$ has the smallest diameter of all the sets D_i , $i = 1, \dots, n$, we have $\text{diam}(\widehat{\mathbb{C}} \setminus D) \geq \text{diam}(D)$. Hence Lemma 4.2 implies that

$$d = \text{diam}(D) = \text{diam}(D) \wedge \text{diam}(\widehat{\mathbb{C}} \setminus D) \leq \text{diam}(\partial D).$$

We can pick points $x_1, x_2, x_3 \in \partial\mathbb{D}$ such that for their image points we have

$$\sigma(x'_i, x'_j) \geq \text{diam}(\partial D)/2 \geq d/2 \quad \text{for } i \neq j. \quad (29)$$

By pre-composing f with a Möbius transformation if necessary, we may assume the points x_1, x_2, x_3 are the third roots of unity. Then

$$\sigma(x_i, x_j) = \sqrt{3} \quad \text{for } i \neq j.$$

Since the sets $D_i, i = 1, \dots, n$, are s -relatively separated, and $D = D_n$ has the smallest diameter of the sets, we have

$$\text{dist}(D_i, D) \geq sd \quad \text{for } i = 1, \dots, n-1.$$

In particular, $N_{sd}(D) \setminus D \subseteq \Omega$. We claim that we can thicken up $\mathbb{D}$ by a definite amount only depending on s and k to a larger set that is mapped into $N_{sd}(D)$ by f . More precisely, we claim that

$$f(N_\delta(\mathbb{D})) \subseteq N_{sd}(D) \quad (30)$$

if $\delta = \delta(s, k) \in (0, 1)$ is suitably chosen.

So assume $\delta > 0$ is a small constant whose precise value will be chosen later. Let $v = 0$, and $u \in N_\delta(\mathbb{D}) \setminus \overline{\mathbb{D}}$ be arbitrary. Let z be the closest point to u on $\partial\mathbb{D}$. Then $\sigma(u, z) < \delta$. By Lemma 3.4 there exists $w \in \{x_1, x_2, x_3\}$ such that

$$\sigma(u, w) \geq \sqrt{3}/2 \geq 1/2 \quad \text{and} \quad \sigma(v', w') \geq d/4.$$

We also have the relations $\sigma(v', z') \leq \text{diam}(D) = d$, $\sigma(v, z) = \sqrt{2} \geq 1$, $\sigma(v, w) \leq \text{diam}(\overline{\mathbb{D}}) = 2$, and $\sigma(z', w') \leq \text{diam}(D) = d$. Since f is η -quasi-Möbius, we obtain

$$\begin{aligned} \frac{\sigma(u', z')}{d} &\leq \frac{\sigma(u', z')}{\sigma(v', z')} \\ &\leq \eta \left(\frac{\sigma(u, z)\sigma(v, w)}{\sigma(v, z)\sigma(u, w)} \right) \frac{\sigma(u', w')}{\sigma(v', w')} \\ &\leq \eta \left(\delta \frac{\sigma(v, w)}{\sigma(u, w)} \right) \frac{\sigma(u', w')}{\sigma(v', w')} \\ &\leq \frac{4}{d} \eta(4\delta)(\sigma(u', z') + \sigma(z', w')) \\ &\leq 4\eta(4\delta) \left(1 + \frac{\sigma(u', z')}{d} \right). \end{aligned} \quad (31)$$

Since $\eta(t) \rightarrow 0$ as $t \rightarrow 0$ we have that $4\eta(4\delta) < 1$ if $\delta > 0$ is small. For such δ we get

$$\frac{\sigma(u', z')}{d} \leq \frac{4\eta(4\delta)}{1 - 4\eta(4\delta)}.$$

Again using $\eta(t) \rightarrow 0$ as $t \rightarrow 0$, this shows that we can choose $\delta = \delta(s, \eta) = \delta(s, k) > 0$ such that the left hand side in the last inequality is less than s . For such δ we have $\text{dist}(p, D) < sd$ whenever $p \in f(N_\delta(\mathbb{D}))$. This gives the desired inclusion (30).

Now define $V = f(N_\delta(\mathbb{D}))$. Then (i) is true, because

$$V \setminus D_n \subseteq N_{sd}(D) \setminus D \subseteq \Omega.$$

To show an inclusion of type (ii), let $v \in \widehat{\mathbb{C}} \setminus N_\delta(\mathbb{D})$ and $z \in \overline{\mathbb{D}}$ be arbitrary. Then $\sigma(v, z) \geq \delta$. We can choose $u \in \overline{\mathbb{D}}$ such that $\sigma(u', z') \geq d/2$. Similarly as above, we can then choose $w \in \{x_1, x_2, x_3\}$ such that

$$\sigma(u, w) \geq 1/2 \quad \text{and} \quad \sigma(v', w') \geq d/4.$$

We also have $\sigma(u, z) \leq 2$, $\sigma(v, w) \leq 2$, and $\sigma(u', w') \leq d$. Then using these estimates, we get

$$\begin{aligned} \frac{d}{2\sigma(v', z')} &\leq \frac{\sigma(u', z')}{\sigma(v', z')} \\ &\leq \eta\left(\frac{\sigma(u, z)\sigma(v, w)}{\sigma(v, z)\sigma(u, w)}\right) \frac{\sigma(u', w')}{\sigma(v', w')} \\ &\leq \eta(8/\delta) \frac{\sigma(u', w')}{\sigma(v', w')} \leq 4\eta(8/\delta). \end{aligned} \quad (32)$$

This shows that for $c := c(s, k) = 1/(8\eta(8/\delta)) > 0$ we have

$$\sigma(v', z') \geq cd,$$

whenever $v \in \widehat{\mathbb{C}} \setminus N_\delta(\mathbb{D})$ and $z \in \overline{\mathbb{D}}$. It follows that

$$\text{dist}(\widehat{\mathbb{C}} \setminus V, D) \geq cd.$$

This implies $N_{cd}(D) \subseteq V$ as desired.

It remains to show (iii). The annulus $N_\delta(\mathbb{D}) \setminus \overline{\mathbb{D}}$ is an A -uniform region with $A = A(\delta) = A(s, k)$. Thus this annulus is ψ -Loewner with ψ only depending on s and k by Proposition 7.1. Since by Proposition 7.2 quasi-Möbius images of Loewner regions are Loewner regions, quantitatively, it follows that $U = V \setminus D = f(N_\delta(\mathbb{D}) \setminus \overline{\mathbb{D}})$ is ϕ -Loewner with ϕ only depending on η, s, k . But

since η was chosen to depend only on s and k , this means that ϕ can also be chosen to depend only on these parameters. $\square$

Proposition 7.5 *Let $n \geq 1$, and Ω be a region in $\widehat{\mathbb{C}}$ such that*

$$\Omega = \widehat{\mathbb{C}} \setminus \bigcup_{i=1}^n D_i,$$

where the sets D_i are pairwise disjoint closed Jordan regions. Suppose that the boundaries ∂D_i are k -quasicircles and the regions D_i are s -relatively separated for $i = 1, \dots, n$. Then Ω is a ϕ -Loewner region with $\phi = \phi_{n,s,k}$ only depending on n , s , and k .

In the proof we need a simple fact about the existence of subcontinua. Namely, if $x \in \widehat{\mathbb{C}}$, $r > 0$, and $E \subseteq \widehat{\mathbb{C}}$ is a continuum with $x \in E$ and $E \setminus \overline{B}(x, r) \neq \emptyset$, then there exists a subcontinuum $E' \subseteq E$ with $x \in E'$, $E' \subseteq \overline{B}(x, r)$, and $E' \cap \partial B(x, r) \neq \emptyset$. Note that then $r \leq \text{diam}(E') \leq 2r$. So every continuum can be “cut to size” near each of its points.

To see that this statement is true let E' be the connected component of $E \cap \overline{B}(x, r)$ containing x . Then E' is a closed subset of E with $x \in E' \subseteq \overline{B}(x, r)$. If we had $E' \cap \partial B(x, r) = \emptyset$, then E' would be relatively open in E and so $E = E' \subseteq B(x, r)$. This is impossible since $E \setminus B(x, r) \neq \emptyset$. So E' is a continuum with the desired properties.

Proof of Proposition 7.5 The proof is by induction on n with s and k fixed. The induction beginning $n = 1$ is covered by Proposition 7.3 (the requirement of s -relative separation is vacuous in this case). For the induction step suppose that $n \geq 2$ and that the statement is true for regions with the stated properties and $n - 1$ complementary components.

We may assume that D_n is the complementary component of Ω with smallest diameter $d := \text{diam}(D_n)$. Let E and F be arbitrary continua in $\overline{\Omega}$ with relative separation $\Delta(E, F) \leq t$ where $t > 0$. We have to show that if ρ is an arbitrary non-negative Borel function on $\widehat{\mathbb{C}}$ with sufficiently small mass $\int \rho^2 d\Sigma < m$, where $m = m(n, s, k, t) > 0$, then there exists a rectifiable path γ in Ω connecting E and F with $\int_{\gamma} \rho ds < 1$.

By induction hypothesis we can find $m_1 = m_1(n, s, k, t) > 0$ such that if

$$\int \rho^2 d\Sigma < m_1,$$

then there exists a rectifiable path α in $\widetilde{\Omega} := \Omega \cup D_n$ that connects E and F and satisfies

$$\int_{\alpha} \rho ds < 1/2. \quad (33)$$

A suitable constant m will be found in the course of the proof. We make the preliminary choice $m = m_1$. Then there exists a rectifiable path α in $\tilde{\Omega}$ connecting E and F satisfying (33). If α stays inside Ω (with the possible exception of its endpoints), we can take $\gamma = \alpha$. So we may assume that α hits D_n . Let U be the Loewner collar around D_n found in Lemma 7.4. The idea now is to remove $\alpha \cap D_n$ from α and to connect suitable pieces of $\alpha \setminus D_n$ by a rectifiable path β in U such that $\int_{\beta} \rho \, ds < 1/2$. A concatenation of β with pieces of α will then give a rectifiable path γ in Ω with $\int_{\gamma} \rho \, ds < 1$ as desired.

For carrying out the details of this argument, we consider several cases. Let $c = c(s, k) > 0$ be the constant from Lemma 7.4 with $N_{sd}(D_n) \setminus D_n \subseteq U$.

Case 1. Neither E nor F is contained in $N_{\frac{1}{3}cd}(D_n)$.

We choose a closed, possibly degenerate, subpath α' of α by starting at the endpoint x of α in E and traveling along α until we first hit $\overline{N_{\frac{1}{6}cd}(D_n)}$ at the point $x' \in \overline{N_{\frac{1}{6}cd}(D_n)}$, say. Since α meets D_n , there exists such a point x' . Then $\alpha' \setminus \{x\} \subseteq \tilde{\Omega} \setminus D_n = \Omega$.

The set $\alpha' \cup E$ is a continuum that contains the point x' , but that is not contained in $N_{\frac{1}{3}cd}(D_n)$ by our assumption in this case. So if we choose $r = \frac{1}{6}cd$, then $(\alpha' \cup E) \setminus B(x', r) \neq \emptyset$. By the statement about the existence of subcontinua discussed before the proof, we can find a continuum $E' \subseteq \alpha' \cup E$ that is contained in $\overline{B}(x', r)$ such that $\text{diam}(E') \geq r = \frac{1}{6}cd$. Then $E' \subseteq N_{cd}(D_n) \cap \tilde{\Omega} \subseteq \overline{U}$.

In the same way, we choose a closed subpath α'' of α with endpoints $y \in F$ and $y' \in \overline{N_{\frac{1}{6}cd}(D_n)}$ such that $\alpha'' \setminus \{y\} \subseteq \Omega$. Again we can find a subcontinuum F' of $\alpha'' \cup F$ that is contained in $N_{cd}(D_n) \cap \tilde{\Omega} \subseteq \overline{U}$ such that $\text{diam}(F') \geq \frac{1}{6}cd$. Then $E', F' \subseteq \overline{U}$ and

$$\text{dist}(E', F') \leq (2c + 1)d \leq (12 + 6/c)(\text{diam}(E') \wedge \text{diam}(F')).$$

The last inequality implies that $\Delta(E', F') \leq C(s, k)$. Since U is ϕ -Loewner with $\phi = \phi_{s,k}$ there exists a constant $m_2 = m_2(s, k) > 0$ with the following property: if we impose the additional condition

$$\int \rho^2 \, d\Sigma < m_2$$

on ρ (as we may), then there exists a rectifiable path β in $U \subseteq \Omega$ with $\int_{\beta} \rho \, ds < 1/2$ that connects E' and F' . The path β will lie in Ω with the possible exception of its endpoints. One endpoint of β lies in $E' \subseteq \alpha' \cup E \subseteq \Omega \cup E$, and one in $F' \subseteq \alpha'' \cup F \subseteq \Omega \cup F$. So by concatenating β with suitable pieces

of α' and α'' , we obtain a rectifiable path γ in Ω with $\int_{\gamma} \rho ds < 1$ that connects E and F .

Case 2. $t(\text{diam}(E) \wedge \text{diam}(F)) \geq \frac{1}{3}cd$.

We choose subpaths α' and α'' of α as in Case 1. Arguing similarly as in this case, we can find continua $E' \subseteq \alpha' \cup E$ and $F' \subseteq \alpha'' \cup F$ with $E', F' \subseteq N_{cd}(D_n) \cap \overline{\Omega} \subseteq \overline{U}$ such that $\text{diam}(E') \geq \frac{1}{3}(\text{diam}(E) \wedge cd)$ and $\text{diam}(F') \geq \frac{1}{3}(\text{diam}(F) \wedge cd)$. Then again we have

$$\text{dist}(E', F') \leq (2c + 1)d,$$

and also

$$\text{diam}(E') \wedge \text{diam}(F') \geq \frac{1}{3}(\text{diam}(E) \wedge \text{diam}(F) \wedge cd) \geq \frac{cd}{9(t \vee 1)}.$$

Hence

$$\Delta(E', F') \leq C(s, k, t).$$

In other words, the relative distance of E' and F' is controlled by s , k , and t . By the Loewner property of U we know that if

$$\int \rho^2 d\Sigma < m_3,$$

where $m_3 = m_3(s, k, t) > 0$, then there exists a rectifiable path β in $U \subseteq \Omega$ with $\int_{\beta} \rho ds < 1/2$ that connects E' and F' . As in Case 1, this leads to a path γ as desired.

Case 3. E or F lies in $N_{\frac{1}{3}cd}(D_n)$, and we have

$$t(\text{diam}(E) \wedge \text{diam}(F)) < \frac{1}{3}cd.$$

We may assume $E \subseteq N_{\frac{1}{3}cd}(D_n)$. Then $E \subseteq N_{cd}(D_n) \cap \overline{\Omega} \subseteq \overline{U}$, and

$$\text{dist}(E, F) \leq t(\text{diam}(E) \wedge \text{diam}(F)) \leq \frac{1}{3}cd.$$

Pick points $x \in E$ and $y \in F$ with $\sigma(x, y) = \text{dist}(E, F)$, and let $r = \frac{1}{3}(\text{diam}(F) \wedge cd)$. There exists a continuum $F' \subseteq F \cap \overline{B}(y, r)$ with $y \in F'$ and $\text{diam}(F') \geq r = \frac{1}{3}(\text{diam}(F) \wedge cd)$. Then $F' \subseteq N_{cd}(D_n) \cap \overline{\Omega} \subseteq \overline{U}$ and $\text{dist}(E, F') = \sigma(x, y) = \text{dist}(E, F)$.

This implies that

$$\text{dist}(E, F') = \text{dist}(E, F) \leq t(\text{diam}(E) \wedge \text{diam}(F)) < \frac{1}{3}cd,$$

and so

$$\text{dist}(E, F') \leq t \text{diam}(E) \wedge t \text{diam}(F) \wedge \frac{1}{3}cd \leq 3(t \vee 1)(\text{diam}(E) \wedge \text{diam}(F')).$$

We conclude that $\Delta(E, F') \leq 3(t \vee 1)$. Since U is Loewner we know that if

$$\int \rho^2 d\Sigma < m_4,$$

where $m_4 = m_4(s, k, t) > 0$, then there exists a rectifiable path β in $U \subseteq \Omega$ with $\int_\beta \rho ds < 1$ that connects E and $F' \subseteq F$. In this case we can take $\gamma = \beta$.

In conclusion, if

$$\int \rho^2 d\Sigma < m,$$

where $m = \min\{m_1, m_2, m_3, m_4\}$, then we can find a rectifiable path γ in Ω that connects E and F and satisfies $\int_\gamma \rho ds < 1$. Since $m > 0$ only depends on n, s, k, t , the statement follows. $\square$

8 Bounds for transboundary modulus

The present chapter is the technical core of the paper. We will prove various bounds for transboundary modulus. We use the chordal metric σ on $\widehat{\mathbb{C}}$ and the spherical measure Σ . We will make repeated use of the relation $\Sigma(B(x, r)) = \Sigma(\overline{B}(x, r)) \sim r^2$ for $x \in \widehat{\mathbb{C}}$ and small enough $r > 0$. Actually, we have

$$\Sigma(B(x, r)) = \Sigma(\overline{B}(x, r)) = \pi r^2 \quad (34)$$

valid for all $x \in \widehat{\mathbb{C}}$ and $0 < r \leq 2 = \text{diam}(\widehat{\mathbb{C}})$.

A set $M \subseteq \widehat{\mathbb{C}}$ is called λ -quasi-round, where $\lambda \geq 1$, if there exist $x_0 \in \widehat{\mathbb{C}}$ and $r \in (0, \text{diam}(\widehat{\mathbb{C}})] = (0, 2]$ such that $\overline{B}(x_0, r/\lambda) \subseteq M \subseteq \overline{B}(x_0, r)$. Note that in this case $\text{diam}(M) \geq r/\lambda$. By Proposition 4.3 every Jordan region whose boundary is a quasicircle is quasi-round, quantitatively.

Proposition 8.1 *Let Ω be a ϕ -Loewner region in $\widehat{\mathbb{C}}$, and $\mathcal{K} = \{K_i : i \in I\}$ a finite collection of pairwise disjoint compact sets in Ω . Suppose that the sets K_i are λ -quasi-round and s -relatively separated for $i \in I$.*

Then there is a non-increasing function $\psi : (0, \infty) \rightarrow (0, \infty)$ that can be chosen only depending on ϕ , λ , and s with the following property: if E and F are arbitrary disjoint continua in $\overline{\Omega}$, then

$$M_{\Omega, \mathcal{K}}(\Gamma(E, F; \Omega)) \geq \psi(\Delta(E, F)).$$

So we get Loewner type bounds for the transboundary modulus in Ω with a Loewner function ψ that only depends on the Loewner function ϕ for classical modulus in Ω , and the parameters λ and s .

For the proof we need two lemmas.

Lemma 8.2 *Under the assumptions of Proposition 8.1 let $A \subseteq \widehat{\mathbb{C}}$ be an arbitrary set, and $t > 0$. Let N be the number of the sets K_i such that $K_i \cap A \neq \emptyset$ and*

$$\text{diam}(K_i) \geq t \text{diam}(A).$$

Then $N \leq C(s, t)$.

This means that an arbitrary set $A \subseteq \widehat{\mathbb{C}}$ can only meet a controlled number of those sets K_i whose diameters are not much smaller than the diameter of A .

Proof For each set K_i that meets A and satisfies $\text{diam}(K_i) \geq t \text{diam}(A)$ pick a point $x_i \in A \cap K_i$. In this way we obtain a collection $\{x_i : i \in I'\}$, $I' \subseteq I$, of distinct points in A . Now if x_i and x_j , $i \neq j$, are points in this collection, then we have

$$\sigma(x_i, x_j) \geq \text{dist}(K_i, K_j) \geq s(\text{diam}(K_i) \wedge \text{diam}(K_j)) \geq st \text{diam}(A).$$

From (34) it easily follows that the number $N = \#I'$ of these points is bounded above by $1 + C/(st)^2$, where C is a universal constant. $\square$

If $M \subseteq \widehat{\mathbb{C}}$ is an arbitrary set, we denote by χ_M its characteristic function.

Lemma 8.3 *For each $\lambda \geq 1$, there exists a constant $C(\lambda) \geq 1$ with the following property: if $\{\overline{B}(x_i, r_i) : i \in I\}$ is a collection of closed disks in $\widehat{\mathbb{C}}$ indexed by a countable index set I , and if $a_i \geq 0$ for $i \in I$, then*

$$\int \left(\sum_{i \in I} a_i \chi_{\overline{B}(x_i, r_i)} \right)^2 d\Sigma \leq C(\lambda) \int \left(\sum_{i \in I} a_i \chi_{\overline{B}(x_i, r_i/\lambda)} \right)^2 d\Sigma.$$

The lemma is a special case of a well-known more general fact. It follows from a duality argument and the L^2 -boundedness of the Hardy-Littlewood maximal operator (see [4, p. 58, Lemma 4.2] for a very similar statement whose proof can easily be adapted to the present situation).

Proof of Proposition 8.1 Let E and F be arbitrary continua in $\overline{\Omega}$ with relative distance $\Delta(E, F) \leq t$ where $t > 0$. It is enough to show that if an arbitrary transboundary mass distribution on Ω has sufficiently small total mass

$$\int_{\Omega \setminus K} \rho^2 d\Sigma + \sum_{i \in I} \rho_i^2 < m,$$

where $m = m(\phi, s, \lambda, t) > 0$, then there exists a rectifiable path γ in Ω connecting E and F with

$$\int_{\gamma \cap (\Omega \setminus K)} \rho \, ds + \sum_{\gamma \cap K_i \neq \emptyset} \rho_i < 1.$$

Here $K = \bigcup_{i \in I} K_i$.

Since each set K_i is λ -quasi-round, we can find a disk $\overline{B}(x_i, r_i)$ with $x_i \in \widehat{\mathbb{C}}$ and $r_i \in (0, 2]$ such that

$$\overline{B}(x_i, r_i/\lambda) \subseteq K_i \subseteq \overline{B}(x_i, r_i).$$

If we have an arbitrary transboundary mass distribution on Ω , we define a density $\tilde{\rho}$ on $\widehat{\mathbb{C}}$ as follows:

$$\tilde{\rho} = \rho + \sum_{i \in I} \frac{\rho_i}{r_i} \chi_{\overline{B}(x_i, 2r_i)}.$$

Here we consider ρ as a function on $\widehat{\mathbb{C}}$ by setting it equal to 0 outside its original domain of definition $\Omega \setminus K$.

Then

$$\begin{aligned} \int \tilde{\rho}^2 \, d\Sigma &\leq 2 \int \rho^2 \, d\Sigma + 2 \int \left(\sum_{i \in I} \frac{\rho_i}{r_i} \chi_{\overline{B}(x_i, 2r_i)} \right)^2 \, d\Sigma \\ &\leq 2 \int \rho^2 \, d\Sigma + C_1(\lambda) \int \left(\sum_{i \in I} \frac{\rho_i}{r_i} \chi_{\overline{B}(x_i, r_i/\lambda)} \right)^2 \, d\Sigma \\ &\leq 2 \int \rho^2 \, d\Sigma + C_1(\lambda) \sum_{i \in I} \frac{\rho_i^2}{r_i^2} \int \chi_{\overline{B}(x_i, r_i/\lambda)} \, d\Sigma \\ &\leq C_2(\lambda) \left(\int \rho^2 \, d\Sigma + \sum_{i \in I} \rho_i^2 \right). \end{aligned}$$

In this estimate we used Lemma 8.3, the fact that the disks $\overline{B}(x_i, r_i/\lambda)$, $i \in I$, are pairwise disjoint, and (34).

Since Ω is ϕ -Loewner, the previous estimate implies that there exists a constant $m_1 = m_1(\phi, \lambda, t) > 0$ with the following property: if we impose the restriction

$$\int_{\Omega \setminus K} \rho^2 \, d\Sigma + \sum_{i \in I} \rho_i^2 < m_1$$

on the transboundary mass distribution (as we may), then there exists a rectifiable path γ in Ω with $\int_{\gamma} \tilde{\rho} ds < 1/2$ that connects E and F .

Using this path γ we define two disjoint subsets I_1 and I_2 of I . Let I_1 be the set of all $i \in I$ such that $K_i \cap \gamma \neq \emptyset$ and $4r_i < \text{diam}(\gamma)$, and I_2 be the set of all $i \in I$ such that $K_i \cap \gamma \neq \emptyset$ and $4r_i \geq \text{diam}(\gamma)$. Note that $I_1 \cup I_2$ is the set of all $i \in I$ with $K_i \cap \gamma \neq \emptyset$.

If $i \in I_1$, then γ meets $K_i \subseteq \overline{B}(x_i, r_i)$, but is not contained in $B(x_i, 2r_i)$. Hence

$$\int_{\gamma} \chi_{\overline{B}(x_i, 2r_i)} ds \geq r_i.$$

This implies

$$\begin{aligned} \int_{\gamma \cap (\Omega \setminus K)} \rho ds + \sum_{i \in I_1} \rho_i &\leq \int_{\gamma \cap (\Omega \setminus K)} \rho ds + \sum_{i \in I_1} \frac{\rho_i}{r_i} \int_{\gamma} \chi_{\overline{B}(x_i, 2r_i)} ds \\ &\leq \int_{\gamma} \tilde{\rho} ds < 1/2. \end{aligned}$$

If $i \in I_2$, then $\text{diam}(K_i) \geq r_i/\lambda \geq \text{diam}(\gamma)/(4\lambda)$. Using Lemma 8.2 for $A = \gamma$, we conclude that $N := \#I_2 \leq C_3 = C_3(s, \lambda)$.

If we impose the additional restriction

$$\int_{\Omega \setminus K} \rho^2 d\Sigma + \sum_{i \in I} \rho_i^2 < m_2$$

on our transboundary mass distribution, where $m_2 = m_2(s, \lambda) = \frac{1}{4C_3^2}$, then $\rho_i < \frac{1}{2C_3}$ for all $i \in I$ and so

$$\sum_{i \in I_2} \rho_i \leq \frac{N}{2C_3} \leq 1/2.$$

It follows that if

$$\int_{\Omega \setminus K} \rho^2 d\Sigma + \sum_{i \in I} \rho_i^2 < m = m(\phi, s, \lambda, t) := \min\{m_1, m_2\},$$

then there exists a rectifiable path γ in Ω connecting E and F with

$$\begin{aligned} \int_{\gamma \cap (\Omega \setminus K)} \rho ds + \sum_{\gamma \cap K_i \neq \emptyset} \rho_i &= \int_{\gamma \cap (\Omega \setminus K)} \rho ds + \sum_{i \in I_1} \rho_i + \sum_{i \in I_2} \rho_i \\ &< 1/2 + 1/2 = 1. \end{aligned}$$

Since $m > 0$ only depends on ϕ , λ , s , and t , the proof is complete. $\square$

Before we formulate the next proposition we will discuss some facts that will be useful for estimating path integrals. Let $\Omega \subseteq \widehat{\mathbb{C}}$ be region, $\pi: \Omega \rightarrow \mathbb{R}$ a continuous map, and $\alpha: I \rightarrow \Omega$ a locally rectifiable path in Ω . If $K \subseteq \widehat{\mathbb{C}}$ is compact and $U \subseteq \widehat{\mathbb{C}}$ is open, then $\pi(\alpha \cap U \cap K)$ is a Borel subset of $\mathbb{R}$. This follows from the fact that both the image set of α and the open set U are countable unions of compact sets. Hence $\pi(\alpha \cap U \cap K)$ is a countable union of compact sets and so indeed a Borel set. In particular, if we denote by m_1 Lebesgue measure on $\mathbb{R}$, then $m_1(\pi(\alpha \cap U \cap K))$ is defined.

If π is a 1-Lipschitz map, i.e., if $|\pi(u) - \pi(v)| \leq \sigma(u, v)$ for all $u, v \in \Omega$, then we have $m_1(\pi(\alpha)) \leq \text{length}(\alpha)$, and more generally

$$m_1(\pi(\alpha \cap U)) \leq \int_{\alpha} \chi_U ds, \quad (35)$$

whenever $U \subseteq \widehat{\mathbb{C}}$ is open. We will use these statements in the proof of the next proposition.

Proposition 8.4 *Let Ω be a region in $\widehat{\mathbb{C}}$, and $\mathcal{K} = \{K_i : i \in I\}$ a finite collection of pairwise disjoint compact sets in Ω . Suppose that the sets K_i are λ -quasi-round and s -relatively separated for $i \in I$.*

Then there is a non-increasing function $\phi: (0, \infty) \rightarrow (0, \infty)$ that can be chosen only depending on λ and s with the following property: if E and F are arbitrary disjoint continua in $\overline{\Omega}$, then

$$M_{\Omega, \mathcal{K}}(\Gamma(E, F; \Omega)) \leq \phi(\Delta(E, F)).$$

Here we cannot guarantee that $\phi(t) \rightarrow 0$ as $t \rightarrow \infty$. The point of the lemma is to have an upper bound for $M_{\Omega, \mathcal{K}}(\Gamma(E, F; \Omega))$ if $\Delta(E, F)$ is small.

Proof Let E and F be arbitrary disjoint continua in $\overline{\Omega}$, $\Gamma = \Gamma(E, F; \Omega)$, and $\Delta(E, F) \geq t > 0$. It suffices to produce an admissible transboundary mass distribution for Γ whose mass can be bounded above by a constant only depending on s , λ , and t .

For this we may assume $d := \text{diam}(E) \leq \text{diam}(F)$. Then $\text{dist}(E, F) \geq td$. Let $K = \bigcup_{i \in I} K_i$ and define

$$\rho(u) = \frac{1}{td} \quad \text{if } u \in N_{td}(E) \cap (\Omega \setminus K)$$

and $\rho(u) = 0$ elsewhere. Moreover, for $i \in I$ set

$$\rho_i = 1 \wedge \left(\frac{\text{diam}(K_i)}{td} \right) \quad \text{if } K_i \cap N_{td}(E) \neq \emptyset$$

and $\rho_i = 0$ otherwise.

We claim that this transboundary mass distribution is admissible for Γ . To see this let $\gamma \in \Gamma$ be an arbitrary path that is locally rectifiable in $\Omega \setminus K$, and consider the map $\pi : \widehat{\mathbb{C}} \rightarrow [0, \infty)$ defined by $u \mapsto \text{dist}(u, E)$. Since γ has an endpoint in E , but leaves the set $N_{td}(E)$, we have

$$\begin{aligned} [0, td) &= \pi(\gamma \cap N_{td}(E)) \\ &\subseteq \{0\} \cup \pi(\gamma \cap N_{td}(E) \cap (\Omega \setminus K)) \cup \bigcup_{i \in I} \pi(\gamma \cap N_{td}(E) \cap K_i) \\ &\subseteq \{0\} \cup \pi(\gamma \cap N_{td}(E) \cap (\Omega \setminus K)) \cup \bigcup_{\gamma \cap N_{td}(E) \cap K_i \neq \emptyset} \pi(N_{td}(E) \cap K_i). \end{aligned} \quad (36)$$

All subsets of $\mathbb{R}$ appearing in these inclusions are Borel sets as follows from the discussion before the statement of the proposition.

Inequality (35) applied to the map π , the set $U = N_{td}(E)$, and the pieces of the path γ in $\Omega \setminus K$ implies that

$$m_1(\pi(\gamma \cap N_{td}(E) \cap (\Omega \setminus K))) \leq \int_{\gamma \cap (\Omega \setminus K)} \chi_{N_{td}(E)} ds = td \int_{\gamma \cap (\Omega \setminus K)} \rho ds.$$

Combining this with (36) we obtain

$$\begin{aligned} 1 &\leq \frac{1}{td} m_1(\pi(\gamma \cap N_{td}(E))) \\ &\leq \frac{1}{td} m_1(\pi(\gamma \cap N_{td}(E) \cap (\Omega \setminus K))) \\ &\quad + \frac{1}{td} \sum_{\gamma \cap N_{td}(E) \cap K_i \neq \emptyset} m_1(\pi(N_{td}(E) \cap K_i)) \\ &\leq \int_{\gamma \cap (\Omega \setminus K)} \rho ds + \frac{1}{td} \sum_{\gamma \cap N_{td}(E) \cap K_i \neq \emptyset} ((td) \wedge \text{diam}(K_i)) \\ &\leq \int_{\gamma \cap (\Omega \setminus K)} \rho ds + \sum_{\gamma \cap K_i \neq \emptyset} \rho_i. \end{aligned}$$

The admissibility of our transboundary mass distribution follows.

To estimate the total mass for our transboundary mass distribution, we define two subsets I_1 and I_2 of I similarly as in the proof of Proposition 8.1. Namely, let I_1 be the set of all $i \in I$ such that $N_{td}(E) \cap K_i \neq \emptyset$ and $\text{diam}(K_i) < td$, and let I_2 be the set of all $i \in I$ such that $N_{td}(E) \cap K_i \neq \emptyset$ and $\text{diam}(K_i) \geq td$.

Since each set K_i is λ -quasi-round, we can find disks $\overline{B}(x_i, r_i)$ with $x_i \in \widehat{\mathbb{C}}$ and $r_i \in (0, 2]$ such that

$$\overline{B}(x_i, r_i/\lambda) \subseteq K_i \subseteq \overline{B}(x_i, r_i).$$

If $i \in I_1$, then $\overline{B}(x_i, r_i/\lambda) \subseteq K_i \subseteq N_{2td}(E)$ and so

$$\bigcup_{i \in I_1} B(x_i, r_i/\lambda) \subseteq N_{2td}(E).$$

Note that the balls in this union are pairwise disjoint and that the set $N_{2td}(E)$ is contained in a ball of radius $(2t+1)d$ centered at any point in E . So using (34) we obtain

$$\begin{aligned} \sum_{i \in I_1} \rho_i^2 &\leq \frac{1}{t^2 d^2} \sum_{i \in I_1} \text{diam}(K_i)^2 \leq \frac{4}{t^2 d^2} \sum_{i \in I_1} r_i^2 \\ &\leq \frac{4\lambda^2}{\pi t^2 d^2} \sum_{i \in I_1} \Sigma(B(x_i, r_i/\lambda)) \\ &= \frac{4\lambda^2}{\pi t^2 d^2} \Sigma\left(\bigcup_{i \in I_1} B(x_i, r_i/\lambda)\right) \\ &\leq \frac{4\lambda^2}{\pi t^2 d^2} \Sigma(N_{2td}(E)) \\ &\leq \frac{4\lambda^2(2t+1)^2}{t^2} = C_1(\lambda, t). \end{aligned}$$

If $i \in I_2$, then $K_i \cap N_{td}(E) \neq \emptyset$ and

$$\text{diam}(K_i) \geq td \geq \frac{t}{2t+1} \text{diam}(N_{td}(E)).$$

Using Lemma 8.2 for $A = N_{td}(E)$, we conclude that $\#I_2 \leq C_2 = C_2(s, t)$. Hence

$$\begin{aligned} \int \rho^2 d\Sigma + \sum_{i \in I} \rho_i^2 &\leq \frac{1}{t^2 d^2} \Sigma(N_{td}(E)) + \sum_{i \in I_1} \rho_i^2 + \sum_{i \in I_2} \rho_i^2 \\ &\leq \pi \frac{(t+1)^2}{t^2} + C_1(\lambda, t) + \sum_{i \in I_2} 1 \\ &\leq \pi \frac{(t+1)^2}{t^2} + C_1(\lambda, t) + C_2(s, t) = C(\lambda, s, t). \end{aligned}$$

The claim follows. $\square$

Let (X, d) be a locally compact metric space, and ν be a Borel measure on X . A measurable set $M \subseteq X$ is called μ -fat (for given (X, d, ν)), where $\mu > 0$, if for all $x \in M$ and all $0 < r \leq \text{diam}(M)$ we have

$$\nu(M \cap B(x, r)) \geq \mu \nu(B(x, r)).$$

In other words, a set M is fat if the intersection of M with every sufficiently small ball centered at a point in M has measure comparable to the measure of the whole ball. The notion of a fat set in the context of conformal mapping theory was introduced in [29, Sect. 2].

A (metric) annulus in a metric space (X, d) is a set $A \subseteq X$ of the form

$$A = A(x; r, R) := \{y \in X : r < d(y, x) < R\},$$

where $x \in X$, $0 < r < R < \text{diam}(X)/2$. Note that by the restriction on R both sets $\overline{B}(x, r)$ and $X \setminus B(x, R)$ are non-empty. We call them the *complementary parts* of $A(x; r, R)$. The *width* w_A of the annulus $A = A(x; r, R)$ is defined as $w_A = \log(R/r)$.

If $K \subseteq X$ is a compact set with $K \cap A \neq \emptyset$, then we define two numbers that describe how the set lies relative to the annulus $A = A(x; r, R)$, namely

$$r_A(K) := \inf_{y \in K \cap A} d(y, x) \quad \text{and} \quad R_A(K) := \sup_{y \in K \cap A} d(y, x).$$

Then $r \leq r_A(K) \leq R_A(K) \leq R$. We define the *width* $w_A(K)$ of K relative to A as

$$w_A(K) = \log(R_A(K)/r_A(K)).$$

If $K \cap A = \emptyset$ it is useful to set $w_A(K) = 0$.

In the following we consider annuli and fat sets in the metric space $(\widehat{\mathbb{C}}, \sigma)$ equipped with the measure Σ . Note that in this space a closed disk $M = \overline{B}(a, R)$ with $a \in \widehat{\mathbb{C}}$ and $0 < R \leq 2$, is μ -fat with $\mu = 1/4$. Indeed, let $x \in M$ and $0 < r \leq \text{diam}(M) \leq 2R$. If $\sigma(a, x) \geq r/2$ we can pick a point $y \in M$ on the minimizing spherical geodesic segment connecting x and a with $\sigma(x, y) = r/2$. If $\sigma(a, x) < r/2$ pick $y = a$. In both cases $B(y, r/2) \subseteq M \cap B(x, r)$ and so

$$\Sigma(M \cap B(x, r)) \geq \Sigma(B(y, r/2)) = \pi r^2/4 = \Sigma(B(x, r))/4.$$

Lemma 8.5 *Let $K_1, \dots, K_n$ be pairwise disjoint μ -fat sets in $(\widehat{\mathbb{C}}, \sigma, \Sigma)$, and suppose that there exists a metric annulus $A \subseteq \widehat{\mathbb{C}}$ with $w_A \geq 1$ such that each set K_i meets both complementary parts of A . Then $n \leq N(\mu) \in \mathbb{N}$.*

So if pairwise disjoint μ -fat sets meet both complementary parts of a sufficiently thick annulus in $\widehat{\mathbb{C}}$, then the number of these sets is bounded by a constant only depending on μ .

Proof Suppose that $A = A(x; r, R)$. Since $w_A \geq 1$, we have $R \geq er \geq 2r$. By our assumption each set K_i meets $B = B(x, r)$ and the complement of $B' = B(x, 2r) \subseteq B(x, R)$. Hence $\text{diam}(K_i) \geq r$. Picking a point $a_i \in K_i \cap B$, we see that

$$\Sigma(B' \cap K_i) \geq \Sigma(B(a_i, r) \cap K_i) \geq \mu \Sigma(B(a_i, r)) = \pi \mu r^2.$$

Since the sets $K_i \cap B'$, $i = 1, \dots, n$, are pairwise disjoint and contained in $B' = B(x, 2r)$, we conclude that the number of these sets is bounded above by

$$\Sigma(B(x, 2r))/(\pi \mu r^2) = 4/\mu^2.$$

So for $N(\mu)$ we can take the smallest integer $\geq 4/\mu^2$. $\square$

For the proof of Theorem 1.1 we are interested in the case where the sets $K_1, \dots, K_n$ are pairwise disjoint closed disks in $\widehat{\mathbb{C}}$. Then $\mu = 1/4$ and the previous proof gives the bound $n \leq 64$. It is not hard to see that if A is sufficiently thick, say $w_A \geq 100$, then actually $n \leq 2$.

Lemma 8.6 *Suppose that the collection $\{K_i : i \in I\}$ consists of pairwise disjoint compact and μ -fat sets in $(\widehat{\mathbb{C}}, \sigma, \Sigma)$. Let $N = N(\mu) \in \mathbb{N}$ be a number as in Lemma 8.5.*

If $A = A(x; r, R)$ is an arbitrary annulus in $\widehat{\mathbb{C}}$ with $w_A \geq 1$, then there exists a subannulus $A' = A(x; r', R') \subseteq A$ and a set $I_0 \subseteq I$ with the following properties:

- (i) $\#I_0 \leq N$.
- (ii) $w_{A'} \geq w_A^{1/3^N}$.
- (iii) $w_{A'}(K_i) \leq w_{A'}^{1/3}$ for all $i \in I \setminus I_0$.

This lemma will be applied when the width of A is every large. It then says that by removing a controlled number of compact sets in the given collection, we can find a subannulus A' of A whose width is not much smaller than the width of A and is much larger than the width relative to A' of the remaining sets in the collection.

Proof If $w_A(K_i) \leq w_A^{1/3}$ for all $i \in I$, we can choose $I_0 = \emptyset$ and $A' = A$.

Otherwise, there exists $i_1 \in I$ such that $w_A(K_{i_1}) \geq w_A^{1/3}$. Let

$$A_1 = A(x; r_A(K_{i_1}), R_A(K_{i_1})).$$

Then K_{i_1} meets both complementary parts of A_1 . This follows from the definitions of $r_A(K_{i_1})$ and $R_A(K_{i_1})$, and the facts that K_{i_1} is compact while A_1 is open. We also have $w_{A_1} = w_A(K_{i_1}) \geq w_A^{1/3}$.

If $w_{A_1}(K_i) \leq w_{A_1}^{1/3}$ for all $i \in I \setminus \{i_1\}$, we choose $I_0 = \{i_1\}$ and $A' = A_1$. Otherwise, there exists $i_2 \in I$, $i_2 \neq i_1$ such that $w_{A_1}(K_{i_2}) \geq w_{A_1}^{1/3}$. Define

$$A_2 = A(x; r_{A_1}(K_{i_2}), R_{A_1}(K_{i_2})).$$

Then A_2 is a subannulus of A_1 with $w_{A_2} \geq w_A^{1/3^2}$ and the sets K_{i_1} and K_{i_2} meet both complementary parts of A_2 .

Continuing in this manner we obtain a sequence of annuli $A_1, \dots, A_k$, and indices $i_1, \dots, i_k$. The process must stop after $k \leq N$ steps, because otherwise we would obtain $N + 1$ distinct μ -fat sets $K_{i_1}, \dots, K_{i_{N+1}}$ that meet both complementary parts of the annulus A_{N+1} . This is impossible by Lemma 8.5, since $w_{A_{N+1}} \geq w_A^{1/3^{N+1}} \geq 1$.

The annulus $A' = A_k$ and the set $I_0 = \{i_1, \dots, i_k\}$ have the desired properties. $\square$

Proposition 8.7 *Let $\mathcal{K} = \{K_i : i \in I\}$ be a finite collection of pairwise disjoint continua in $\widehat{\mathbb{C}}$. Suppose that the sets K_i are μ -fat sets in $(\widehat{\mathbb{C}}, \sigma, \Sigma)$ for $i \in I$, and let $N = N(\mu) \in \mathbb{N}$ be a number as in Lemma 8.5.*

Then there exists a function $\psi : (0, \infty) \rightarrow (0, \infty)$ with

$$\lim_{t \rightarrow \infty} \psi(t) = 0$$

that can be chosen only depending on μ and satisfies the following property: if E and F are arbitrary disjoint continua in $\widehat{\mathbb{C}} \setminus \bigcup_{i \in I} \text{int}(K_i)$ with $\Delta(E, F) \geq 12$, then there exists a set $I_0 \subseteq I$ with $\#I_0 \leq N$ such that for the transboundary modulus in the open set $\Omega' = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I_0} K_i$ with respect to the collection $\mathcal{K}' = \{K_i : i \in I \setminus I_0\}$ we have

$$M_{\Omega', \mathcal{K}'}(\Gamma(E, F; \Omega')) \leq \psi(\Delta(E, F)).$$

In general, Ω' will only be an open subset of $\widehat{\mathbb{C}}$ and not necessarily a region. The definitions of the path family $\Gamma(E, F; \Omega')$ and of the transboundary modulus $M_{\Omega', \mathcal{K}'}(\Gamma(E, F; \Omega'))$ for an open set Ω' are exactly the same as for regions. Note that

$$E, F \subseteq \widehat{\mathbb{C}} \setminus \bigcup_{i \in I} \text{int}(K_i) = \overline{\widehat{\mathbb{C}} \setminus \bigcup_{i \in I} K_i} \subseteq \overline{\widehat{\mathbb{C}} \setminus \bigcup_{i \in I_0} K_i} = \overline{\Omega'}.$$

To explain what the proposition means suppose that E and F are continua in $\widehat{\mathbb{C}} \setminus \bigcup_{i \in I} \text{int}(K_i)$ whose relative distance is large. Consider the family Γ of all paths in $\widehat{\mathbb{C}}$ that connect E and F . Then in general the transboundary modulus of Γ in $\widehat{\mathbb{C}}$ with respect to the collection $\{K_i : i \in I\}$ need not be

small. The reason is that there could be some sets K_i that are very close to both E and F and serve as a “bridge” between E and F . If there are many such bridges, the transboundary modulus of Γ can be large even if E and F have large relative distance. The proposition says that if we impose a uniform fatness condition on the sets K_i , and remove some elements K_i from our collection, then the transboundary modulus of the family of paths connecting E and F in the complement of the discarded sets behaves in the expected way; namely, it is uniformly small if the relative separation of E and F is large. The sets K_i that we have to remove from the collection may depend on E and F , but their number is uniformly bounded only depending on the fatness parameter μ .

Using the remark following Lemma 8.5 one can show that if the sets K_i are round disks and $\Delta(E, F)$ is large enough, one has to discard at most two disks in order to get a modulus bound of the desired type.

The restriction $\Delta(E, F) \geq 12$ in Proposition 8.7 is not very essential and one can prove a more general version. For this one has to find an appropriate bound for $M_{\Omega', \mathcal{K}'}(\Gamma(E, F; \Omega'))$ also for small $\Delta(E, F) > 0$. This can be done by an argument very similar to the proof of Proposition 8.4. The present version of Proposition 8.7 will be sufficient for our purpose.

In the proof of this proposition we need a variant of inequality (35). To formulate it, let (X, d) be a metric space, $x \in X$, $\pi: X \setminus \{x\} \rightarrow \mathbb{R}$ be the map defined by $u \in X \setminus \{x\} \mapsto \pi(u) = \log d(u, x)$, and α be a locally rectifiable path in $X \setminus \{x\}$. Then

$$m_1(\pi(\alpha)) \leq \int_{\alpha} \frac{ds}{d(x, \cdot)}, \quad (37)$$

where integration is with respect to arclength and m_1 again denotes 1-dimensional Lebesgue measure.

One can easily reduce this statement to the case when $\alpha: [0, L] \rightarrow X \setminus \{x\}$ is a rectifiable path in arclength parametrization, where $L = \text{length}(\alpha)$. By considering a suitable subpath and reversing orientation of the path if necessary one can further assume that $p = \alpha(0)$ is a point on α with minimal distance to x , and $q = \alpha(L)$ a point with maximal distance. Then $\pi(\alpha) = [\log d(x, p), \log d(x, q)]$, and so

$$\begin{aligned} \int_{\alpha} \frac{ds}{d(x, \cdot)} &\geq \int_0^L \frac{ds}{d(x, p) + s} = \log \left(1 + \frac{L}{d(x, p)} \right) \\ &\geq \log \left(\frac{d(x, q)}{d(x, p)} \right) = m_1(\pi(\alpha)) \end{aligned}$$

as desired.

Proof of Proposition 8.7 Let E and F be disjoint continua in $\widehat{\mathbb{C}} \setminus \bigcup_{i \in I} \text{int}(K_i)$ with $\Delta(E, F) = t \geq 12$. We may assume that $\text{diam}(E) \leq \text{diam}(F)$. Pick a point $x \in E$, and define $r = 2 \text{diam}(E)$ and $R = \text{dist}(E, F)/2$. Then $R/r = t/4 \geq 3$. Consider the annulus $A = A(x; r, R)$. Then E is contained in $B(x, r)$ and F in the complement of $B(x, R)$. So the annulus A separates the sets E and F . Moreover,

$$w_A = \log(R/r) = \log(t/4) \geq 1.$$

Then we can find a subannulus $A' = A(x; r', R')$ of A and a set $I_0 \subseteq I$ as in Lemma 8.6.

We define $\Omega' = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I_0} K_i$, and consider the transboundary modulus of the path family $\Gamma(E, F; \Omega')$ in Ω' with respect to the collection $\mathcal{K}' = \{K_i : i \in I \setminus I_0\}$. We set $K' := \bigcup_{i \in I \setminus I_0} K_i$.

We have to find a bound for $M_{\Omega', \mathcal{K}'}(\Gamma(E, F; \Omega'))$ depending on t and μ that is small if t is large. We define a transboundary mass distribution as follows. We let

$$\rho(u) = \frac{1}{w_{A'} \sigma(u, x)} \quad \text{for } u \in A' \cap (\Omega' \setminus K')$$

and $\rho(u) = 0$ elsewhere. Moreover, we let

$$\rho_i = w_{A'}(K_i)/w_{A'} \quad \text{for } i \in I \setminus I_0 \text{ with } K_i \cap A' \neq \emptyset,$$

and $\rho_i = 0$ for all other $i \in I \setminus I_0$.

We claim that this transboundary mass distribution is admissible for $\Gamma(E, F; \Omega')$. To see this let $\gamma \in \Gamma(E, F; \Omega')$ be an arbitrary path that is locally rectifiable in $\Omega' \setminus K'$. Since A' is a subannulus of A , it also separates E and F . Hence γ meets both complementary parts of A' , and so there exists an open subpath α of γ that lies in A' and connects the components of the boundary of A' . Obviously,

$$\int_{\gamma \cap (\Omega' \setminus K')} \rho \, ds + \sum_{i \in I \setminus I_0, K_i \cap \gamma \neq \emptyset} \rho_i \geq \int_{\alpha \cap (\Omega' \setminus K')} \rho \, ds + \sum_{i \in I \setminus I_0, K_i \cap \alpha \neq \emptyset} \rho_i.$$

We want to show that the right hand side of this inequality is bounded below by 1.

Let π be the map on $\overline{A'}$ to the interval $[\log r', \log R']$ defined by $u \mapsto \pi(u) := \log \sigma(u, x)$. Then

$$(\log r', \log R') = \pi(\alpha) \subseteq \pi(\alpha \cap (\Omega' \setminus K')) \cup \bigcup_{i \in I \setminus I_0, \alpha \cap K_i \neq \emptyset} \pi(A' \cap K_i). \quad (38)$$

By using (37) for $d = \sigma$ and the pieces of α in $\Omega' \setminus K'$, we see that

$$\int_{\alpha \cap (\Omega' \setminus K')} \rho \, ds \geq \frac{1}{w_{A'}} m_1(\pi(\alpha \cap (\Omega' \setminus K'))).$$

We also have

$$\rho_i \geq \frac{1}{w_{A'}} m_1(\pi(A' \cap K_i)) \quad \text{for all } i \in I \setminus I_0,$$

and so (38) implies

$$\begin{aligned} & \int_{\alpha \cap (\Omega' \setminus K')} \rho \, ds + \sum_{i \in I \setminus I_0, K_i \cap \alpha \neq \emptyset} \rho_i \\ & \geq \frac{1}{w_{A'}} m_1(\pi(\alpha \cap (\Omega' \setminus K'))) + \frac{1}{w_{A'}} \sum_{i \in I \setminus I_0, \alpha \cap K_i \neq \emptyset} m_1(\pi(A' \cap K_i)) \\ & \geq \frac{1}{w_{A'}} m_1((\log r', \log R')) = 1. \end{aligned}$$

The admissibility of our transboundary mass distribution follows.

To obtain mass bounds for our transboundary mass distribution, we first note that

$$\int_{A'} \frac{d\Sigma(u)}{\sigma(u, x)^2} = \int_{r'}^{R'} \frac{d(\Sigma(B(x, v)))}{v^2} = 2\pi \int_{r'}^{R'} \frac{dv}{v} = 2\pi \log(R'/r') = 2\pi w_{A'}. \quad (39)$$

Hence for the density part of the mass we get

$$\int_{\Omega' \setminus K'} \rho^2 \, d\Sigma \leq \frac{1}{w_{A'}^2} \int_{A'} \frac{d\Sigma(u)}{\sigma(u, x)^2} = \frac{2\pi}{w_{A'}}. \quad (40)$$

To estimate the mass of the discrete part, we consider two subsets I_1 and I_2 of $I \setminus I_0$. Namely, let I_1 be the set of all $i \in I \setminus I_0$ such that $A' \cap K_i \neq \emptyset$ and $w_{A'}(K_i) \leq \log 2$, and I_2 be the set of all $i \in I \setminus I_0$ such that $A' \cap K_i \neq \emptyset$ and $w_{A'}(K_i) > \log 2$. Since $\rho_i = 0$ for all $i \in I \setminus I_0$ with $A' \cap K_i = \emptyset$, we have

$$\sum_{i \in I \setminus I_0} \rho_i^2 = \sum_{i \in I_1} \rho_i^2 + \sum_{i \in I_2} \rho_i^2. \quad (41)$$

For $i \in I_1 \cup I_2$ let $r_i := r_{A'}(K_i)$ and $R_i := R_{A'}(K_i)$. For these i we then have

$$\rho_i = \frac{1}{w_{A'}} \log(R_i/r_i)$$

and $\text{diam}(K_i) \geq (R_i - r_i)$.

Since K_i is connected, we can find a point $a_i \in A' \cap K_i$ with $\sigma(a_i, x) = \frac{1}{2}(r_i + R_i)$. Let $B_i := B(a_i, \frac{1}{2}(R_i - r_i)) \subseteq A'$. The disk B_i is centered at a point in K_i and has a radius not exceeding the diameter of K_i . Hence the μ -fatness of K_i gives

$$\Sigma(A' \cap K_i) \geq \Sigma(B_i \cap K_i) \geq \mu \Sigma(B_i) = \frac{\pi}{4} \mu (R_i - r_i)^2, \quad (42)$$

and so

$$\int_{A' \cap K_i} \frac{d\Sigma(u)}{\sigma(u, x)^2} \geq \frac{\Sigma(A' \cap K_i)}{R_i^2} \geq \pi \mu \frac{(R_i - r_i)^2}{4R_i^2}. \quad (43)$$

Now if $i \in I_1$, then $R_i \leq 2r_i$ and so

$$\log(R_i/r_i) = \log\left(1 + \frac{R_i - r_i}{r_i}\right) \leq \frac{R_i - r_i}{r_i} \leq 2 \frac{R_i - r_i}{R_i}.$$

These inequalities imply

$$\begin{aligned} \sum_{i \in I_1} \rho_i^2 &= \frac{1}{w_{A'}^2} \sum_{i \in I_1} \log(R_i/r_i)^2 \\ &\leq \frac{16}{\pi \mu w_{A'}^2} \sum_{i \in I_1} \int_{A' \cap K_i} \frac{d\Sigma(u)}{\sigma(u, x)^2} \\ &\leq \frac{16}{\pi \mu w_{A'}^2} \int_{A'} \frac{d\Sigma(u)}{\sigma(u, x)^2} \leq \frac{32}{\mu w_{A'}}. \end{aligned} \quad (44)$$

If $i \in I_2$, then $R_i > 2r_i$ and so (43) shows that

$$\int_{A' \cap K_i} \frac{d\Sigma(u)}{\sigma(u, x)^2} \geq \frac{\pi}{16} \mu.$$

This implies that

$$\frac{\pi}{16} \mu \cdot \#I_2 \leq \sum_{i \in I_2} \int_{A' \cap K_i} \frac{d\Sigma(u)}{\sigma(u, x)^2} \leq \int_{A'} \frac{d\Sigma(u)}{\sigma(u, x)^2} \leq 2\pi w_{A'}.$$

Hence

$$\#I_2 \leq \frac{32}{\mu} w_{A'}.$$

By choice of I_0 according to Lemma 8.6, we have

$$w_{A'}(K_i) \leq w_{A'}^{1/3}$$

for all $i \in I \setminus I_0$. Using this with the upper bound on $\#I_2$ we conclude

$$\begin{aligned} \sum_{i \in I_2} \rho_i^2 &= \frac{1}{w_{A'}^2} \sum_{i \in I_2} w_{A'}(K_i)^2 \leq \frac{1}{w_{A'}^2} \cdot \#I_2 \cdot w_{A'}^{2/3} \\ &\leq \frac{32}{\mu w_{A'}^{1/3}}. \end{aligned} \quad (45)$$

By choice of A and A' we have $w_{A'} \geq w_A^{1/3N} = \log(t/4)^{1/3N} \geq 1$. Combining this with (40), (44), and (45), we arrive at the bound

$$\begin{aligned} M_{\Omega', \mathcal{K}'}(\Gamma(E, F; \Omega')) &\leq \int_{\Omega' \setminus K'} \rho^2 d\Sigma + \sum_{i \in I \setminus I_0} \rho_i^2 \leq \frac{2\pi}{w_{A'}} + \frac{32}{\mu w_{A'}} + \frac{32}{\mu w_{A'}^{1/3}} \\ &\leq \frac{C(\mu)}{w_{A'}^{1/3}} \leq \frac{C(\mu)}{\log(t/4)^{1/3N+1}}. \end{aligned}$$

Since $N = N(\mu)$ this gives the desired uniform bound in μ and t that becomes small if t becomes large. $\square$

Remark 8.8 The previous proposition holds in greater generality. Namely, suppose that we have a region $U \subseteq \widehat{\mathbb{C}}$ equipped with a path metric d induced by a conformal length element $ds_\lambda = \lambda ds$ and an associated measure ν such that $d\nu = \lambda^2 d\Sigma$ as in Remark 6.1. Suppose also that there exists a constant $C_0 \geq 1$ such that

$$\frac{1}{C_0} r^2 \leq \nu(B_d(a, r)) \leq C_0 r^2 \quad (46)$$

whenever $a \in U$ and $0 < r \leq \text{diam}_d(U)$. Then an analog of Proposition 8.7 holds in the metric measure space (U, d, ν) (instead of $(\widehat{\mathbb{C}}, \sigma, \Sigma)$) with a constant $N = N(\mu, C_0)$ and a function $\psi = \psi_{\mu, C_0}$.

Indeed, it is clear that versions of Lemmas 8.5 and 8.6 are true in this greater generality with a constant $N = N(\mu, C_0)$. Based on this, the proof of Proposition 8.7 can easily be adapted by changing the metric σ to d and the measure Σ to ν . All inequalities will remain valid up to an adjustment of the multiplicative constants. The upper mass bound in (46) is used to derive an inequality for the analog of the integral on the left hand side in (39) for sufficiently thick annuli A' . The bound will be a multiple of $w_{A'}$ with a suitable constant depending on C_0 . The lower mass bound (46) is used in the proof of Lemma 8.5 and in (42). Actually, in both cases we only need the lower mass bound for disks $B_d(a, r)$ with $r \leq \sup_{i \in I} \text{diam}(K_i)$.

We will later formulate a specific case explicitly in Proposition 11.5.

9 Classical uniformization

In this section we discuss some facts related to classical uniformization. The main result is Theorem 9.12. It can be derived from the remark in [30, p. 412] on periodic uniformization. We will give a different proof based on the methods developed in [30]. We will use some standard facts from complex analysis such as Montel's Theorem, the Argument Principle, etc. See, for example, [27] for precise statements and general background.

We consider *finitely connected* regions $U \subseteq \widehat{\mathbb{C}}$, i.e., regions with finitely many complementary components. The region is called *labeled* if its complementary components are labeled by the numbers $0, \dots, n$, i.e., if a bijection between the set of complementary components and the set $\{0, \dots, n\}$ has been specified. Here we assume that there are $n + 1 \geq 2$ complementary components. Two labeled regions are considered equal if the underlying sets are the same and the labels on complementary components agree. If U is a labeled region, then we denote the component of the complement with label i by $\widehat{\partial}_i U$ and the boundary of this component by $\partial_i U$. Then we have

$$\partial U = \partial_0 U \cup \dots \cup \partial_n U.$$

Let $f: U \rightarrow V$ be a conformal map between labeled regions U and V . Then the number of complementary components of U and V is the same and the map f induces a bijection ϕ on $\{0, \dots, n\}$ with the following property: for all $i = 0, \dots, n$ and all sequences (z_k) in U with $z_k \rightarrow \partial_i U$ we have $f(z_k) \rightarrow \partial_{\phi(i)} V$ (see [13, Sect. 15.3], and in particular [13, p. 81, Proposition 15.3.2]). The map f is *label-preserving* if ϕ is the identity on $\{0, \dots, n\}$.

A complementary or boundary component of a region U is called *degenerate* or *non-degenerate* depending on whether it consists of one or of more than one point. If $f: U \rightarrow V$ is a label-preserving conformal map between labeled regions, then for each $i = 0, \dots, n$ the component $\widehat{\partial}_i U$ is degenerate if and only if $\widehat{\partial}_i V$ is degenerate; this easily follows from the fact that points are removable singularities for bounded analytic functions.

Lemma 9.1 *Let $f: U \rightarrow V$ be a label-preserving conformal map between labeled regions U and V in $\widehat{\mathbb{C}}$ with finitely boundary components. If $\partial_0 U$ and $\partial_0 V$ are Jordan curves, then there exists a unique extension of f to a homeomorphism from $U \cup \partial_0 U$ onto $V \cup \partial_0 V$.*

This follows from [13, p. 83, Theorem 15.3.6(b)] (note that the points on $\partial_0 U$ and $\partial_0 V$ are *simple* boundary points of U and V , respectively; see [13, p. 52, Definition 14.5.9] and [13, p. 53, Corollary 14.5.11]). See also Remark 9.4 below where an outline of the proof will be given. Lemma 9.1 implies that if all boundary components of U and V are Jordan curves or

degenerate, then f extends to a homeomorphism from $\overline{U}$ onto $\overline{V}$ (see also [13, p. 82, Theorem 15.3.4]).

We need a statement similar in spirit to Lemma 9.1 on uniform convergence of sequences of conformal maps “up to the boundary”. It relies on some equicontinuity result for boundary maps which will be derived from the following well-known fact.

Lemma 9.2 (Wolff’s lemma) *Let $U \subseteq \mathbb{C}$ be open, $z_0 \in \mathbb{C}$, and $f: U \rightarrow \mathbb{C}$ be a conformal map with $f(U) \subseteq \mathbb{D}$. For $r > 0$ let $\gamma_r = U \cap \{z \in \mathbb{C} : |z - z_0| = r\}$.*

Then for all $0 < t < 1$ there exists $s \in (t, \sqrt{t})$ such that

$$\text{length}_{\mathbb{C}}(f(\gamma_s)) \leq \frac{2\pi}{\sqrt{\log(1/t)}}.$$

Note that $\gamma_r = U \cap \{z \in \mathbb{C} : |z - z_0| = r\}$ is a circle or consists of a countable collection of open circular arcs. In the statement $\text{length}_{\mathbb{C}}(f(\gamma_s))$ denotes the total Euclidean length of the images under f (recall from Sect. 2 that the subscript $\mathbb{C}$ refers to the Euclidean metric on $\mathbb{C}$).

For the proof of Lemma 9.2 see [25, p. 20, Proposition 2.2].

A *crosscut* in an open Jordan region $D \subseteq \widehat{\mathbb{C}}$ is an arc α whose interior points lie in D and whose endpoints lie on ∂D . A crosscut α in D *separates* two points $p \in D$ and $q \in \partial D$ if every path $\phi: [0, 1] \rightarrow \widehat{\mathbb{C}}$ with $p = \phi(0)$, $q = \phi(1)$, and $\phi([0, 1)) \subseteq D$ meets α .

The set of points on ∂D separated by a crosscut α in D from a given point $p \in D \setminus \alpha$ is equal to one of the subarcs γ of ∂D with the same endpoints as α . In the special case $D = \mathbb{D}$ and $p = 0$ a simple argument shows there is a constant $c_0 > 0$ such that if $\text{diam}_{\mathbb{C}}(\alpha) \leq c_0$, then γ is the smaller arc on $\partial \mathbb{D}$ with the same endpoints as α , and so $\text{diam}_{\mathbb{C}}(\gamma) \leq \text{diam}_{\mathbb{C}}(\alpha)$. It is not hard to see that $c_0 = 1$ is the sharp constant in this statement.

Based on this and the Schönflies Theorem one can show that if $D \subseteq \widehat{\mathbb{C}}$ is an arbitrary open Jordan region, and $p \in D$, then for every $\epsilon > 0$ there exists $\delta > 0$ such that for every crosscut α in D with $p \notin \alpha$ and $\text{diam}(\alpha) < \delta$ we have $\text{diam}(\gamma) < \epsilon$ for the arc γ of points on ∂D separated by α from p .

Lemma 9.3 (Equicontinuity of boundary maps) *Let $r \in (0, 1)$,*

$$A = \{z \in \mathbb{C} : r < |z| \leq 1\},$$

and let $\mathcal{F}$ be the family of all homeomorphisms f on A that are conformal on $\text{int}(A)$ so that $f(A) \subseteq \overline{\mathbb{D}}$, $f(\partial \mathbb{D}) = \partial \mathbb{D}$, and 0 is contained in the bounded component of $\mathbb{C} \setminus f(\text{int}(A))$.

Then the family $\{f|_{\partial\mathbb{D}} : f \in \mathcal{F}\}$ of boundary maps is equicontinuous with respect to the Euclidean metric. Moreover, every sequence in $\mathcal{F}$ has a subsequence that converges to a function in $\mathcal{F}$ uniformly on compact subsets of A .

The existence of a subsequence that converges uniformly on compact subsets of $\text{int}(A)$ immediately follows from Montel's Theorem. The point here is that we get uniform convergence "up to the boundary" $\partial\mathbb{D}$, i.e., on compact subsets of A .

Note that for $f \in \mathcal{F}$ the set $f(\text{int}(A))$ has two complementary components. One is equal to the complement of $\mathbb{D}$ while the other is a compact subset of $\mathbb{D}$.

Proof Let $\epsilon > 0$, $z_0, z_1 \in \partial\mathbb{D}$, and $f \in \mathcal{F}$ be arbitrary. We may assume that $\epsilon < 1/10$.

Suppose that $\delta > 0$ and $|z_1 - z_0| < \delta$. Lemma 9.2 implies that if $\delta > 0$ is small enough only depending on ϵ , then there exists a crosscut α in $\mathbb{D}$ that lies in A , separates the points z_0 and z_1 from each point on the circle $\{z \in \mathbb{C} : |z| = r\} \subseteq \partial A$ and satisfies $\text{length}_{\mathbb{C}}(f(\alpha)) < \epsilon$. Then $\beta = f(\alpha)$ is also a crosscut in $\mathbb{D}$, and it separates $f(z_0), f(z_1) \in \partial\mathbb{D}$ from each point in the bounded component of $\mathbb{C} \setminus f(A)$, and hence from 0 by our hypotheses. Since $\text{length}_{\mathbb{C}}(\beta) < \epsilon < 1/10$, this implies that $f(z_0)$ and $f(z_1)$ lie on the smaller subarc of $\partial\mathbb{D}$ determined by the endpoints of β . This arc has diameter bounded by the diameter of β . Hence

$$|f(z_0) - f(z_1)| \leq \text{length}_{\mathbb{C}}(\beta) < \epsilon.$$

The equicontinuity of the family of boundary maps follows.

Let (f_n) be an arbitrary sequence in $\mathcal{F}$. Since the sequence is uniformly bounded, Montel's Theorem implies that there exists a subsequence that converges uniformly on compact subsets of $\text{int}(A)$. By the first part of the proof, we know that the maps $f_n|_{\partial\mathbb{D}}$ are equicontinuous. Hence by passing to a further subsequence, we may assume that our subsequence converges uniformly on $\partial\mathbb{D}$.

Replacing our original sequence by such a subsequence, we may assume that (f_n) converges uniformly on compact subsets of $\text{int}(A)$ and uniformly on $\partial\mathbb{D}$. We claim that this implies that (f_n) converges uniformly on compact subsets of A . Indeed, if $K \subseteq A$ is an arbitrary compact set, then there exists $r < r' < 1$ such that

$$K \subseteq A' = \{z \in \mathbb{C} : r' \leq |z| \leq 1\}.$$

The circle $\{z \in \mathbb{C} : |z| = r'\}$ is a compact subset of $\text{int}(A)$, so the convergence of our sequence (f_n) is uniform on this set. Moreover, (f_n) converges uniformly on $\partial\mathbb{D}$. The Maximum Principle implies that the sequence converges uniformly on A' and hence on K .

Let f be the limit function of the sequence (f_n) . Then f is continuous on A . Moreover, by Hurwitz's Theorem f is either constant or a conformal map on $\text{int}(A)$. Here the former case is impossible, because we have $f_n(\partial\mathbb{D}) = \partial\mathbb{D}$ and so $f(\partial\mathbb{D}) = \partial\mathbb{D}$; indeed, if $y \in \partial\mathbb{D}$ is arbitrary, and x is any sublimit of a sequence (x_n) in $\partial\mathbb{D}$ with $f_n(x_n) = y$ for each n , then $f(x) = y$.

The set $f(\text{int}(A))$ is a region in $\mathbb{D}$. It follows from the Argument Principle that every point in $\mathbb{D}$ that is sufficiently close to $\partial\mathbb{D}$ lies in $f(\text{int}(A))$. Hence one of the boundary components of $f(\text{int}(A))$ is $\partial\mathbb{D}$, and so Lemma 9.1 implies that f is a homeomorphism on A . Since the value 0 is not attained by any of the functions f_n , the function f does not attain 0 either. This implies that 0 is contained in the bounded component of $\mathbb{C} \setminus f(\text{int}(A))$. We conclude that $f \in \mathcal{F}$. $\square$

Remark 9.4 We referred to Lemma 9.1 in the proof of the previous lemma. By using similar ideas as in the previous proof based on Lemma 9.2 and the fact on crosscuts in Jordan regions mentioned before Lemma 9.3, one can actually easily give a proof of Lemma 9.1. One first shows the uniform continuity of the map f in Lemma 9.1 on a (topological) annulus $A \subseteq U$ with $\partial_1 U \subseteq \partial A$. This implies that f has a continuous extension to $\partial_1 U$. This extension satisfies $f(\partial_1 U) \subseteq \partial_1 V$. The map $f|_{\partial_1 U}$ is a homeomorphism of $\partial_1 U$ onto $\partial_1 V$, because an inverse map can be obtained by applying the same argument to f^{-1} on V .

A region $V \subseteq \widehat{\mathbb{C}}$ is called a *circle domain* if its complementary components are round, possibly degenerate, disks. The following theorem is one of the landmarks of classical uniformization theory.

Theorem 9.5 (Koebe's uniformization theorem) *Let $U \subseteq \widehat{\mathbb{C}}$ be a region with finitely many complementary components. Then there exists a conformal map $f: U \rightarrow V$ of U onto a circle domain V . The map f is unique up to postcomposition with an orientation-preserving Möbius transformation.*

See [13, p. 106, Theorem 15.7.9] for the existence, and [13, p. 102, Proposition 15.7.5] for the uniqueness statement (note that in [13] this is only formulated for regions U whose complementary components are non-degenerate, but our more general version can easily be derived from this).

If we equip $\mathbb{C}$ with the Euclidean metric, then the cyclic group Γ generated by the translation $z \mapsto z + 2\pi i$, where i is the imaginary unit, acts on $\mathbb{C}$ by isometries. The Riemannian quotient $\mathbb{C}/\Gamma$ is isometric to the infinite cylinder $Z = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ with the Riemannian metric induced from $\mathbb{R}^3$. The exponential function induces an isometry of $\mathbb{C}/\Gamma \cong Z$ with $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Here $\mathbb{C}^*$ is equipped with the *flat metric* $d_{\mathbb{C}^*}$ induced by the length element

$$ds_{\mathbb{C}^*} = \frac{|dz|}{|z|}.$$

We use terminology for sets in $\mathbb{C}^*$ that is suggested by this identification of $\mathbb{C}^*$ with the cylinder Z .

A *finite $\mathbb{C}^*$ -cylinder* is a set A of the form

$$A = \{z \in \mathbb{C} : r < |z| < R\},$$

where $0 < r < R$. We denote by $\partial_i A = \{z \in \mathbb{C} : |z| = r\}$ its *inner*, and by $\partial_o A = \{z \in \mathbb{C} : |z| = R\}$ its *outer* boundary component. The *height* h_A of the finite $\mathbb{C}^*$ -cylinder A is the quantity $h_A = \log(R/r)$. A $\mathbb{C}^*$ -square Q is a set of the form

$$Q = \{\rho e^{it} : \alpha \leq t \leq \beta \text{ and } r \leq \rho \leq R\},$$

where $\alpha < \beta$, $\beta - \alpha < 2\pi$, $0 < r < R$, and $\beta - \alpha = \log(R/r)$. The *side length* $\ell(Q)$ of Q is defined as

$$\ell(Q) = \beta - \alpha = \log(R/r).$$

Note that $0 < \ell(Q) < 2\pi$. We call the point $p_Q = \sqrt{rR}e^{i(\alpha+\beta)/2}$ the *center* of Q . Sometimes it is useful to allow the case of *degenerate* $\mathbb{C}^*$ -squares, where $r = R$, $\alpha = \beta$, and $\ell(Q) = 0$. Then Q only consists of the point p_Q .

In the following we fix $n \in \mathbb{N}$. We denote by $\mathcal{S}$ the set of labeled regions $U \subseteq \mathbb{C}^*$ that can be written as

$$U = \mathbb{D} \setminus (Q_1 \cup \dots \cup Q_n) \quad (47)$$

where the sets $Q_1, \dots, Q_n$ are pairwise disjoint subsets of $\mathbb{D}$ such that $Q_1, \dots, Q_{n-1}$ are $\mathbb{C}^*$ -squares, and Q_n is a closed Euclidean disk centered at 0. Here we allow degenerate squares and disks, i.e., sets consisting of only one point. The complementary components of U as in (47) are the sets

$$\widehat{\mathbb{C}} \setminus \mathbb{D}, Q_1, \dots, Q_n.$$

We assume they are labeled by $0, \dots, n$ in this order.

Similarly, we denote by $\mathcal{C}$ the set of all labeled regions $V \subseteq \mathbb{D}$ that can be written as

$$V = \mathbb{D} \setminus (C_1 \cup \dots \cup C_n), \quad (48)$$

where $C_1, \dots, C_n$ are pairwise disjoint closed Euclidean disks contained in $\mathbb{D}$ such that C_n has center 0. Again we allow degenerate disks C_i consisting of only one point. Moreover, we assume that the complementary components

$$\widehat{\mathbb{C}} \setminus \mathbb{D}, C_1, \dots, C_n$$

of V are labeled by $0, \dots, n$, respectively.

There are natural identifications of the spaces $\mathcal{S}$ and $\mathcal{C}$ with certain (relatively) open and connected subsets S and C of $\mathbb{D}^{n-1} \times [0, \infty)^n$, respectively. Indeed, if a region $U \in \mathcal{S}$ is written as in (47), we let it correspond to the point

$$x = (p_1, \dots, p_{n-1}, r_1, \dots, r_n) \in \mathbb{D}^{n-1} \times [0, \infty)^n,$$

where $p_i \in \mathbb{D}$ is the center and $r_i \geq 0$ is the sidelength of the $\mathbb{C}^*$ -square Q_i for $i = 1, \dots, n-1$, and $r_n \geq 0$ is the radius of Q_n . It is clear that the correspondence $U \leftrightarrow x$ gives a bijection of the space $\mathcal{S}$ and an open subset S of $\mathbb{D}^{n-1} \times [0, \infty)^n$. The set S is path-connected and hence connected. Indeed, if $x \in S$ is arbitrary, then we can get a path in S connecting x to a basepoint in S by performing the following procedure on the region U corresponding to x : we shrink the complementary components of U distinct from $\widehat{\mathbb{C}} \setminus \mathbb{D}$ to points, and then move these points in $\mathbb{D}$ to prescribed positions while avoiding collisions of the points.

Similarly, if $V \in \mathcal{C}$ is written as in (48), we let it correspond to the point

$$y = (q_1, \dots, q_{n-1}, s_1, \dots, s_n) \in \mathbb{D}^{n-1} \times [0, \infty)^n,$$

where $q_i \in \mathbb{D}$ is the center and $s_i \geq 0$ is the radius of the disk C_i for $i = 1, \dots, n-1$, and $s_n \geq 0$ is the radius of C_n . Again we get a bijection of the space $\mathcal{C}$ and an open and connected subset C of $\mathbb{D}^{n-1} \times [0, \infty)^n$.

We need a criterion when a sequence (x_k) in one of the sets S or C has a convergent subsequence with a limit in the set. For this the only obstacle for sequences in C is when the corresponding regions have complementary components that get close to each other. For sequences in S there is the additional obstacle that some of the complementary $\mathbb{C}^*$ -squares of the regions may “wrap around” the cylinder $\mathbb{C}^*$ and have sidelengths approaching 2π . The following lemma gives a simple condition that prevents these phenomena.

In the proof we use *Hausdorff convergence* of sets. We remind the reader of the definition of this concept. Let (A_k) be a sequence of closed subsets of a metric space (X, d) . We say that the sequence (A_k) *Hausdorff converges* to another closed set $A \subseteq X$, written as $A_k \rightarrow A$ as $k \rightarrow \infty$, if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $A \subseteq N_\epsilon(A_k)$ and $A_k \subseteq N_\epsilon(A)$ whenever $k > N$. We will use this for subsets of $X = \widehat{\mathbb{C}}$. Unless otherwise specified, d will then be the chordal metric on $\widehat{\mathbb{C}}$. If the sets under consideration are contained in a compact subset of $\mathbb{C}$, one can alternatively use the Euclidean metric.

Lemma 9.6 (Subconvergence criterion) *Let (x_k) be a sequence in S (or C), and U_k be the labeled region in S (or C) corresponding to x_k for $k \in \mathbb{N}$. Suppose that there exist pairwise disjoint closed Jordan regions $D_1, \dots, D_n \subseteq \mathbb{D}$ such that $\partial_i U_k \subseteq D_i$ for all $i = 1, \dots, n$ and all $k \in \mathbb{N}$. Then the sequence (x_k) has a subsequence that converges to a point in S (or C).*

Proof We will only prove the statement if $n \geq 2$ and (x_k) is a sequence in S . The cases when $n = 1$ or when (x_k) is a sequence in C are similar and easier.

Note that 0 is contained in each of the sets $\widehat{\partial}_n U_k$, and so $0 \in D_n$. For each $k \in \mathbb{N}$ the boundary component $\widehat{\partial}_1 U_k$ of U_k is a $\mathbb{C}^*$ -square contained in $D_1 \subseteq \mathbb{D}$. By passing to a subsequence if necessary, we may assume that for $k \rightarrow \infty$ the centers of the $\mathbb{C}^*$ -squares $\partial_1 U_k$ converge to a point $c_1 \in D_1$ and their sidelengths converge to a number $l_1 \in [0, 2\pi]$. We claim that $l_1 < 2\pi$.

For otherwise, $l_1 = 2\pi$. Then a limiting argument shows that the circle $\{z \in \mathbb{C} : |z| = |c_1|\}$ is contained in D_1 . Since D_1 is a Jordan region, this implies that $\{z \in \mathbb{C} : |z| \leq |c_1|\} \subseteq D_1$ and so $0 \in D_1$. On the other hand, $0 \in D_n$. Since $n \neq 1$, and D_1 and D_n are disjoint by hypothesis we get a contradiction. So $l_1 < 2\pi$.

Let Q_1 be the (possibly degenerate) $\mathbb{C}^*$ -square with center c_1 and sidelength l_1 . Then $\widehat{\partial}_1 U_k \rightarrow Q_1$ as $k \rightarrow \infty$ in the sense of Hausdorff convergence, and $Q_1 \subseteq D_1$.

A similar argument shows that by passing to successive subsequences if necessary, we may assume that $\partial_i U_k \rightarrow Q_i$ as $k \rightarrow \infty$, where $Q_i \subseteq D_i$ is a $\mathbb{C}^*$ -square for $i = 1, \dots, n-1$, and a closed disk centered at 0 for $i = n$. Since the sets $D_1, \dots, D_n$ are pairwise disjoint subsets of $\mathbb{D}$, the same is true for the sets $Q_1, \dots, Q_n$. It follows that $U = \mathbb{D} \setminus (Q_1 \cup \dots \cup Q_n)$ is a region in S , where the complementary components

$$\widehat{\mathbb{C}} \setminus \mathbb{D}, Q_1, \dots, Q_n$$

of U are labeled by the numbers $0, \dots, n$ in this order. If $x \in S$ is the point corresponding to U , then it is clear that (x_k) subconverges to x . $\square$

We define a map $\eta: S \rightarrow C$ as follows. Let $x \in S$ be arbitrary, and $U \in S$ be the labeled region corresponding to x . By Koebe's Uniformization Theorem there exists a conformal map f of U onto a circle domain V , unique up to post-composition with a Möbius transformation. We label the complementary components of V so that f is label-preserving. By post-composing f by a Möbius transformation we may assume that the boundary component of V with label 0 is the unit circle $\partial\mathbb{D}$ and that the one with label n is of the form $\{z \in \mathbb{C} : |z| = s\}$ with $0 \leq s < 1$. By the remark following Lemma 9.1 the map f has an extension, also denoted f , to a homeomorphism from $\overline{U}$ to $\overline{V}$. By post-composing f by a suitable rotation, we may also assume that $f(1) = 1$. So f is normalized such that

$$f(\partial\mathbb{D}) = \partial\mathbb{D},$$

$$f(\partial_n U) = \partial_n V = \{w \in \mathbb{C} : |w| = s\}, \quad \text{where } 0 \leq s < 1, \text{ and } f(1) = 1.$$

Note that with these normalizations the map f is uniquely determined, and the labeled region $V = f(U)$ lies in $\mathcal{C}$. We let $y \in \mathcal{C}$ be the point corresponding to V , and set $\eta(x) := y$.

Our goal is to show that η is surjective. We need some preparation.

Lemma 9.7 *For $k \in \mathbb{N} \cup \{\infty\}$ let $\varphi_k: \partial\mathbb{D} \rightarrow J_k := \varphi_k(\partial\mathbb{D}) \subseteq \widehat{\mathbb{C}}$ be homeomorphisms such that $\varphi_k \rightarrow \varphi_\infty$ uniformly on $\partial\mathbb{D}$ as $k \rightarrow \infty$. For $k \in \mathbb{N}$ let $M_k \subseteq \widehat{\mathbb{C}} \setminus J_k$ be a set whose points are separated by J_k from a basepoint $p \in \widehat{\mathbb{C}} \setminus \bigcup_{k \in \mathbb{N} \cup \{\infty\}} J_k$. If*

$$\delta := \liminf_{k \rightarrow \infty} \text{dist}(M_k, J_\infty) > 0,$$

then J_∞ separates the points in M_k from p for all large enough k .

Here we say that a Jordan curve $J \subseteq \widehat{\mathbb{C}}$ separates two points $a, b \in \widehat{\mathbb{C}}$ if a and b lie in different complementary components of J .

Proof We may assume that $p = \infty$. Then J_k is a Jordan curve in $\mathbb{C}$ for all $k \in \mathbb{N} \cup \{\infty\}$. One of the two closed Jordan regions in $\widehat{\mathbb{C}}$ bounded by J_k contains ∞ . Let $D_k \subseteq \mathbb{C}$ be the other one, and let α_k be the loop defined by $\alpha_k(t) = \varphi_k(e^{it})$ for $t \in [0, 2\pi]$. Then a point $a \in \mathbb{C} \setminus J_k$ is separated from $p = \infty$ by J_k if and only if the winding number of the loop α_k around a is non-zero; indeed, this winding number is ± 1 for points in $\text{int}(D_k)$ depending on the orientation of α_k , and 0 for points in $\mathbb{C} \setminus D_k$.

By our hypotheses we have $M_k \subseteq \mathbb{C} \setminus N_{\delta/2}(J_\infty)$ for large enough k , say for $k \geq k_1$. Moreover, since $\varphi_k \rightarrow \varphi_\infty$ as $k \rightarrow \infty$ uniformly on $\partial\mathbb{D}$, for all large enough k , say for $k \geq k_2$, the loop α_k lies in $N_{\delta/2}(J_\infty)$ and is homotopic to α_∞ in $N_{\delta/2}(J_\infty)$. Then for $k \geq k_2$ the winding numbers of α_k and α_∞ around any point in $\mathbb{C} \setminus N_{\delta/2}(J_\infty)$ are the same. By our hypotheses the winding number of α_k around any point in M_k is ± 1 . Hence for $k \geq k_1 \vee k_2$ the winding number of α_∞ around any point $a \in M_k$ is ± 1 , and so J_∞ separates a from p . $\square$

Lemma 9.8 *The map η is continuous.*

Proof Let (x_k) be an arbitrary sequence in S with $x_k \rightarrow x_\infty \in S$ as $k \rightarrow \infty$. Define $y_k = \eta(x_k)$ and $y_\infty = \eta(x_\infty)$. We will show that there exist a subsequence (y_{k_l}) of (y_k) such that $y_{k_l} \rightarrow y_\infty$ as $l \rightarrow \infty$.

Since (x_k) is arbitrary, this fact can then also be applied to any subsequence of (x_k) . Hence for every subsequence of (y_k) there exists a “subsubsequence” that converges to y_∞ . This implies that the sequence (y_k) itself converges to y_∞ , and the continuity of η follows.

It remains to produce the subsequence (y_{k_l}) with $y_{k_l} \rightarrow y_\infty$. For $k \in \mathbb{N} \cup \{\infty\}$ let $U_k \in \mathcal{S}$ be the labeled region corresponding to $x_k \in S$, $V_k \in \mathcal{C}$ be the

labeled region corresponding to $y_k \in C$, and for $k \in \mathbb{N}$ let $f_k: \overline{U}_k \rightarrow \overline{V}_k$ be the homeomorphism as in the definition of η .

Since $x_k \rightarrow x_\infty$, every compact subset of $U_\infty \cup \partial\mathbb{D}$ lies in $U_k \cup \partial\mathbb{D}$ for sufficiently large k . In particular, the map f_k is defined on each such set for sufficiently large k .

Using the second part of Lemma 9.3 together with Montel's theorem, we can find a subsequence of the sequence (f_k) that converges uniformly on compact subsets of $U_\infty \cup \partial\mathbb{D}$. By replacing our original sequence by this subsequence, we may assume that (f_k) itself has this convergence property, and that the limit map f is a homeomorphism on $U_\infty \cup \partial\mathbb{D}$ that is conformal on U_∞ , and satisfies $f(\partial\mathbb{D}) = \partial\mathbb{D}$ and $f(1) = 1$.

Let $D_1, \dots, D_n \subseteq \mathbb{D}$ be pairwise disjoint closed Jordan regions with

$$\widehat{\partial}_i U_\infty \subseteq \text{int}(D_i)$$

for $i = 1, \dots, n$. Such Jordan regions can be found by slightly enlarging the complementary components $\widehat{\partial}_1 U_\infty, \dots, \widehat{\partial}_n U_\infty$ of U_∞ . For each $i = 1, \dots, n$ and all large enough $k \in \mathbb{N}$ we then have $\widehat{\partial}_i U_k \subseteq \text{int}(D_i)$, and so ∂D_i separates the points in $\widehat{\partial}_i U_k$ from the point 1.

Let

$$U = \mathbb{D} \setminus (D_1 \cup \dots \cup D_n).$$

Then $\overline{U} \subseteq U_\infty \cup \partial\mathbb{D}$. The image $V = f(U)$ can be written as

$$V = \mathbb{D} \setminus (E_1 \cup \dots \cup E_n),$$

where $E_1, \dots, E_n \subseteq \mathbb{D}$ are pairwise disjoint closed Jordan regions with $f(\partial D_i) = \partial E_i$ for $i = 1, \dots, n$.

Since $f_k \rightarrow f$ uniformly on compact subsets of U_∞ , the Argument Principle implies that if $K \subseteq U_\infty$ is compact, then $f(K) \subseteq f_k(U_k)$ for all large enough k . In particular, a small neighborhood of $\partial E_i = f(\partial D_i)$ will lie in $V_k = f_k(U_k)$ if k is large enough. Hence we can choose $\delta > 0$ such that $\text{dist}(\partial E_i, \widehat{\partial}_i V_k) \geq \delta$ for all $i = 1, \dots, n$ and all large enough k . We also have $\widehat{\partial}_i U_k \subseteq \text{int}(D_i)$ and $\partial D_i \subseteq U_k$ for large enough k . Then $f_k(\partial D_i)$ separates the points in $\widehat{\partial}_i V_k$ from the point 1. Since $f_k \rightarrow f$ uniformly on ∂D_i , it follows from Lemma 9.7 that $\partial E_i = f(\partial D_i)$ separates the points in $\widehat{\partial}_i V_k$ from 1 for all $i = 1, \dots, n$ and all large enough k . For such k we then have $\widehat{\partial}_i V_k \subseteq E_i$ for all $i = 1, \dots, n$. By Lemma 9.6 we may pass to another subsequence if necessary and assume that $y_k \rightarrow \tilde{y}_\infty \in C$.

The proof will be complete if we can show that $\tilde{y}_\infty = y_\infty$. Let $\tilde{V}_\infty \in \mathcal{C}$ be the labeled region corresponding to $\tilde{y}_\infty$. By definition of the map η it is enough to show that f is a label-preserving conformal map of U_∞ onto $\tilde{V}_\infty$. For then f has the right normalization and so $\tilde{y}_\infty = \eta(x_\infty) = y_\infty$.

It is clear that $\widehat{\partial}_i \widetilde{V}_\infty \subseteq E_i$ for $i = 1, \dots, n$. If we let D_i shrink to $\widehat{\partial}_i U_\infty$, then the corresponding Jordan region E_i shrinks to $\widehat{\partial}_i f(U_\infty)$. Hence $\widehat{\partial}_i V_\infty \subseteq \widehat{\partial}_i f(U_\infty)$ for $i = 1, \dots, n$, and so $f(U_\infty) \subseteq \widetilde{V}_\infty$. These inclusions show that if in addition $f(U_\infty) \supseteq \widetilde{V}_\infty$, then f is a label-preserving conformal map between U_∞ and $\widetilde{V}_\infty$ as desired.

In order to establish $f(U_\infty) \supseteq \widetilde{V}_\infty$, we repeat the argument in the first part of the proof for the sequence (f_k^{-1}) . Passing to a subsequence if necessary, we may assume that it converges uniformly on compact subsets of $\widetilde{V}_\infty \cup \partial\mathbb{D}$ to a conformal map g . The same proof as above then shows that $g(\widetilde{V}_\infty) \subseteq U_\infty$. Hence the map $f \circ g$ is well-defined on $\widetilde{V}_\infty$ and by uniform convergence we have

$$f(g(w)) = \lim_{k \rightarrow \infty} f_k(f_k^{-1}(w)) = w$$

for $w \in \widetilde{V}_\infty$. This implies that $f(U_\infty) \supseteq \widetilde{V}_\infty$ as desired. $\square$

A map between topological spaces X and Y is called *proper* if the preimage of each compact subset of Y is a compact subset of X .

Lemma 9.9 *The map η is proper.*

Proof We claim that every point $y \in C$ has a neighborhood $N \subseteq C$ such that $\eta^{-1}(N)$ is relatively compact in S . Given this claim every compact set $K \subseteq C$ can be covered by finitely many such neighborhoods $N_1, \dots, N_m$. Then

$$\eta^{-1}(K) \subseteq \eta^{-1}(N_1) \cup \dots \cup \eta^{-1}(N_m)$$

is relatively compact. The set $\eta^{-1}(K)$ is also closed, since K is closed and η is continuous. Hence $\eta^{-1}(K)$ is compact as desired.

It remains to prove the claim. Let $y \in C$ be arbitrary, and let $V \in \mathcal{C}$ be the labeled region corresponding to y . By enlarging the complementary components of V with labels $1, \dots, n$ slightly, we can find a neighborhood N of y in C and closed Jordan regions $D'_i \subseteq \text{int}(D_i) \subseteq D_i \subseteq \mathbb{D}$ for $i = 1, \dots, n$ with the following properties: the regions $D_1, \dots, D_n$ are pairwise disjoint, and if $\widetilde{V} \in \mathcal{C}$ is any region corresponding to a point in N , then $\widehat{\partial}_i \widetilde{V} \subseteq D'_i$ and for all $i = 1, \dots, n$. In particular, if $\Omega = \mathbb{D} \setminus (D'_1 \cup \dots \cup D'_n)$, then $\partial D_1 \cup \dots \cup \partial D_n \subseteq \Omega \subseteq \widetilde{V}$.

Now let (x_k) be an arbitrary sequence in $\eta^{-1}(N)$, let $U_k \in \mathcal{S}$ be the labeled region corresponding to x_k and f_k be the map on $\overline{U}_k$ as in the definition of η for $k \in \mathbb{N}$. Then $g_k = f_k^{-1}$ is defined on $\Omega \cup \partial\mathbb{D}$ by choice of N .

Lemma 9.6 and Montel's theorem imply that by passing to a subsequence if necessary, we may assume that the sequence (g_k) converges to a map g uniformly on compact subsets of $\Omega \cup \partial\mathbb{D}$. The map g is a homeomorphism on $\Omega \cup \partial\mathbb{D}$, is conformal on Ω , and we have $g(\partial\mathbb{D}) = \partial\mathbb{D}$ and $g(1) = 1$. Let

E_i for $i = 1, \dots, n$ be the closed Jordan region in $\mathbb{D}$ bounded by the Jordan curve $g(\partial D_i) \subseteq \mathbb{D}$. Then the regions $E_1, \dots, E_n$ are pairwise disjoint.

Similarly as in the proof of Lemma 9.8 one can show that $\hat{\partial}_i U_k \subseteq E_i$ for all $i = 1, \dots, n$ and all large enough k . By Lemma 9.6 the sequence (x_k) subconverges to a point in $\mathcal{S}$. Hence $\eta^{-1}(N)$ is precompact. $\square$

Lemma 9.10 *Let A and B be (relatively) open and connected subsets of $\mathbb{C}^m \times [0, \infty)^n$, where $m, n \in \mathbb{N}$. Assume that*

$$A_0 = \{(p, 0) \in A : p \in \mathbb{C}^m, 0 \in \mathbb{R}^n\} = A \cap (\mathbb{C}^m \times \{(0, \dots, 0)\}) \neq \emptyset$$

and that $\eta: A \rightarrow B$ is a proper and continuous map satisfying the following conditions: for arbitrary $x = (p_1, \dots, p_m, r_1, \dots, r_n) \in A$ and $\eta(x) = (q_1, \dots, q_m, s_1, \dots, s_n) \in B$ we require that

- (i) *if $r_i = 0$ for some $i = 1, \dots, n$, then $s_i = 0$,*
- (ii) *if $r_1 = \dots = r_n = 0$, then $p_1 = q_1, \dots, p_m = q_m$, and $s_1 = \dots = s_n = 0$.*

Then the map η is surjective.

This is a special case of [30, 3.4 Degree Lemma, p. 407].

Lemma 9.11 *The map $\eta: S \rightarrow C$ is surjective.*

Proof If $n = 1$ this is clear, because then $S = C$ and η is the identity map. If $n \geq 2$, we apply Lemma 9.10 with $A = S$, $B = C$, and $m = n - 1 \geq 1$. Then obviously $A_0 \neq \emptyset$, as the points in this set correspond to the regions U in $\mathcal{S}$ with degenerate complementary components $\hat{\partial}_i U$, $i = 1, \dots, n$. The map η is continuous by Lemma 9.8 and proper by Lemma 9.9. Condition (i) in Lemma 9.10 follows from that fact that the map f as in the definition of η will send a degenerate complementary component of a region in $\mathcal{S}$ to a degenerate complementary components of a region in $\mathcal{C}$. Moreover, if all the complementary components except the one with label 0 are degenerate, then f is the identity map. This implies condition (ii) in Lemma 9.10. $\square$

Theorem 9.12 *Let $\Omega \subseteq \hat{\mathbb{C}}$ be a region with $n + 1 \geq 2$ complementary components, one of which is non-degenerate. Then there exists a conformal map of Ω onto a region U of the form*

$$U = \mathbb{D} \setminus (Q_1 \cup \dots \cup Q_n),$$

where $Q_1, \dots, Q_n$ are pairwise disjoint subsets of $\mathbb{D}$ such that $Q_1, \dots, Q_{n-1}$ are (possibly degenerate) $\mathbb{C}^$ -squares, and Q_n is a closed (possibly degenerate) Euclidean disk centered at 0.*

Proof Koebe's Uniformization Theorem implies that there is a conformal map of Ω onto a circle domain $V \in \mathcal{C}$. Since the map $\eta: S \rightarrow C$ introduced above is surjective, V , and hence also Ω , is conformally equivalent to a region $U \in \mathcal{S}$. $\square$

If all the complementary components of Ω are non-degenerate, the same is true for the region U in the previous theorem. Combining this with Lemma 9.1, we get the following statement.

Corollary 9.13 *Let $n \geq 1$, and suppose that $D_0, \dots, D_n$ are pairwise disjoint closed Jordan regions in $\widehat{\mathbb{C}}$. Then there exist a finite $\mathbb{C}^*$ -cylinder A , pairwise disjoint $\mathbb{C}^*$ -squares $Q_1, \dots, Q_{n-1} \subseteq A$, and a homeomorphism $f: \overline{\Omega} \rightarrow \overline{U}$, where*

$$\Omega = \widehat{\mathbb{C}} \setminus (D_0 \cup \dots \cup D_n) \quad \text{and} \quad U = A \setminus (Q_1 \cup \dots \cup Q_{n-1}),$$

that is conformal on Ω and maps ∂D_0 to $\partial_i A$ and ∂D_n to $\partial_o A$.

As we will see in Corollary 11.3, the map f in the previous statement is unique up to post-composition with a Euclidean similarity fixing the origin.

10 Proof of the main result

We start with a definition. A set $\Omega \subseteq \widehat{\mathbb{C}}$ is called λ -LLC for $\lambda \geq 1$ (LLC stands for *linearly locally connected*) if the following two conditions are satisfied:

- (λ -LLC₁): If $a \in \Omega$, $r > 0$, and $x, y \in \Omega \cap B(a, r)$, $x \neq y$, then there exists a continuum $E \subseteq \Omega \cap B(a, \lambda r)$ with $x, y \in E$.
- (λ -LLC₂): If $a \in \Omega$, $r > 0$, and $x, y \in \Omega \cap (\widehat{\mathbb{C}} \setminus B(a, r))$, $x \neq y$, then there exists a continuum $E \subseteq \Omega \cap (\widehat{\mathbb{C}} \setminus B(a, r/\lambda))$ with $x, y \in E$.

Lemma 10.1 *Every finitely connected circle domain $V \subseteq \widehat{\mathbb{C}}$ is 1-LLC.*

Proof Let V be as in the statement and $B \subseteq \widehat{\mathbb{C}}$ be an arbitrary open or closed disk. It suffices to show that $B \cap V$ is path-connected. Indeed, if this is true, then it follows that V is 1-LLC₁. Noting that the complement of an open disk in $\widehat{\mathbb{C}}$ (as in the LLC₂-condition) is a closed disk (centered at the point antipodal to the center of the original disk), we will also have that V is 1-LLC₂.

We first show that $B \cap \overline{V}$ is path-connected. Let $x, y \in B \cap \overline{V}$ be arbitrary. We can connect x and y by a path α in B . If D is one of the disks which form the complementary components of V , then $B \cap \partial D$ is a connected set (this is an elementary geometric fact where it is important that B and D are round disks). So if α meets $\text{int}(D)$, then we can replace a subpath of α by a path in

$B \cap \partial D$, so that the new path lies in B , connects x and y , but is disjoint from $\text{int}(D)$. By repeating this procedure for the other complementary components of V , we finally obtain a path β in B that connects x and y and avoids the interior of each complementary component of V . Then β is a path in $B \cap \overline{V}$ connecting x and y .

To show that $B \cap V$ is also path-connected, let again $x, y \in B \cap V$ be arbitrary. By slightly enlarging the radii of the complementary components of V , we can find a finitely connected circle domain $V' \subseteq \widehat{\mathbb{C}}$ with $x, y \in V'$ and $\overline{V'} \subseteq V$. Then by the first part of the proof there exists a path β in $B \cap \overline{V'} \subseteq B \cap V$ connecting x and y . The path-connectedness of $B \cap V$ follows. $\square$

A Schottky set is a set $T \subseteq \widehat{\mathbb{C}}$ that can be written as

$$T = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I} B_i,$$

where $\{B_i : i \in I\}$ is a collection of pairwise disjoint round open disks. One can define the notion of a Schottky set similarly for subsets of higher-dimensional spheres. This concept was introduced in [8] (with the additional requirement $\#I \geq 3$). By [8, Proposition 2.2] every Schottky set is 1-LLC (the condition $\#I \geq 3$ is irrelevant for this conclusion). Since the closure of every circle domain is a Schottky set, one can derive Lemma 10.1 from this result. We included a complete proof of this lemma for the convenience of the reader. See also the related Lemma 11.1 below.

The following theorem is the main ingredient in the proof of Theorem 1.1. It is of independent interest.

Theorem 10.2 *Let $U = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I} D_i$ be a finitely connected region whose complementary components D_i are closed Jordan regions that are s -relatively separated and whose boundaries ∂D_i are k -quasicircles for $i \in I$. Assume that $0, 1, \infty \in U$.*

If $f : U \rightarrow V$ is a conformal map of U onto a circle domain V with $f(0) = 0$, $f(1) = 1$, and $f(\infty) = \infty$, then f is η -quasisymmetric with η only depending on s and k .

Proof The map f extends uniquely to a homeomorphism between $\overline{U}$ and $\overline{V}$ (see the discussion after Lemma 9.1). Moreover, we can further extend this map (non-uniquely) to a homeomorphism on $\widehat{\mathbb{C}}$ (this follows from Remark 5.4 as in the proof of Lemma 5.5(iii)). We keep denoting this homeomorphism on $\widehat{\mathbb{C}}$ by f , and use a prime to denote image points under f , i.e., $a' = f(a)$ for $a \in \widehat{\mathbb{C}}$.

Since all subsets of $\mathbb{C}$ are N_0 -doubling with a universal constant N_0 , by Proposition 3.2 it suffices to show that $f|_U$ is H -weakly quasisymmetric with $H = H(s, k)$. So we have to show that there exists a constant

$H = H(s, k)$ with the following property: if $x, y, z \in U$ are arbitrary, then $\sigma(x, y) \leq \sigma(x, z)$ implies that $\sigma(x', y') \leq H\sigma(x', z')$.

Assume on the contrary that for some points $x, y, z \in U$ with $\sigma(x, y) \leq \sigma(x, z)$ we have $\sigma(x', y') > H\sigma(x', z')$ for some large $H \gg 1$. We will show that this leads to a contradiction if H is chosen large enough depending only on s and k .

Note that under our assumption the points x, y, z must be distinct. Since V is a finitely connected circle domain, this set 1-LLC by Lemma 10.1. So we can find a continuum $E' \subseteq V$ with $x', z' \in E'$ such that

$$\text{diam}(E') \leq 3\sigma(x', z').$$

The points $0, 1, \infty$ have mutual distance bounded below by $\sqrt{2} \geq 1$. So by Lemma 3.4 we can find a point $u = u' \in \{0, 1, \infty\}$ such that $\sigma(u, y) \geq 1/2$ and $\sigma(u', x') \geq 1/2$. Since $\sigma(x', y') \leq \text{diam}(\widehat{\mathbb{C}}) = 2$, we then have $\sigma(u', x') \geq \frac{1}{4}\sigma(x', y')$, and so $u' \notin B(x', \frac{1}{4}\sigma(x', y'))$. Again using the 1-LLC-property of V , this allows us to find a continuum $F' \subseteq V$ with $y', u' \in F'$ such that

$$F' \cap B\left(x', \frac{1}{4}\sigma(x', y')\right) = \emptyset.$$

Then assuming that $H \geq 24$ we have

$$\text{dist}(E', F') \geq \frac{1}{4}\sigma(x', y') - 3\sigma(x', z') \geq \frac{1}{8}\sigma(x', y') \geq \frac{H}{8}\sigma(x', z')$$

and

$$\text{diam}(E') \wedge \text{diam}(F') \leq 3\sigma(x', z').$$

Therefore,

$$\Delta(E', F') \geq \frac{H}{24}.$$

Let $K_i = f(D_i)$ for $i \in I$. Then $\{K_i : i \in I\}$ is the collection of complementary components of V . Since V is a circle domain, every set K_i is a closed round disk. Round disks are μ -fat in $(\widehat{\mathbb{C}}, \sigma, \Sigma)$ with $\mu = 1/4$ (see the discussion before Lemma 8.5). If $N = N(1/4)$ the corresponding integer as provided by Lemma 8.5 (N is a universal constant; as we have seen, one can take $N = 64$, or even $N = 2$), then Proposition 8.7 allows us the following conclusion. There exists a universal non-increasing function $\psi : (0, \infty) \rightarrow (0, \infty)$ such that $\lim_{t \rightarrow \infty} \psi(t) = 0$ with the following property: for some set $I_0 \subseteq I$ with $\#I_0 \leq N$ we have that

$$M_{\Omega', \mathcal{K}'}(\Gamma(E', F'; \Omega')) \leq \psi(\Delta(E', F')) \leq \psi(H/24),$$

where $\Omega' = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I_0} K_i$ and transboundary modulus is with respect to the collection $\mathcal{K}' = \{K_i : i \in I \setminus I_0\}$.

Define $E = f^{-1}(E')$ and $F = f^{-1}(F')$. Then E and F are continua in U containing the sets $\{x, z\}$ and $\{y, u\}$, respectively. Then

$$\text{diam}(F) \geq \sigma(y, u) \geq \frac{1}{2} \geq \frac{1}{4} \text{diam}(E),$$

and

$$\text{dist}(E, F) \leq \sigma(x, y) \leq \sigma(x, z) \leq \text{diam}(E) \leq 4(\text{diam}(E) \wedge \text{diam}(F)).$$

It follows that

$$\Delta(E, F) \leq 4. \quad (49)$$

Since $\#I_0$ can be bounded by the universal constant N , it follows from Proposition 7.5 that the region $\Omega = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I_0} D_i$ is ϕ -Loewner with ϕ only depending on s and k . Combining this with Proposition 4.3, Proposition 8.1, and (49), we see that there is a positive constant $m = m(s, k) > 0$ such that

$$M_{\Omega, \mathcal{K}}(\Gamma(E, F; \Omega)) \geq m,$$

where the transboundary modulus in Ω is with respect to the collection $\mathcal{K} = \{D_i : i \in I \setminus I_0\}$.

Now $f(\Gamma(E, F; \Omega)) = \Gamma(E', F'; \Omega')$ and f is conformal on the set $\Omega \setminus \bigcup_{i \in I \setminus I_0} D_i = U$. Hence

$$M_{\Omega, \mathcal{K}}(\Gamma(E, F; \Omega)) = M_{\Omega', \mathcal{K}'}(\Gamma(E', F'; \Omega'))$$

by invariance of transboundary modulus (see the discussion after Lemma 6.3) and our estimates give

$$m \leq \psi(H/24).$$

Since ψ is a fixed function with $\psi(t) \rightarrow 0$ as $t \rightarrow \infty$, this leads to a contradiction if H is larger than a constant depending on s and k . $\square$

Note that the homeomorphic extension $f: \overline{U} \rightarrow \overline{V}$ of the map in the previous theorem is also an η -quasisymmetry with the same function η as for the map $f|_U$. This follows from the distortion estimates for f on U by a simple limiting argument.

The previous proof is somewhat technical and it is worthwhile to summarize the main ideas of the argument. If the map f does not have the desired quasisymmetry property, then, as we have seen, one can find continua E and F in U with controlled relative distance such that the relative distance of the

image continua E' and F' in V is large. To get a contradiction one wants to consider a suitable family Γ of paths connecting E and F , and its image family Γ' . Since $\Delta(E, F) \lesssim 1$, one hopes to find Γ so that the modulus of this family is not too small, while $\Delta(E', F') \gg 1$ should imply that the modulus of Γ' is small. Conformal invariance of modulus will then give the desired contradiction.

The obvious first choice $\Gamma = \Gamma(E, F; U)$ cannot serve this purpose. Even though the complementary components of U are uniformly relatively separated, by restricting oneself to paths *in* U , it is possible to obtain a very sparse family whose modulus is not uniformly bounded below by a constant only depending on the relevant parameters s and k . Using this family $\Gamma(E, F; U)$ in combination with Proposition 7.5, one can actually show that f is η -quasisymmetric, where η will depend on s and k , but also on the number of complementary components of U (for which we have no control).

To get a larger path family one should allow the paths to run through the “holes” (i.e., the complementary components) of U and $\Gamma = \Gamma(E, F; \widehat{\mathbb{C}})$ seems like a better choice. It is clear that then one has to use transboundary modulus to get the necessary modulus invariance. Proposition 8.1 applied to the Loewner domain $\Omega = \widehat{\mathbb{C}}$ and the family of all complementary components of U , then actually gives a uniform lower bound for the transboundary boundary modulus of $\Gamma = \Gamma(E, F; \widehat{\mathbb{C}})$. Unfortunately, the corresponding transboundary modulus of the image family $\Gamma' = \Gamma(E', F'; \widehat{\mathbb{C}})$ need not be small due to possible complementary components of V that serve as “bridges” between E' and F' . As discussed after Proposition 8.7, one can remedy this problem by disallowing the paths to run through certain holes that have to be selected depending on E' and F' , but whose number is bounded by a universal constant N . Accordingly, in the previous proof we considered the family $\Gamma = \Gamma(E, F; \Omega)$, where $\Omega = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I_0} D_i$, and its image family Γ' . The paths in these families are allowed to pass through holes except through those labeled by $i \in I_0$. By Proposition 8.7 the family Γ' has small transboundary modulus. Even though we have no control which elements are in I_0 , we have a uniform upper bound $\#I_0 \leq N$. So Proposition 7.5 allows us to conclude that Ω is ϕ -Loewner with a function ϕ only depending on s and k (the number $n = \#I_0$ of complementary components of Ω does not enter as it is uniformly bounded). Together with Proposition 8.1 this leads to a lower bound for the transboundary modulus of Γ . The crucial point in this argument is that all quantities that are relevant in the upper and lower estimates can be controlled by the parameters s and k . Hence f will be an η -quasisymmetry with $\eta = \eta_{s,k}$.

Another subtlety in the previous proof is the initial choice of the continua E' and F' . For the family $\Gamma(E', F'; \Omega')$ to be defined, we need $E', F' \subseteq \overline{\Omega'}$. On the other hand, we do not know in advance which set $\Omega' = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I_0} K_i$ will be, because this region depends on the choice of I_0 . Hence we choose

E' and F' as subsets of V , because this set, and hence also E' and F' , are contained in *all* regions Ω' that can possibly appear.

Proof of Theorem 1.1 Suppose that $\mathcal{S} = \{S_i : i \in I\}$ is a collection of s -relatively separated k -quasicircles bounding pairwise disjoint closed Jordan regions D_i . Note that by the remark following Lemma 4.8 the regions $D_i, i \in I$, are also s -relatively separated.

It is clear that the index set I is at most countable. So if I is infinite, we may assume that $I = \mathbb{N}$. Let $T = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I} \text{int}(D_i)$. We first want to show there exists an η -quasi-Möbius map $f: T \rightarrow T'$ of T onto a Schottky set T' , i.e., T' is the complement of a collection of pairwise disjoint round open disks. Here we can choose f so that its distortion function η only depends on s and k .

The set T contains three distinct points that do not lie on any of the quasicircles $S_i = \partial D_i, i \in I$. This follows from Lemma 5.5(iv). Note that if $I = \mathbb{N}$ then we can apply this lemma, since $\text{diam}(D_i) \rightarrow 0$ as $i \rightarrow \infty$. This was shown in the first part of the proof of Proposition 5.1 and was derived from the fact that the sets D_i are λ -quasi-round with $\lambda = \lambda(k) \geq 1$ (Proposition 4.3).

If we apply any Möbius transformation to our collection $\mathcal{S}$, then the new collection will consist of s' -relatively separated k' -quasicircles, where s' only depends on s and k' only depends on k (Corollary 4.7). In this way we may reduce ourselves to the case where T contains the points $0, 1, \infty$ and none of these points lies on any quasicircle S_i .

If I is finite, then there exists a conformal map f of the finitely connected region $U = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I} D_i$ onto a circle domain V such that $f(0) = 0, f(1) = 1, f(\infty) = \infty$. By Theorem 10.2 the map f is η -quasisymmetric with η only depending on s and k . The map f extends uniquely to a homeomorphism of $\overline{U} = T$ onto the Schottky set $T' := \overline{V}$ (this was pointed out in the proof of Theorem 10.2), and the extended map f is an η -quasisymmetry on $\overline{U} = T$ (see the remark after the proof of Theorem 10.2).

If I is infinite, and so $I = \mathbb{N}$, then for each $n \in \mathbb{N}$ we consider the finitely connected region $U_n = \widehat{\mathbb{C}} \setminus \bigcup_{i=1}^n D_i$. Then $0, 1, \infty \in U_n$ for all $n \in \mathbb{N}$, and so again there exist an η -quasisymmetric map f_n of U_n onto the closure $\overline{V}_n$ of a circle domain V_n such that $f_n(0) = 0, f_n(1) = 1, f_n(\infty) = \infty$. Here η depends only on s and k , but not on n . Note that $\bigcap_{n \in \mathbb{N}} \overline{U}_n = T$. Since the maps $f_n|_T, n \in \mathbb{N}$, are normalized and η -quasisymmetric, the sequence (f_n) subconverges to an η -quasisymmetric embedding $f: T \rightarrow \widehat{\mathbb{C}}$, i.e., there exists a subsequence (f_{n_k}) of (f_n) that converges to f uniformly on T (Lemma 3.3).

We claim that $f(T)$ is a Schottky set. To see this note that if $i \in \mathbb{N}$ is arbitrary, then S_i is the boundary of the complementary component D_i of U_n for $n \geq i$. Hence $f_n(S_i)$ is a circle for $n \geq i$. Since $f_{n_k} \rightarrow f$ uniformly on T , it follows that the Jordan curve $f(S_i)$ is the Hausdorff limit of a sequence of circles. Therefore, $f(S_i)$ must be a circle itself. By Lemma 5.5(iii), the circles

$f(S_i)$, $i \in I$, bound pairwise disjoint closed disks D'_i such that $T' = f(T) = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I} D'_i$. Hence T' is a Schottky set.

So both when I is finite or infinite we showed that there exists an η -quasisymmetric map of T onto a Schottky set T' where $\eta = \eta_{s,k}$. Then f is $\tilde{\eta}$ -quasi-Möbius with $\tilde{\eta}$ only depending on η and hence only on s and k (Proposition 3.1(ii)). By Proposition 5.1 we can extend f to an H -quasiconformal homeomorphism on $\widehat{\mathbb{C}}$ with H only depending on $\tilde{\eta}$ and k and hence only on s and k . The map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is the desired quasiconformal map that sends each quasicircle S_i to a round circle. $\square$

The next example shows that we cannot omit the condition of uniform relative separation in Theorem 1.1.

Example 10.3 For a set $M \subseteq \mathbb{C}$ and $a \in \mathbb{C}$, $b > 0$, let $a + bM := \{a + bz : z \in M\}$. Define $Q = [-1, 1] \times [-1, 1] \subseteq \mathbb{R}^2 \cong \mathbb{C}$ and

$$Q_{2i-1} = 2^{-i} + 8^{-i}Q \quad \text{and} \quad Q_{2i} = 2^{-i} + (2 + 1/i)8^{-i} + 8^{-i}Q$$

for $i \in \mathbb{N}$. Then the sequence Q_i , $i \in \mathbb{N}$, consists of pairwise disjoint Euclidean squares lined up on the positive real axis. The main point is that for $i \rightarrow \infty$ the distance $(1/i)8^{-i}$ of Q_{2i-1} and Q_{2i} goes to 0 faster than the sidelength $2 \cdot 8^{-i}$ of these squares.

The sets $S_i = \partial Q_i$, $i \in I$, are uniform quasicircles. We claim that there is no quasiconformal map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ such that $f(S_i)$ is a round circle for each $i \in \mathbb{N}$. Indeed suppose there is such a map. By precomposing f by a Euclidean similarity that maps Q to Q_{2i-1} and post-composing f by a Möbius transformation we obtain a sequence of (f_i) of H -quasiconformal maps on $\widehat{\mathbb{C}}$ such that $f_i(\partial Q) = \partial \mathbb{D}$, $f_i(1) = 1$, $f_i(i) = i$, $f_i(-1) = -1$, and such that $f_i(\partial D_i)$ is a circle where $D_i = (2 + 1/i) + Q$ for $i \in \mathbb{N}$. Here H does not depend on i and so the sequence is *uniformly quasiconformal*. Hence (f_i) has a convergent subsequence that converges uniformly to a quasiconformal map g on $\widehat{\mathbb{C}}$ (a suitably normalized sequence of uniformly quasiconformal maps on $\widehat{\mathbb{C}}$ has subsequence that converges uniformly to a quasiconformal map as a sublimit; see [33, Sects. 19–21 and 37]. The existence of the quasiconformal sublimit g can also easily be derived from Proposition 3.1 and Lemma 3.3).

Then we have $g(\partial Q) = \partial \mathbb{D}$. Moreover, since $\partial D_i \rightarrow 2 + \partial Q$ in the Hausdorff sense and $f_i(D_i)$ is a circle for each $i \in \mathbb{N}$, the set $g(2 + \partial Q)$ is also a circle. Since the squares Q and $2 + Q$ share a common side, and $g(\partial Q) = \partial \mathbb{D}$, we conclude $g(2 + \partial Q) = \partial \mathbb{D}$. This is impossible, since g is a homeomorphism and so two distinct sets in $\widehat{\mathbb{C}}$ cannot have the same image.

Finally, we give an example showing that in Theorem 1.1 one cannot drop the assumption that the quasicircles bound pairwise disjoint Jordan regions.

Example 10.4 We define a collection S_i , $i \in \mathbb{N}_0$, of uniformly relatively separated uniform quasicircles as follows. For S_0 we pick any quasicircle in $\widehat{\mathbb{C}}$ that is not a round circle; to be specific, let $S_0 = \partial Q$, where $Q = [-1, 1] \times [-1, 1] \subseteq \mathbb{R}^2 \cong \mathbb{C}$. All other curves S_i , $i \in \mathbb{N}$, will be round circles that bound closed disks D_i that are pairwise disjoint and disjoint from S_0 . We can choose the circles S_i , $i \in \mathbb{N}$, such that the family S_i , $i \in \mathbb{N}_0$ (including S_0), is uniformly relatively separated and such that the set $T = \widehat{\mathbb{C}} \setminus \bigcup_{i \in \mathbb{N}} \text{int}(D_i)$ has spherical measure zero. One can obtain such disks D_i and circles $S_i = \partial D_i$ by a procedure that successively scoops out disks from the two complementary components of S_0 . This is essentially identical to the construction in the proof of Theorem 1.3 in [8, p. 435], so we omit the details.

Now suppose that there was a quasiconformal (and hence quasiasymmetric) map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ such that $f(S_i)$ is a round circle for each $i \in \mathbb{N}_0$. Then $f|T$ is a quasiasymmetric map of the Schottky set T onto the Schottky set

$$T' = \widehat{\mathbb{C}} \setminus \bigcup_{i \in \mathbb{N}} \text{int}(f(D_i)).$$

Since T has spherical measure zero, the map $f|T$ is identical to the restriction of a Möbius transformation [8, Theorem 1.1]. Since $S_0 \subseteq T$ and $f(S_0)$ is a round circle, this implies that S_0 must be a round circle itself. This is a contradiction showing that a map f as stipulated does not exist.

11 Extremal metrics for transboundary modulus

In this section we solve an extremal problem for transboundary modulus and prove a related variant of Theorem 1.1. We employ the terminology for sets in the cylinder $\mathbb{C}^* \cong Z$ introduced in the beginning of Sect. 9. We denote by $d_{\mathbb{C}^*}$ the flat metric on $\mathbb{C}^*$ induced by the length element

$$ds_{\mathbb{C}^*} = \frac{|dz|}{|z|},$$

and by $A_{\mathbb{C}^*}$ the corresponding measure on $\mathbb{C}^*$ induced by the volume element

$$dA_{\mathbb{C}^*}(z) = \frac{dm_2(z)}{|z|^2}.$$

Here m_2 is 2-dimensional Lebesgue measure. Note that if Q is a $\mathbb{C}^*$ -square with sidelength $\ell(Q)$, then $A_{\mathbb{C}^*}(Q) = \ell(Q)^2$. The length of a locally rectifiable path α in $\mathbb{C}^*$ with respect to the metric $d_{\mathbb{C}^*}$ is given by

$$\text{length}_{\mathbb{C}^*}(\alpha) := \int_{\alpha} \frac{|dz|}{|z|}.$$

For $z_0 \in \mathbb{C}^*$ and $r > 0$ we define

$$B_{\mathbb{C}^*}(z_0, r) = \{z \in \mathbb{C}^* : d_{\mathbb{C}^*}(z_0, z) < r\}.$$

Recall that the height h_A of a finite $\mathbb{C}^*$ -cylinder $A = \{z \in \mathbb{C} : r < |z| < R\}$ is given by $h_A = \log(R/r)$.

To motivate our next result we will discuss some background on extremal problems for classical modulus (for more details see [23, Chap. I], for example). Suppose $Q \subseteq \mathbb{C}$ is a quadrilateral, i.e., a closed Jordan region with four distinguished points on its boundary. These points divide ∂Q into four arcs. Let E and F be two of these arcs that are “opposite to each other” on ∂Q (i.e., non-adjacent and separated by the other two arcs) and consider the path family $\Gamma = \Gamma(E, F; \text{int}(Q))$ of all paths in $\text{int}(Q)$ connecting E and F . It is well-known how to compute $\text{mod}(\Gamma)$ (at least in principle); namely, map Q by a conformal map to a Euclidean rectangle such that the vertices of Q and R correspond to each other under the map. If E and F corresponds to sides of R with length a , and the other two arcs on ∂Q to sides of R with length b , then

$$\text{mod}(\Gamma) = a/b.$$

Moreover, if f is the conformal map of Q onto R , then the unique (up to changes on sets of measure zero) extremal density ρ for $\text{mod}(\Gamma)$ is $\rho(z) = |f'(z)|/b$ if we use the Euclidean metric as base metric on Q . This easily follows from conformal invariance of modulus, and the fact that if $Q = R$, then $\rho \equiv 1/b$ is the extremal density. The main point here is that rectangles are “extremal regions” for this type of modulus problem.

To give another example, suppose $D_0, D_1 \subseteq \widehat{\mathbb{C}}$ are disjoint closed Jordan regions. Consider the (topological) annulus $V = \widehat{\mathbb{C}} \setminus (D_0 \cup D_1)$, and the family $\Gamma = \Gamma(\partial D_0, \partial D_1; V)$ of all paths in V connecting the boundary components ∂D_0 and ∂D_1 of the annulus. In this case the extremal regions for $\text{mod}(\Gamma)$ are finite $\mathbb{C}^*$ -cylinders. One can find a $\mathbb{C}^*$ -cylinder A that is conformally equivalent to V . Then

$$\text{mod}(\Gamma) = \frac{2\pi}{h_A}.$$

Moreover, the essentially unique extremal density for $\text{mod}(\Gamma(\partial_i A, \partial_o A; A))$ is $\rho \equiv 1/h_A$ (with the flat metric on $\mathbb{C}^*$ as base metric), and using a conformal map between V and A one can easily identify the extremal density for $\text{mod}(\Gamma)$.

In this section we are interested in similar results for transboundary modulus. Suppose $D_0, \dots, D_{n+1} \subseteq \widehat{\mathbb{C}}$ are pairwise disjoint closed Jordan regions, and $V = \widehat{\mathbb{C}} \setminus (D_0 \cup D_{n+1})$. We consider the transboundary modulus $M_{V, \mathcal{K}}(\Gamma)$, where $\mathcal{K} = \{D_1, \dots, D_n\}$ and Γ is the family of all paths in V connecting ∂D_0

and ∂D_{n+1} . So the paths have their endpoints on ∂D_0 and ∂D_{n+1} , but they do not meet D_0 and D_{n+1} otherwise, and they may pass through all the other Jordan regions $D_1, \dots, D_n \subseteq V$. For the transboundary mass distribution we may put weights on the elements in $\mathcal{K}$, i.e., on $D_1, \dots, D_n$, but not on D_0 and D_{n+1} . The density of the transboundary mass distribution will be defined on

$$\Omega = V \setminus (D_1 \cup \dots \cup D_n) = \widehat{\mathbb{C}} \setminus (D_0 \cup \dots \cup D_{n+1}). \quad (50)$$

As we will see (cf. the discussion after the proof of Proposition 11.2) the extremal region (corresponding to Ω) is of the form

$$U = A \setminus (Q_1 \cup \dots \cup Q_n), \quad (51)$$

where A is a finite $\mathbb{C}^*$ -cylinder and $Q_1, \dots, Q_n$ are pairwise disjoint $\mathbb{C}^*$ -squares in A .

We first require a lemma.

Lemma 11.1 *Let A be a finite $\mathbb{C}^*$ -cylinder, and $Q_1, \dots, Q_n$ be pairwise disjoint $\mathbb{C}^*$ -squares in A . Then any two points*

$$x, y \in T := \overline{A} \setminus \bigcup_{i=1}^n \text{int}(Q_i)$$

can be joined by a rectifiable path β in T with

$$\text{length}_{\mathbb{C}^*}(\beta) \leq 2d_{\mathbb{C}^*}(x, y).$$

Suppose in addition that

$$\ell(Q_i) \leq 2\pi - \epsilon_0 \quad \text{for } i = 1, \dots, n,$$

where $0 < \epsilon_0 < 2\pi$. Then for all $z_0 \in T$ and all $r_0 \in (0, \epsilon_0/2]$ the set $T \setminus B_{\mathbb{C}^}(z_0, r_0)$ is path-connected.*

Proof Let $x, y \in T$ be arbitrary. We connect x and y by a geodesic segment α in $\overline{A}$ with respect to the metric $d_{\mathbb{C}^*}$. The path α is contained in T except for finitely many pairwise disjoint (open) subarcs γ of α , where each subarc γ is contained in one of the $\mathbb{C}^*$ -squares Q_i and has its endpoints in ∂Q_i . An elementary geometric argument shows that one of the two subarcs of ∂Q_i with the same endpoints as γ has length bounded by $2\text{length}_{\mathbb{C}^*}(\gamma)$. If we replace each subarc γ by such an arc in the boundary of one of the $\mathbb{C}^*$ -squares, then we obtain a path β in T with endpoints x and y , and with $\text{length}_{\mathbb{C}^*}(\beta) \leq 2\text{length}_{\mathbb{C}^*}(\alpha) = 2d_{\mathbb{C}^*}(x, y)$.

For the proof of the second part of the lemma we may assume that z_0 lies on the positive real axis. Since $r_0 < \pi$, the disk $B_0 := B_{\mathbb{C}^*}(z_0, r_0)$ does not

contain any point on the negative real axis. For the path-connectedness of $T \setminus B_0$ it suffices to show that by paths in $T \setminus B_0$ we can connect any point $x \in T \setminus B_0$ to a point $x' \in T \setminus B_0$ on the negative real axis, and that we can connect any two points $u, v \in T \setminus B_0$ on the negative real axis.

To prove the first statement, we may assume that $x = \rho e^{i\theta} \in T \setminus B_0$, where $\rho \geq |z_0|$ and $0 \leq \theta \leq \pi$. We then connect x to the negative real axis by a “zig-zag” path α in $T \setminus B_0$ as follows. We start at x and move counter-clockwise along the circle $\{z \in \mathbb{C} : |z| = \rho\}$ towards the negative real axis as long as we remain in T . If we hit one of the $\mathbb{C}^*$ -squares Q_i , then we run radially along the boundary $\partial Q_i \subseteq T$ with increasing Euclidean distance to 0 until we hit a “corner” v of Q_i . Then we turn and continue counter-clockwise on the circle $\{z \in \mathbb{C} : |z| = |v|\}$ towards the negative real axis, etc. It is clear that in this way we obtain a path α in T that connects x to a point $x' \in T$ on the negative real axis. Moreover, α has a parametrization of the form $t \mapsto \alpha(t) = r(t)e^{i\varphi(t)}$, where r is a non-decreasing function with values $\geq \rho$ and $\varphi(t)$ a non-decreasing function with values in the interval $[\theta, \pi]$. Since $\rho \geq |z_0|$ and $\theta \in [0, \pi]$, this implies that for each point $p \in \alpha$ we have

$$d_{\mathbb{C}^*}(p, z_0) \geq d_{\mathbb{C}^*}(\rho e^{i\theta}, z_0) = d_{\mathbb{C}^*}(x, z_0) \geq r_0.$$

Hence $\alpha \cap B_0 = \emptyset$, and so $\alpha \subseteq T \setminus B_0$ as desired.

For the proof of the second statement that points in $T \setminus B_0$ on the negative real axis can be connected by a suitable path in $T \setminus B_0$, we start with the following observation. Suppose one of the $\mathbb{C}^*$ -squares Q_i meets the negative real axis and suppose γ is the maximal subinterval of the negative real axis contained in Q_i . Note that the endpoints of γ lie in ∂Q_i . Then one of the two subarcs of ∂Q_i with the same endpoints as γ is disjoint from B_0 . This easily follows from the fact that if $\varphi \leq \pi \leq \psi$ and $(\psi - \varphi) + \epsilon_0 \leq 2\pi$, then one of the sets

$$S_1 = \{re^{it} : r > 0 \text{ and } \varphi \leq t \leq \pi\}$$

and

$$S_2 = \{re^{it} : r > 0 \text{ and } \pi \leq t \leq \psi\}$$

is disjoint from the set

$$S = \{re^{it} : r > 0 \text{ and } -\epsilon_0/2 < t < \epsilon_0/2\} \supseteq B_0.$$

Note that this applies in our situation by our assumption that $\ell(Q_i) + \epsilon_0 \leq 2\pi$ for each $i = 1, \dots, n$.

Now if $u, v \in T \setminus B_0$ are two points on the negative real axis, then we can find a path β in $T \setminus B_0$ connecting these points as follows. Let α be the subinterval of the negative real axis with endpoints u and v . Then $\alpha \cap B_0 = \emptyset$. The interval α is contained in T except for finitely many pairwise disjoint

(open) subintervals γ of α , where each subinterval γ is contained in one of the $\mathbb{C}^*$ -squares Q_i and has its endpoints in ∂Q_i . By what we have seen above, we can replace each such arc γ by a subarc of $\partial Q_i \subseteq T$ that is disjoint from B_0 and has the same endpoints as γ . In this way we obtain the desired path β in $T \setminus B_0$ connecting u and v . $\square$

Proposition 11.2 *Let A be a finite $\mathbb{C}^*$ -cylinder, and $\mathcal{K} = \{Q_i : i = 1, \dots, n\}$ be a finite (possibly empty) family of pairwise disjoint (possibly degenerate) $\mathbb{C}^*$ -squares in A . Define $\Gamma = \Gamma(\partial_i A, \partial_o A; A)$ and $K = Q_1 \cup \dots \cup Q_n$. Then*

$$M_{A, \mathcal{K}}(\Gamma) = \frac{2\pi}{h_A}.$$

Moreover, the essentially unique extremal admissible transboundary mass distribution for Γ consisting of a Borel function ρ on $A \setminus K$, and discrete weights $\rho_i \geq 0$ for $i = 1, \dots, n$ such that

$$\int_{A \setminus K} \rho^2 dA_{\mathbb{C}^*} + \sum_{i=1}^n \rho_i^2 = M_{A, \mathcal{K}}(\Gamma) = \frac{2\pi}{h_A}, \quad (52)$$

is given by $\rho(z) = 1/h_A$ for $z \in A \setminus K$, and $\rho_i = \ell(Q_i)/h_A$ for $i = 1, \dots, n$.

The underlying base metric here (see Remark 6.4) is the flat metric on $\mathbb{C}^*$. Essential uniqueness means that if we have another admissible transboundary mass distribution for Γ with (52), then $\rho(z) = 1/h_A$ for *almost every* $z \in A \setminus K$, and $\rho_i = \ell(Q_i)/h_A$ for $i = 1, \dots, n$.

Proof Suppose that $A = \{z \in \mathbb{C} : r < |z| < R\}$, where $0 < r < R$. Then $h_A = \log(R/r)$. Let $\rho(z) = 1/h_A$ for $z \in A \setminus K$ and $\rho_i = \ell(Q_i)/h_A$ for $i = 1, \dots, n$. We claim that this transboundary mass distribution is admissible for the family Γ . Let $\gamma \in \Gamma$ be an arbitrary path that is locally rectifiable in $A \setminus K$. We may assume that γ is parametrized by the interval $[0, 1]$ and that $\gamma(0) \in \partial A_i$ and $\gamma(1) \in \partial A_o$. By definition of Γ we have $\gamma((0, 1)) \subseteq A$.

Let $\pi : \bar{A} \rightarrow [\log r, \log R]$ be the map $z \mapsto \pi(z) := \log |z|$. Then

$$(\log r, \log R) \subseteq \pi(\gamma \cap (A \setminus K)) \cup \bigcup_{\gamma \cap Q_i \neq \emptyset} \pi(\gamma \cap Q_i). \quad (53)$$

Note that (37) implies that

$$\int_{\gamma \cap (A \setminus K)} \rho ds_{\mathbb{C}^*} = \frac{1}{h_A} \int_{\gamma \cap (A \setminus K)} \frac{|dz|}{|z|} \geq \frac{1}{h_A} m_1(\pi(\gamma \cap (A \setminus K))).$$

We also have $\rho_i = \frac{1}{h_A} m_1(\pi(Q_i))$ for $i = 1, \dots, n$, and so by (53) we obtain

$$\begin{aligned} & \int_{\gamma \cap (A \setminus K)} \rho \, ds_{\mathbb{C}^*} + \sum_{\gamma \cap Q_i \neq \emptyset} \rho_i \\ & \geq \frac{1}{h_A} m_1(\pi(\gamma \cap (A \setminus K))) + \frac{1}{h_A} \sum_{\gamma \cap Q_i \neq \emptyset} m_1(Q_i) \\ & \geq \frac{1}{h_A} m_1((\log r, \log R)) = 1. \end{aligned}$$

The admissibility of our transboundary mass distribution follows.

We conclude that

$$\begin{aligned} M_{A, \mathcal{K}}(\Gamma) & \leq \int_{A \setminus K} \rho^2 \, dA_{\mathbb{C}^*} + \sum_{i=1}^n \rho_i^2 \\ & = \frac{1}{h_A^2} A_{\mathbb{C}^*}(A \setminus K) + \frac{1}{h_A^2} \sum_{i=1}^n A_{\mathbb{C}^*}(Q_i) \\ & = \frac{1}{h_A^2} A_{\mathbb{C}^*}(A) = \frac{2\pi}{h_A}. \end{aligned}$$

To get an inequality in the other direction, suppose that we have an admissible transboundary mass distribution for the family Γ consisting of a density ρ on $A \setminus K$, and discrete weights $\rho_i \geq 0$ for $i = 1, \dots, n$. For each $\varphi \in [0, \pi]$ the path $\alpha_\varphi: [\log r, \log R] \rightarrow \bar{A}$ defined by $\alpha_\varphi(t) := te^{i\varphi}$ for $t \in [\log r, \log R]$ belongs to Γ . Hence for each $\varphi \in [0, 2\pi]$ we have

$$\int_{\alpha_\varphi \cap (A \setminus K)} \rho \, ds_{\mathbb{C}^*} + \sum_{\alpha_\varphi \cap Q_i \neq \emptyset} \rho_i \geq 1.$$

Integrating this over φ , using Fubini's theorem, and the Cauchy-Schwarz inequality, we arrive at

$$\begin{aligned} 2\pi & \leq \int_{A \setminus K} \rho \, dA_{\mathbb{C}^*} + \sum_{i=1}^n \ell(Q_i) \rho_i \\ & \leq A_{\mathbb{C}^*}(A \setminus K)^{1/2} \left(\int_{A \setminus K} \rho^2 \, dA_{\mathbb{C}^*} \right)^{1/2} + \sum_{i=1}^n \ell(Q_i) \rho_i \end{aligned} \quad (54)$$

$$\leq \left(A_{\mathbb{C}^*}(A \setminus K) + \sum_{i=1}^n \ell(Q_i)^2 \right)^{1/2} \left(\int_{A \setminus K} \rho^2 \, dA_{\mathbb{C}^*} + \sum_{i=1}^n \rho_i^2 \right)^{1/2} \quad (55)$$

$$= (2\pi h_A)^{1/2} \left(\int_{A \setminus K} \rho^2 dA_{\mathbb{C}^*} + \sum_{i=1}^n \rho_i^2 \right)^{1/2}.$$

Hence

$$\int_{A \setminus K} \rho^2 dA_{\mathbb{C}^*} + \sum_{i=1}^n \rho_i^2 \geq \frac{2\pi}{h_A}$$

for every transboundary mass distribution that is admissible for Γ . This implies the other desired inequality $M_{A, \mathcal{K}}(\Gamma) \geq 2\pi/h_A$.

If we have an admissible transboundary mass distribution satisfying (52), then we must have equality in (54) and (55). Equality in (55) implies that there exists $\lambda > 0$ such that $\int_{A \setminus K} \rho^2 dA_{\mathbb{C}^*} = \lambda^2 A_{\mathbb{C}^*}(A \setminus K)$ and $\rho_i = \lambda \ell(Q_i)$ for $i = 1, \dots, n$. Hence $\lambda = 1/h_A$ by (52), and so $\int_{A \setminus K} \rho^2 dA_{\mathbb{C}^*} = A_{\mathbb{C}^*}(A \setminus K)/h_A^2$ and $\rho_i = \ell(Q_i)/h_A$ for $i = 1, \dots, n$. This and equality in (54) give

$$\int_{A \setminus K} \rho^2 dA_{\mathbb{C}^*} = \frac{1}{h_A^2} A_{\mathbb{C}^*}(A \setminus K) = \frac{1}{h_A} \int_{A \setminus K} \rho dA_{\mathbb{C}^*},$$

and so $\rho = 1/h_A$ almost everywhere on $A \setminus K$. $\square$

A general criterion for a density to be extremal for the modulus of a given path family is due to Beurling (see [2, Theorem 4.4, p. 61]). It is easy to extend this condition to a criterion for the extremality of a transboundary mass distribution. Based on this one can give a proof of Proposition 11.2 that is slightly more streamlined (but uses essentially the same ideas).

Combining Proposition 11.2 with Corollary 9.13 and invariance of transboundary modulus, one can immediately give a solution to the problem discussed in the beginning of this section. If the setup is as before Lemma 11.1, then we map the region Ω in (50) to a region of the form U as in (51) by a conformal map f . The map f has a unique extension to a homeomorphism from $\overline{\Omega}$ onto $\overline{U}$, and a further (non-unique) extension as a homeomorphism on $\widehat{\mathbb{C}}$. We assume that $f(\partial D_0) = \partial_i A$, $f(\partial D_{n+1}) = \partial_o A$, and that the labeling of the other complementary components is such that $f(D_i) = Q_i$ for $i = 1, \dots, n$. Then $f(V) = A$ and $f(\Gamma) = \Gamma(\partial_i A, \partial_o A; A)$. Hence

$$M_{V, \mathcal{K}}(\Gamma) = \frac{2\pi}{h_A}.$$

Moreover, based on the last part of Proposition 11.2 one can easily identify the essentially unique extremal transboundary mass distribution for $M_{V, \mathcal{K}}(\Gamma)$ (we leave this to the reader).

We record another application of Proposition 11.2.

Corollary 11.3 *The map f in Corollary 9.13 is unique up to a post-composition by a map of the form $z \mapsto az$, $a \in \mathbb{C}^*$.*

Proof Let $n \in \mathbb{N}_0$, A and A' be finite cylinders, $Q_1, \dots, Q_n$ pairwise disjoint $\mathbb{C}^*$ -squares in A , and $Q'_1, \dots, Q'_n$ pairwise disjoint $\mathbb{C}^*$ -squares in A' . Let $U = A \setminus (Q_1 \cup \dots \cup Q_n)$, $V = A' \setminus (Q'_1 \cup \dots \cup Q'_n)$, and suppose that $g: \overline{U} \rightarrow \overline{V}$ is a homeomorphism that is a conformal map on U with $g(U) = V$, and satisfies $g(\partial_i A) = \partial_i A'$ and $f(\partial_o A) = \partial_o A'$. It suffices to show that there exists $a \in \mathbb{C}^*$ such that $g(z) = az$ for all $z \in U$. We extend g (non-uniquely) to a homeomorphism from $\overline{A}$ onto $\overline{A'}$, which we also denote by g .

Let $\Gamma = \Gamma(\partial_i A, \partial_o A; A)$, and $\Gamma' = \Gamma(\partial_i A', \partial_o A'; A')$. Then $\Gamma' = g(\Gamma)$. By invariance of transboundary modulus and Proposition 11.2 we have

$$\frac{2\pi}{h_A} = M_{A, \mathcal{K}}(\Gamma) = M_{A', \mathcal{K}'}(\Gamma') = \frac{2\pi}{h_{A'}},$$

where $\mathcal{K} = \{Q_1, \dots, Q_n\}$ and $\mathcal{K}' = \{Q'_1, \dots, Q'_n\}$.

As we have seen in the proof of Proposition 11.2, the transboundary mass distribution consisting of the density $\rho' = 1/h_{A'}$ on V and the weights $\rho'_i = \ell(Q'_i)/h_{A'}$ is admissible for the modulus $M_{A', \mathcal{K}'}(\Gamma')$ and has minimal total mass. As in the proof of Lemma 6.3 (using the flat metric $d_{\mathbb{C}^*}$ on $\mathbb{C}^*$ instead of the spherical metric) one sees that the transboundary mass distribution consisting of the density

$$\rho(z) = \frac{|zg'(z)|}{h_{A'}|g(z)|} \quad \text{for } z \in U$$

and the weights $\rho_i = \ell(Q'_i)/h_{A'}$ is admissible for the modulus $M_{A, \mathcal{K}}(\Gamma)$. Since $M_{A, \mathcal{K}}(\Gamma) = M_{A', \mathcal{K}'}(\Gamma')$ by invariance of transboundary modulus, this implies that this transboundary mass distribution is also extremal for the modulus $M_{A, \mathcal{K}}(\Gamma)$. The uniqueness statement in Proposition 11.2 implies that $z \mapsto |zg'(z)|/|g(z)|$ is a constant function on U . Hence the function $z \mapsto zg'(z)/g(z)$ is also constant on U , say $zg'(z)/g(z) \equiv c$ on U , where $c \in \mathbb{C}$. Suppose that $\partial_i A = \{z \in \mathbb{C} : |z| = r\}$, where $r > 0$. Since g maps the circle $\partial_i A$ to the circle $\partial_i A'$, the map g has an analytic extension to a neighborhood of $\partial_i A$ by the Schwarz reflection principle and it follows that $zg'(z)/g(z) = c$ for $z \in \partial_i A$. Let α be the path $t \in [0, 2\pi] \mapsto \alpha(t) := re^{it}$. Then we have

$$\frac{1}{2\pi i} \int_{g \circ \alpha} \frac{dw}{w} = \frac{1}{2\pi i} \int_{\alpha} \frac{g'(z)}{g(z)} dz = \frac{c}{2\pi i} \int_{\alpha} \frac{dz}{z} = c.$$

On the other hand, the expression of the left-hand side represents the winding number of the path $g \circ \alpha$ around 0. Note that $g \circ \alpha$ is a parametrization of the

circle $\partial_i A'$, the map $g|_{\partial_i A}$ is injective, and 0 lies “on the left” of the oriented path $g \circ \alpha$ since g is orientation-preserving. Thus, this winding number is equal to 1 and so $c = 1$. This implies that the function $z \mapsto g(z)/z$ has vanishing derivative on U , and so there exists a constant $a \in \mathbb{C}^*$ with $g(z) = az$ for $z \in U$ as desired. $\square$

Lemma 11.4 *In Corollary 9.13 suppose in addition that the Jordan curves $\partial D_0, \dots, \partial D_n$ are s -relatively separated k -quasicircles, and that*

$$\text{diam}(\partial D_0) \wedge \text{diam}(\partial D_n) \geq d > 0.$$

Then there exist constants $C_1 = C_1(s, k) > 0$, $C_2 = C_2(s, k, d) > 0$, and $\epsilon_0 = \epsilon_0(s, k, d) \in (0, 2\pi)$ such that

$$C_1 \leq h_A \leq C_2, \quad (56)$$

and

$$\ell(Q_i) \leq 2\pi - \epsilon_0 \quad \text{for all } i = 1, \dots, n-1. \quad (57)$$

Proof Let $V = \widehat{\mathbb{C}} \setminus (D_0 \cup D_n)$, $\mathcal{K} = \{D_i : i = 1, \dots, n-1\}$, and $\Gamma = \Gamma(\partial D_0, \partial D_n; V)$. We can extend the map f in Corollary 9.13 (non-uniquely) to a homeomorphism from $\overline{V}$ onto $\overline{A}$. By the properties of the map f we then have

$$f(\Gamma) = \Gamma(\partial_i A, \partial_o A; A).$$

Hence by invariance of transboundary modulus and Proposition 11.2 we get

$$M_{V, \mathcal{K}}(\Gamma) = 2\pi / h_A.$$

This shows that in order to establish inequality (56), it suffices to show that $M_{V, \mathcal{K}}(\Gamma)$ is bounded below by a positive constant only depending on s , k and d , and bounded above by a constant only depending on s and k .

To produce the first bound note that by Lemma 4.8 the regions $D_0, \dots, D_n$ are also s -relatively separated. So by Proposition 7.5 the region $V = \widehat{\mathbb{C}} \setminus (D_0 \cup D_n)$ is ϕ -Loewner, where $\phi = \phi_{s, k}$. Moreover, for the continua ∂D_0 and ∂D_n we have

$$\Delta(\partial D_0, \partial D_n) \leq 2/d.$$

Since the continua in $\mathcal{K}$ are s -relatively separated, and also λ -quasi-round with $\lambda = \lambda(k)$ by Proposition 4.3, it follows from Proposition 8.1 that $M_{V, \mathcal{K}}(\Gamma) \geq C(s, k, d) > 0$ as desired.

To produce an inequality in the opposite direction, note that

$$\Delta(\partial D_0, \partial D_n) \geq s,$$

since ∂D_0 and ∂D_n are s -relatively separated. Hence by Proposition 8.4,

$$M_{V,\mathcal{K}}(\Gamma) \leq C(s, k).$$

The first part of the theorem follows.

To prove the second part of the proposition consider one of the $\mathbb{C}^*$ -squares $Q_1, \dots, Q_{n-1}$, say Q_1 . Under the map f it corresponds to one of the Jordan regions $D_1, \dots, D_{n-1}$, say to D_1 . Let $V' = \widehat{\mathbb{C}} \setminus (D_0 \cup D_1 \cup D_n)$. Then again by Proposition 7.5 the region V' is ϕ -Loewner with $\phi = \phi_{s,k}$. We can again invoke Proposition 8.1 and the invariance of transboundary modulus to conclude that

$$M_{U,\mathcal{L}}(\Gamma(\partial_i A, \partial_o A; U)) = M_{V',\mathcal{K}'}(\Gamma(\partial D_0, \partial D_n; V')) \geq C(s, k, d) > 0.$$

Here $U = A \setminus Q_1$, $\mathcal{L} = \{Q_2, \dots, Q_{n-1}\}$, and $\mathcal{K}' = \{D_2, \dots, D_{n-1}\}$.

On the other hand, suppose that $A = \{z \in \mathbb{C} : r < |z| < R\}$. Without loss of generality we may assume that

$$Q_1 = \{se^{it} : r' \leq s \leq R', t \in [\alpha, 2\pi - \alpha]\},$$

where $r < r' < R' < R$ and $\alpha \in (0, \pi)$. Then $\ell(Q_1) = 2(\pi - \alpha) = \log(R'/r')$. We have to show that α cannot be smaller than a positive constant only depending on s, k , and d .

Note that every path $\gamma \in \Gamma = \Gamma(\partial_i A, \partial_o A; U)$ lies in the complement of Q_1 and meets both circles $\{z \in \mathbb{C} : |z| = r'\}$ and $\{z \in \mathbb{C} : |z| = R'\}$. Hence γ passes through the channel

$$M = \{se^{it} : r' < s < R', t \in (-\alpha, \alpha)\}$$

meeting “bottom” and “top”. We use this fact to produce a transboundary mass distribution for $M_{U,\mathcal{L}}(\Gamma(\partial_i A, \partial_o A; U))$ that has small mass if α is small.

We use the flat metric on $\mathbb{C}^*$ as base metric and set

$$\rho(u) = 1/\ell(Q_1) \quad \text{for } u \in M \cap U',$$

and $\rho = 0$ elsewhere on U' , where

$$U' = U \setminus (Q_2 \cup \dots \cup Q_{n-1}) = A \setminus (Q_1 \cup \dots \cup Q_{n-1}).$$

Moreover, for $i \in \{2, \dots, n-1\}$ we set

$$\rho_i = \ell(Q_i)/\ell(Q_1) \quad \text{if } Q_i \cap M \neq \emptyset$$

and $\rho_i = 0$ otherwise. By considerations very similar to the ones in the proof of Proposition 11.2 one can show that this transboundary mass distribution is admissible for Γ .

A $\mathbb{C}^*$ -square Q that meets M and is disjoint from Q_1 must satisfy $\ell(Q) < 2\alpha$. This implies

$$Q \subseteq \tilde{M} := \{se^{it} : r'e^{-2\alpha} < s < R'e^{2\alpha}, -\alpha < t < \alpha\}.$$

Hence

$$\begin{aligned} \int_{U'} \rho^2 dA_{\mathbb{C}^*} + \sum_{i=2}^{n-1} \rho_i^2 &\leq \frac{1}{\ell(Q_1)^2} \left(A_{\mathbb{C}^*}(M \cap U') + \sum_{Q_i \cap M \neq \emptyset} A_{\mathbb{C}^*}(Q_i) \right) \\ &\leq \frac{1}{\ell(Q_1)^2} A_{\mathbb{C}^*}(\tilde{M}) = \frac{\alpha(\pi + \alpha)}{(\pi - \alpha)^2}, \end{aligned}$$

and so

$$0 < C(s, k, d) \leq M_{U, \mathcal{L}}(\Gamma) \leq \frac{\alpha(\pi + \alpha)}{(\pi - \alpha)^2}.$$

This shows that $\alpha \geq c(s, k, d) > 0$ as desired. $\square$

Proposition 11.5 *There exists a number $N \in \mathbb{N}$, and a function $\psi : (0, \infty) \rightarrow (0, \infty)$ with*

$$\lim_{t \rightarrow \infty} \psi(t) = 0$$

satisfying the following property: if $\mathcal{K} = \{Q_i : i \in I\}$ is a collection of pairwise disjoint $\mathbb{C}^$ -squares $Q_i \subseteq \mathbb{C}^*$, and if E and F are arbitrary disjoint continua in $\mathbb{C}^* \setminus \bigcup_{i \in I} \text{int}(Q_i)$ with $\Delta_{\mathbb{C}^*}(E, F) \geq 12$, then there exists a set $I_0 \subseteq I$ with $\#I_0 \leq N$ such that for the transboundary modulus of the path family $\Gamma(E, F; \Omega')$ in the region $\Omega' = \mathbb{C}^* \setminus \bigcup_{i \in I_0} Q_i$ with respect to the collection $\mathcal{K}' = \{Q_i : i \in I \setminus I_0\}$ we have*

$$M_{\Omega', \mathcal{K}'}(\Gamma(E, F; \Omega')) \leq \psi(\Delta_{\mathbb{C}^*}(E, F)).$$

Here $\Delta_{\mathbb{C}^*}(E, F)$ denotes (in accordance with our convention from Sect. 2) the relative distance of E and F with respect to the flat metric $d_{\mathbb{C}^*}$ on $\mathbb{C}^*$. Note that if E and F are as in the statement, then

$$E, F \subseteq \mathbb{C}^* \setminus \bigcup_{i \in I} \text{int}(Q_i) \subseteq \mathbb{C}^* \setminus \bigcup_{i \in I_0} \text{int}(Q_i) = \overline{\Omega'}.$$

Proof The proposition immediately follows from Remark 8.8. We have to check the relevant conditions in this remark. For the mass bounds in the metric measure space $(\mathbb{C}^*, d_{\mathbb{C}^*}, A_{\mathbb{C}^*})$ note that if $a \in \mathbb{C}^*$, then we have

$$A_{\mathbb{C}^*}(B_{\mathbb{C}^*}(a, r)) \leq \pi r^2$$

for all $r > 0$, and

$$A_{\mathbb{C}^*}(B_{\mathbb{C}^*}(a, r)) = \pi r^2$$

for all $r \leq \pi$. The last equality implies that

$$A_{\mathbb{C}^*}(B_{\mathbb{C}^*}(a, r)) \geq \frac{\pi}{5} r^2$$

for all $r \leq \sup\{\text{diam}_{\mathbb{C}^*}(Q) : Q \text{ is a } \mathbb{C}^*\text{-square}\} = \pi\sqrt{5}$. So we get the relevant upper and lower mass bounds.

Moreover, it is clear that a $\mathbb{C}^*$ -square Q in $(\mathbb{C}^*, d_{\mathbb{C}^*}, A_{\mathbb{C}^*})$ is μ -fat for some universal constant $\mu > 0$. To produce an explicit (non-sharp) constant μ let $x \in Q$ and $0 < r \leq \text{diam}_{\mathbb{C}^*}(Q) \leq \sqrt{2}\ell(Q)$ be arbitrary. If $0 \leq s \leq \ell(Q)/2$, then $Q \cap B_{\mathbb{C}^*}(x, s)$ contains at least a “quarter” of the disk $B_{\mathbb{C}^*}(x, s)$. If we apply this for $s = r/(2\sqrt{2}) \leq \ell(Q)/2 \leq \pi$, we obtain

$$\begin{aligned} A_{\mathbb{C}^*}(Q \cap B_{\mathbb{C}^*}(x, r)) &\geq A_{\mathbb{C}^*}(Q \cap B_{\mathbb{C}^*}(x, s)) \geq \frac{1}{4} A_{\mathbb{C}^*}(B_{\mathbb{C}^*}(x, s)) \\ &= \frac{\pi}{4} s^2 = \frac{\pi}{32} r^2 \geq \frac{1}{32} A_{\mathbb{C}^*}(B_{\mathbb{C}^*}(x, r)). \end{aligned}$$

So we can take $\mu = 1/32$. □

Proposition 11.6 *In Corollary 9.13 suppose in addition that the Jordan curves $\partial D_0, \dots, \partial D_n$ are s -relatively separated k -quasicircles, and that*

$$\text{diam}(\partial D_0) \wedge \text{diam}(\partial D_n) \geq d > 0.$$

Then f is an η -quasisymmetric map from $\overline{\Omega}$ equipped with the chordal metric to $\overline{U}$ equipped with flat metric on $\mathbb{C}^$. Here η only depends on s, k , and d .*

Proof The proof of the theorem is very similar to the proof of Theorem 10.2. Note that both metric spaces $(\widehat{\mathbb{C}}, \sigma)$ and $(\mathbb{C}^*, d_{\mathbb{C}^*})$ are doubling, and so every subset of one of these spaces is N_0 -doubling, where N_0 is a universal constant. So by Proposition 3.2 it is enough to show that on $\overline{\Omega}$ the map f is weakly H -quasisymmetric with $H = H(s, k, d)$. We can extend f (non-uniquely) to a homeomorphism from $\widehat{\mathbb{C}} \setminus (\text{int}(D_0) \cup \text{int}(D_n))$ onto $\overline{A}$. We will use the notation $u' := f(u)$ for $u \in \widehat{\mathbb{C}} \setminus (\text{int}(D_0) \cup \text{int}(D_n))$.

To reach a contradiction assume that for some points $x, y, z \in \overline{\Omega}$ with $\sigma(x, y) \leq \sigma(x, z)$ we have $d_{\mathbb{C}^*}(x', y') > H d_{\mathbb{C}^*}(x', z')$ for some large $H \gg 1$. Then the points x, y, z are distinct. We want to find continua E' and F' in

$$\overline{U} = \overline{A} \setminus (\text{int}(Q_1) \cup \dots \cup \text{int}(Q_{n-1}))$$

whose relative distance is large, but for which the relative distance of the preimages E and F is controlled.

By Lemma 11.1 we can find a continuum $E' \subseteq \overline{U}$ connecting x' and z' such that

$$\text{diam}_{\mathbb{C}^*}(E') \leq 2d_{\mathbb{C}^*}(x', z').$$

The choice of F' is more involved. Since the sets ∂D_0 and ∂D_n are s -relatively separated, we have

$$\text{dist}(\partial D_0, \partial D_n) \geq s(\text{diam}(\partial D_0) \wedge \text{diam}(\partial D_n)) \geq sd.$$

Hence y must have distance $\geq sd/2$ to one of the sets ∂D_0 and ∂D_n , say to ∂D_0 . Then

$$\text{dist}(y, \partial D_0) \geq sd/2.$$

Let $\epsilon_0 = \epsilon_0(s, k, d) \in (0, 2\pi)$ be as (57). Then

$$\text{diam}_{\mathbb{C}^*}(B_{\mathbb{C}^*}(x, \epsilon_0/4)) \leq \epsilon_0/2 < \pi = \text{diam}_{\mathbb{C}^*}(\partial_i A).$$

Hence there exists a point $u \in \partial D_0$ such that for its image point we have $u' \in \partial_i A \setminus B_{\mathbb{C}^*}(x, \epsilon_0/4)$. Note that then

$$\sigma(u, y) \geq sd/2. \quad (58)$$

By (56) we have

$$d_{\mathbb{C}^*}(x', y') \leq \text{diam}_{\mathbb{C}^*}(\overline{A}) \leq (\pi + h_A) \leq (\pi + C_2(s, k, d)) =: C_3(s, k, d).$$

If we define $c_4 := c_4(s, k, d) = \epsilon_0/(4C_3) < 1$, then $c_4 d_{\mathbb{C}^*}(x', y') \leq \epsilon_0/4$, and so both points u' and y' lie outside the ball $B_{\mathbb{C}^*}(x', c_4 d_{\mathbb{C}^*}(x', y'))$. By Lemma 11.1 can find a continuum $F' \subseteq \overline{U}$ connecting y' and u' such that

$$F' \cap B_{\mathbb{C}^*}(x', c_4 d_{\mathbb{C}^*}(x', y')) = \emptyset.$$

Combining this with the diameter bound for E' , we see (as in the proof of Theorem 10.2) that if $H \geq C_5(s, k, d)$, then for the relative distance of E' and F' with respect to the metric $d_{\mathbb{C}^*}$ we have

$$\Delta_{\mathbb{C}^*}(E', F') \geq H/C_6 \geq 12,$$

where $C_6 = C_6(s, k, d)$.

Define $E = f^{-1}(E')$ and $F = f^{-1}(F')$. Then E and F are continua in $\overline{\Omega}$ containing the sets $\{x, z\}$ and $\{y, u\}$, respectively. Then $\text{dist}(E, F) \leq \sigma(x, y)$. Using (58) we get,

$$\text{diam}(E) \wedge \text{diam}(F) \geq \sigma(x, z) \wedge \sigma(y, u) \geq \sigma(x, z) \wedge (sd/2).$$

Hence

$$\Delta(E, F) \leq \frac{\sigma(x, y)}{\sigma(x, z) \wedge (sd/2)} \leq 1 \vee (4/(sd)) =: C_7(s, d). \quad (59)$$

Let $N \in \mathbb{N}$ and $\psi: (0, \infty) \rightarrow (0, \infty)$ with $\lim_{t \rightarrow \infty} \psi(t) = 0$ be as in Proposition 11.5. Then for some set $I_0 \subseteq I := \{1, \dots, n-1\}$ with $\#I_0 \leq N$ we have that

$$M_{W, \mathcal{K}'}(\Gamma(E', F'; W)) \leq \psi(\Delta_{\mathbb{C}^*}(E', F')) \leq \psi(H/C_6),$$

where $W = \mathbb{C}^* \setminus \bigcup_{i \in I_0} Q_i$ and transboundary modulus is with respect to the collection $\mathcal{K}' = \{Q_i : i \in I \setminus I_0\}$. If $V' := A \setminus \bigcup_{i \in I_0} Q_i$, then $U \subseteq V' \subseteq W$, and $\Gamma(E', F'; V') \subseteq \Gamma(E', F'; W)$, and so

$$M_{V', \mathcal{K}'}(\Gamma(E', F'; V')) \leq M_{W, \mathcal{K}'}(\Gamma(E', F'; W)) \leq \psi(H/C_6).$$

Define $V = \widehat{\mathbb{C}} \setminus (D_0 \cup D_n \cup \bigcup_{i \in I_0} D_i)$. Note that by Lemma 4.8 the complementary components D_i , $i \in I_0 \cup \{0, n\}$, of V are s -relatively separated. Since $\#I_0$ can be bounded by the universal constant N , it follows from Proposition 7.5 that the region $V = \widehat{\mathbb{C}} \setminus (D_0 \cup D_n \cup \bigcup_{i \in I_0} D_i)$ is ϕ -Loewner with ϕ only depending on s and k . Combining this with (59) and Proposition 8.1, we see that there is a positive constant $C_8 = C_8(s, k, d) > 0$ such that

$$M_{V, \mathcal{K}}(\Gamma(E, F; V)) \geq C_8,$$

where the transboundary modulus in V is with respect to the collection $\mathcal{K} = \{D_i : i \in I \setminus I_0\}$.

Our (extended) map f is a homeomorphism from $\overline{V}$ onto $\overline{V}'$, and a conformal map from $V \setminus \bigcup_{i \in I \setminus I_0} D_i = \Omega$ onto $V' \setminus \bigcup_{i \in I \setminus I_0} Q_i = U$. Moreover, $f(\Gamma(E, F; V)) = \Gamma(E', F'; V')$, and so invariance of transboundary modulus gives

$$M_{V, \mathcal{K}}(\Gamma(E, F; V)) = M_{V', \mathcal{K}'}(\Gamma(E', F'; V')).$$

Hence our estimates lead to the inequality

$$C_8 \leq \psi(H/C_6).$$

Since ψ is a fixed function with $\psi(t) \rightarrow 0$ as $t \rightarrow \infty$, this leads to a contradiction if H is larger than a constant only depending on s , k , and d . $\square$

Theorem 11.7 *Let $I = \{0, \dots, n\}$, where $n \geq 1$, or $I = \mathbb{N}_0$. Suppose that $\{D_i : i \in I\}$ is a collection of pairwise disjoint closed Jordan regions whose boundaries ∂D_i , $i \in I$, form a family of uniformly relatively separated uniform quasicircles. Then there exists a finite $\mathbb{C}^*$ -cylinder A , pairwise disjoint*

$\mathbb{C}^*$ -squares $Q_i \subseteq A$ for $i \in I \setminus \{0, 1\}$, and a quasimetric homeomorphism $f: T \rightarrow T'$, where

$$T = \widehat{\mathbb{C}} \setminus \bigcup_{i \in I} \text{int}(D_i) \quad \text{and} \quad T' = \overline{A} \setminus \bigcup_{i \in I \setminus \{0, 1\}} \text{int}(Q_i), \quad (60)$$

that maps ∂D_0 to $\partial_o A$ and ∂D_1 to $\partial_i A$. Here T and T' are equipped with the restriction of the chordal metric and the flat metric on $\mathbb{C}^*$, respectively.

Proof If I is finite, then the statement follows from Proposition 11.6.

If $I = \mathbb{N}_0$, for each $n \in \mathbb{N}$ we consider the finitely connected region $\Omega_n = \widehat{\mathbb{C}} \setminus \bigcup_{i=0}^n D_i$. Then $\bigcap_{n \in \mathbb{N}} \overline{\Omega}_n = T$. By Proposition 11.6 there exists an η -quasimetric embedding f_n of $\overline{\Omega}_n$ into the closure $\overline{A}_n$ of a finite $\mathbb{C}^*$ -cylinder A_n such that $f_n(\partial D_0) = \partial_o A_n$, $f_n(\partial D_1) = \partial_i A_n$, and such that the complementary components of $f_n(\Omega_n)$ in A_n are $\mathbb{C}^*$ -squares. Here the distortion function η does not depend on n . Postcomposing f_n with a suitable dilation $z \mapsto \lambda z$, $\lambda \neq 0$, which does not affect η , we may in addition assume that $\partial_o A_n = \partial \mathbb{D}$ for all $n \in \mathbb{N}$.

By Lemma 3.3 the sequence (f_n) subconverges on T to an η -quasimetric embedding $f: T \rightarrow \mathbb{C}^*$, i.e., there exists a subsequence (f_{n_l}) of (f_n) that converges to f uniformly on T . Since ∂D_i is the boundary of the complementary component D_i of Ω_n for $n \geq i$ and $f_{n_k} \rightarrow f$ uniformly, it follows that for fixed $i \in \mathbb{N}_0$ the Jordan curve $f(\partial D_i)$ is the Hausdorff limit of the sets $f_{n_l}(\partial D_i)$ as $l \rightarrow \infty$. Therefore, $f(\partial D_0) = \partial \mathbb{D}$. Since $f_n(\partial D_1) = \{z \in \mathbb{C} : |z| = r_n\}$ with $r_n \in (0, 1)$ for $n \geq 1$, it follows that $f(\partial D_1) = \{z \in \mathbb{C} : |z| = r\}$ for some $0 < r \leq 1$. Since f is an embedding, we have $0 < r < 1$.

By a similar consideration it follows $f(\partial D_i) = \partial Q_i$ for $i \geq 2$, where Q_i is a (non-degenerate) $\mathbb{C}^*$ -square. Here $Q_i \cap Q_j = \emptyset$ for $i \neq j$. Indeed, it is clear that $\text{int}(Q_i) \cap \text{int}(Q_j) = \emptyset$, because Q_i and Q_j can be written as Hausdorff limits of sequences of $\mathbb{C}^*$ -squares, where corresponding $\mathbb{C}^*$ -squares in the sequences have empty intersection. Moreover, $\partial Q_i \cap \partial Q_j = f(\partial D_i) \cap f(\partial D_j) = \emptyset$ for $i \neq j$, because f is an embedding.

Let $A \subseteq \mathbb{C}^*$ be the finite cylinder with $\partial_o A = \partial \mathbb{D}$ and

$$\partial_i A = f(\partial D_1) = \{z \in \mathbb{C} : |z| = r\}.$$

Since $f_n(\overline{\Omega}_n) \subseteq \overline{A}_n$ for all $n \geq 1$, and $\overline{A}_{n_l} \rightarrow \overline{A}$ as $l \rightarrow \infty$, we have $T' = f(T) \subseteq \overline{A}$. Since f is an embedding, this implies that the $\mathbb{C}^*$ -squares Q_i , $i \geq 2$, lie in A . As follows from Lemma 5.5(iii), we have $f(T) \subseteq Q_i$ or $f(T) \subseteq \overline{A} \setminus Q_i$. Here the former case is impossible as $f(\partial D_j) = \partial Q_j$ has empty intersection with Q_i for $j \neq i$. Putting this all together, Lemma 5.5(iii) shows that $T' = f(T)$ can be written as in (60). $\square$

As follows from Proposition 11.6 and the previous proof, the statement in Theorem 11.7 is quantitative in the following sense: if the collection ∂D_i , $i \in I$, consists of s -relatively separated k -quasicircles, and $\text{diam}(\partial D_0) \wedge \text{diam}(\partial D_1) \geq d > 0$, then one can find an η -quasisymmetric map f with $\eta = \eta_{s,k,d}$. The dependence on d here is unavoidable. This can be seen as follows (in the ensuing argument we leave some details to the reader).

Suppose we could always choose $\eta = \eta_{s,k}$. Then for each $n \in \mathbb{N}$ we can find an η -quasisymmetric map f_n (with η independent of n) mapping the closure of the finite $\mathbb{C}^*$ -cylinder $A_n = \{z \in \mathbb{C} : 1/n < |z| < 1\}$ equipped with the chordal metric to the closure of a finite $\mathbb{C}^*$ -cylinder $A'_n = \{z \in \mathbb{C} : r_n < |z| < 1\}$ equipped with the flat metric such that $f_n(\partial \mathbb{D}) = \partial \mathbb{D}$. One can then pass to a sublimit (this does not follow directly from Lemma 3.3, but from the methods of its proof) which produces a quasisymmetric embedding f of $\overline{\mathbb{D}} \setminus \{0\}$ equipped with the chordal metric into $\overline{\mathbb{D}} \setminus \{0\}$ equipped with the flat metric. This map f also satisfies $f(\partial \mathbb{D}) = \partial \mathbb{D}$.

Since $\overline{\mathbb{D}} \setminus \{0\}$ has finite diameter in the chordal metric, its image set $f(\overline{\mathbb{D}} \setminus \{0\})$ must have finite diameter in the flat metric. Since f is a quasisymmetry, this implies that f is uniformly continuous and so it has a continuous extension as a map from $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}} \setminus \{0\}$ (note that 0 is “infinitely far away” in the flat metric). This is impossible for topological reasons. Namely, since the Jordan curve $\partial \mathbb{D}$ is contractible in $\mathbb{D}$, its image $\partial \mathbb{D} = f(\partial \mathbb{D})$ is contractible in $f(\overline{\mathbb{D}}) \subseteq \overline{\mathbb{D}} \setminus \{0\}$, and hence in $\mathbb{C}^*$. This is absurd.

12 Sierpiński carpets and carpet modulus

The *standard Sierpiński carpet* T is defined as follows. Let $T_0 = [0, 1] \times [0, 1] \subseteq \mathbb{R}^2 \cong \mathbb{C}$ be the unit square in $\mathbb{C}$. We subdivide T_0 into nine subsquares of equal sidelength and remove the interior of the “middle” square. The resulting set T_1 is the union of eight non-overlapping closed squares of Euclidean sidelength $1/3$. On each of these squares we perform an operation similar to the one that was used to construct T_1 from T_0 . Continuing successively in this manner, we obtain a nested sequence of compact sets $T_0 \supset T_1 \supset T_2 \supset \dots$ such that T_n consists of 8^n non-overlapping squares of sidelength $1/3^n$. Now T is defined as $T = \bigcap_{n \in \mathbb{N}_0} T_n$.

A (*Sierpiński*) *carpet* is a topological space homeomorphic to the standard Sierpiński carpet. A metric space X is a carpet if and only if it is a locally connected continuum that is planar, has topological dimension 1, and has no local cut-points [36]. Here X is called *planar* if it is homeomorphic to a subset of $\widehat{\mathbb{C}}$. A *local cut point* in X is a point $p \in X$ such that for all sufficiently small neighborhoods U of p the set $U \setminus \{p\}$ is not connected.

A set $T \subseteq \widehat{\mathbb{C}}$ is a carpet if and only if $\text{int}(T) = \emptyset$ and it can be written as

$$T = \widehat{\mathbb{C}} \setminus \bigcup_{i \in \mathbb{N}_0} \text{int}(D_i), \quad (61)$$

where the sets D_i , $i \in \mathbb{N}_0$, form a collection of pairwise disjoint closed Jordan regions in $\widehat{\mathbb{C}}$ with $\text{diam}(D_i) \rightarrow 0$ as $i \rightarrow \infty$ [36].

A Jordan curve S in a carpet T is called a *peripheral circle* if $T \setminus S$ is a connected set. The peripheral circles of a carpet as in (61) are precisely the Jordan curves ∂D_i , $i \in \mathbb{N}_0$. In particular, the collection of the peripheral circles of the standard Sierpiński carpet T consists of the boundary ∂T_0 of the unit square and the boundaries of the squares that were successively removed from T_0 in the construction of T .

A carpet $T \subseteq \widehat{\mathbb{C}}$ is called *round* if its peripheral circles are round circles. This is true if and only if the Jordan regions D_i in the representation of T as in (61) are round disks. So every round carpet is a Schottky set (see Sect. 10). Hence it follows from [8, Theorem 1.1] that round carpets of spherical measure zero are *rigid* in the following sense: if $T \subseteq \widehat{\mathbb{C}}$ is a round carpet of spherical measure zero and $f: T \rightarrow T'$ is a quasiconformal map of T onto another round carpet $T' \subseteq \widehat{\mathbb{C}}$, then f is the restriction of a Möbius transformation to T .

Proof of Corollary 1.2 Let T be a carpet as in the statement. Then T can be written as in (61). By Theorem 1.1 there exists a quasiconformal map $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ such that $f(\partial D_i)$ is a round circle for each $i \in \mathbb{N}_0$. Hence we can write $T' = f(T)$ as

$$T' = \widehat{\mathbb{C}} \setminus \bigcup_{n \in \mathbb{N}_0} \text{int}(D'_n),$$

where the sets $D'_i = f(D_i)$ are pairwise disjoint closed disks. Since $\text{int}(T) = \emptyset$, we also have $\text{int}(T') = \emptyset$, and so T' is a round carpet. By Proposition 3.1 the map f is a quasiconformal map, and hence also its restriction $f|_T: T \rightarrow T'$. The existence part of the statement follows.

Suppose in addition that T has spherical measure zero. Since quasiconformal maps on $\widehat{\mathbb{C}}$ preserve such sets (see [33, Definition 24.6 and Theorem 33.2]), the round carpet T' is also a set of spherical measure zero. Let $g: T \rightarrow \tilde{T}$ be another quasiconformal map onto a round carpet $\tilde{T} \subseteq \widehat{\mathbb{C}}$. Then $g \circ f^{-1}$ is a quasiconformal map of T' onto $\tilde{T}$. Since round carpets of measure zero are rigid, the map $g \circ f^{-1}$ is the restriction of a Möbius transformation, and so g is equal to f post-composed with a Möbius transformation. So we also have the uniqueness part of the statement, and the proof is complete. $\square$

Let $T \subseteq \widehat{\mathbb{C}}$ be a carpet, and $f: T \rightarrow \widehat{\mathbb{C}}$ be an embedding. Then $T' = f(T)$ is also a carpet, and f induces a bijection between the peripheral circles of T

and T' . It was shown in the proof of Lemma 5.5(iii) that there exists a homeomorphism $F: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ with $F|_T = f$. We call f *orientation-preserving* if F is orientation-preserving (with respect to the standard orientation on $\widehat{\mathbb{C}}$). This does not depend on the choice of the homeomorphic extension F of f . In more intuitive terms, f is orientation-preserving if the following condition is true: if we orient each peripheral circle S of T so that T lies “on the left” of S , then the induced orientation on the peripheral circle $S' = f(S)$ of $T' = f(T)$ is such that T' lies “on the left” of S' .

Proof of Theorem 1.5 Let $T \subseteq \mathbb{C}$ be a carpet as in the statement. As we have seen in the proof of Corollary 1.2, there exists a quasiconformal map $g: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ such that $T' = g(T)$ is a round carpet of measure zero. If $f: T \rightarrow T$ is a quasisymmetry, then $f' = g \circ f \circ g^{-1}|_{T'}$ is also a quasisymmetry. Since T' is rigid, it follows that $f' = F'|_{T'}$ is the restriction of a Möbius transformation $F': \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$. Suppose in addition that f is orientation-preserving. Then the same is true for f' and hence for F' .

If f has three distinct fixed points, the same is true for f' and for F' . So F' is the identity map on $\widehat{\mathbb{C}}$, which implies that f is the identity on T .

Similarly, if f fixes three distinct peripheral circles of T setwise, then f' fixes three distinct peripheral circles of T' setwise. Since the peripheral circles of T' are round circles, it follows that F' fixes three disjoint round circles setwise. Moreover, these circles bound pairwise disjoint disks. Since F' is an orientation-preserving Möbius transformation, F' must be the identity map on $\widehat{\mathbb{C}}$. Hence f is the identity on T . $\square$

Proof of Theorem 1.6 This is a special case of Theorem 11.7. $\square$

Let $T \subseteq \widehat{\mathbb{C}}$ be a carpet represented as in (61), and Γ be a collection of paths in $\widehat{\mathbb{C}}$. We define the *carpet modulus* of Γ with respect to T , denoted by $\mathcal{M}_T(\Gamma)$, as follows. Let $\rho_i \geq 0$ for $i \in \mathbb{N}_0$. We call the weight sequence $(\rho_i)_{i \in \mathbb{N}_0}$ *admissible* for Γ (with given T) if there exists a family $\Gamma_0 \subseteq \Gamma$ with $\text{mod}(\Gamma_0) = 0$ such that

$$\sum_{\gamma \cap D_i \neq \emptyset} \rho_i \geq 1 \quad \text{for all } \gamma \in \Gamma \setminus \Gamma_0.$$

Then

$$\mathcal{M}_T(\Gamma) := \inf_{(\rho_i)} \sum_{i \in \mathbb{N}_0} \rho_i^2,$$

where the infimum is taken over all weight sequences (ρ_i) that are admissible for Γ . An admissible weight sequence for which this infimum is attained is called *extremal*.

It is essential here to allow the exceptional family Γ_0 with vanishing modulus in the classical sense. Of course, one could define carpet modulus by requiring the inequality in the admissibility condition for *all* $\gamma \in \Gamma$, but this leads to a notion of carpet modulus that is not very interesting. Our notion of carpet modulus is useful for studying the quasiconformal geometry of carpets, since it is related to the geometry of the carpet and is invariant under quasiconformal maps.

Proposition 12.1 (Quasiconformal invariance of carpet modulus) *Let $T \subseteq \widehat{\mathbb{C}}$ be a carpet, Γ a family of paths in $\widehat{\mathbb{C}}$, and $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ a quasiconformal map. Then*

$$\mathcal{M}_T(\Gamma) = \mathcal{M}_{f(T)}(f(\Gamma)).$$

Proof Note that $T' = f(T)$ is also a carpet. If T is represented as in (61), then

$$T' = \widehat{\mathbb{C}} \setminus \bigcup_{i \in \mathbb{N}_0} \text{int}(D'_i),$$

where $D'_i = f(D_i)$ for $i \in \mathbb{N}_0$. Moreover, we have $\gamma \cap D_i \neq \emptyset$ for $\gamma \in \Gamma$ if and only if $f(\gamma) \cap f(D_i) \neq \emptyset$.

A quasiconformal map preserves the modulus of a path family up to a fixed multiplicative constant (Proposition 6.2). So if $\Gamma_0 \subseteq \Gamma$ and $\text{mod}(\Gamma_0) = 0$, then $\text{mod}(f(\Gamma_0)) = 0$. This implies that if (ρ_i) is an admissible weight sequence for Γ with respect to the carpet T , then it is also admissible for $\Gamma' = f(\Gamma)$ with respect to the carpet T' . Hence $\mathcal{M}_{T'}(\Gamma') \leq \mathcal{M}_T(\Gamma)$. Applying the same argument to the quasiconformal map f^{-1} , we get an inequality in the other direction. Hence $\mathcal{M}_{T'}(\Gamma') = \mathcal{M}_T(\Gamma)$ as desired. $\square$

The crucial point in the previous proof was that while quasiconformal maps only preserve the moduli of general path families up to a multiplicative constant, they preserve the modulus of a path family with vanishing modulus.

Suppose T is a carpet as in (61). Consider the path family

$$\Gamma = \Gamma_o(\partial D_0, \partial D_1; \widehat{\mathbb{C}} \setminus (D_0 \cup D_1))$$

of all open paths in the topological annulus $\widehat{\mathbb{C}} \setminus (D_0 \cup D_1)$ connecting its boundary components ∂D_0 and ∂D_1 . We are interested in finding $\mathcal{M}_T(\Gamma)$. The next statement shows that with suitable assumptions on T the answer is very similar to the answer to the corresponding question for transboundary modulus studied in Sect. 11. A subtlety here is that it is better to consider the family of *open* paths Γ instead of the family of closed paths $\Gamma' = \Gamma(\partial D_0, \partial D_1; \widehat{\mathbb{C}} \setminus (D_0 \cup D_1))$. In contrast to the paths in Γ , the paths in Γ' meet D_0 and D_1 , so one obtains more admissible sequences by putting

non-zero weights on D_0 and D_1 . By choosing the weights $1/2$ on D_0 and D_1 , and all other weights equal to 0, for example, one gets trivial inequalities such as $\mathcal{M}_T(\Gamma') \leq 1/2$ which do not reflect the geometry of T .

Corollary 12.2 *Let $T \subseteq \mathbb{C}$ be a carpet of spherical measure zero whose peripheral circles are uniformly relatively separated uniform quasicircles. Suppose T is represented as in (61) and $f: T \rightarrow T'$ is a quasisymmetric map as in Theorem 1.6 with*

$$T' = \overline{A} \setminus \bigcup_{i \geq 2} \text{int}(Q_i),$$

where A a finite $\mathbb{C}^*$ -cylinder and the sets Q_i , $i \geq 2$, are pairwise disjoint $\mathbb{C}^*$ -squares in A , and we have $f(\partial D_0) = \partial_o A$ and $f(\partial D_1) = \partial_i A$. Let

$$\Gamma = \Gamma_o(\partial D_0, \partial D_1; \widehat{\mathbb{C}} \setminus (D_0 \cup D_1)).$$

Then

$$\mathcal{M}_T(\Gamma) = \frac{2\pi}{h_A}.$$

Moreover, a unique extremal weight sequence $(\rho_i)_{i \in \mathbb{N}_0}$ for $\mathcal{M}_T(\Gamma)$ exists and is given by

$$\rho_0 = \rho_1 = 0 \quad \text{and} \quad \rho_i = \ell(Q_i)/h_A \quad \text{for } i \geq 2. \quad (62)$$

Proof Since the metric $d_{\mathbb{C}^*}$ and the spherical metric are comparable on $\overline{A}$, the map f is a quasisymmetric and hence also a quasi-Möbius embedding from T into $\widehat{\mathbb{C}}$ (equipped with the chordal metric). By Proposition 5.1 it has an extension to quasiconformal map $F: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$. Since T has measure zero and quasiconformal maps preserve such sets, the set $T' = f(T) = F(T)$ has spherical measure zero. Hence also $A_{\mathbb{C}^*}(T') = 0$ and so we have

$$\sum_{i \geq 2} \ell(Q_i)^2 = \sum_{i \geq 2} A_{\mathbb{C}^*}(Q_i) = A_{\mathbb{C}^*}(A) = 2\pi h_A. \quad (63)$$

Note that $\Gamma' := F(\Gamma) = \Gamma_o(\partial_i A, \partial_o A; A)$. So by quasiconformal invariance of carpet modulus (Proposition 12.1) for the first part of the statement it suffices to show that

$$\mathcal{M}_{T'}(\Gamma') = \frac{2\pi}{h_A}.$$

Now the argument is very similar to the proof of Proposition 11.2. We can write $A = \{z \in \mathbb{C} : r < |z| < R\}$, where $0 < r < R$. Then the closures of the complementary components of the carpet T' in $\widehat{\mathbb{C}}$ are the sets $\widehat{\mathbb{C}} \setminus B_{\mathbb{C}}(0, r)$, $\overline{B_{\mathbb{C}}}(0, r)$, and Q_i , $i \geq 2$. They are labeled by 0, 1, and i , respectively. We

define a corresponding weight sequence $(\rho_i)_{i \in \mathbb{N}_0}$ by $\rho_0 = \rho_1 = 0$ and $\rho_i = \ell(Q_i)/h_A$ for $i \geq 2$.

We claim that this weight sequence is admissible for the modulus $M_{T'}(\Gamma')$. To see this let $\Gamma_0 \subseteq \Gamma'$ be the family of all paths $\alpha \in \Gamma'$ that are not locally rectifiable or are locally rectifiable and satisfy

$$\text{length}(\alpha \cap T') := \int_{\alpha} \chi_{T'} ds > 0.$$

Since T' has measure zero, we have $\text{mod}(\Gamma_0) = 0$. Indeed, the function ρ defined by $\rho(z) = \infty$ for $z \in T'$ and $\rho(z) = 0$ for $z \in \widehat{\mathbb{C}} \setminus T'$ is an admissible density for Γ_0 with $\int \rho^2 d\Sigma = 0$.

Now let $\alpha \in \Gamma \setminus \Gamma_0$ be arbitrary. If π is the map of A to the interval $(\log r, \log R)$ defined by $z \mapsto \pi(z) := \log |z|$, then

$$(\log r, \log R) = \pi(\alpha) \subseteq \pi(\alpha \cap T') \cup \bigcup_{\alpha \cap Q_i \neq \emptyset} \pi(\alpha \cap Q_i). \quad (64)$$

All subsets of $\mathbb{R}$ appearing in the last inclusion are Borel sets, and hence measurable. Since $\alpha \notin \Gamma_0$, this path is locally rectifiable and we have $\text{length}(\alpha \cap T') = 0$. Since π is Lipschitz (it is 1-Lipschitz if A is equipped with flat metric, and hence also Lipschitz with respect to the chordal metric) this implies that $m_1(\pi(\alpha \cap T')) = 0$. Hence

$$\begin{aligned} \sum_{\alpha \cap Q_i \neq \emptyset} \rho_i &= \frac{1}{h_A} \sum_{\alpha \cap Q_i \neq \emptyset} \ell(Q_i) = \frac{1}{h_A} \sum_{\alpha \cap Q_i \neq \emptyset} m_1(\pi(Q_i)) \\ &\geq \frac{1}{h_A} m_1(\pi(\alpha \cap T')) + \frac{1}{h_A} \sum_{\alpha \cap Q_i \neq \emptyset} m_1(\pi(\alpha \cap Q_i)) \\ &\geq \frac{1}{h_A} m_1((\log r, \log R)) = 1. \end{aligned}$$

The admissibility of our weight sequence follows, and we conclude that

$$\begin{aligned} \mathcal{M}_{T'}(\Gamma') &\leq \sum_{i \in \mathbb{N}_0} \rho_i^2 = \frac{1}{h_A^2} \sum_{i \geq 2} \ell(Q_i)^2 \\ &= \frac{1}{h_A^2} \sum_{i \geq 2} A_{\mathbb{C}^*}(Q_i) = \frac{1}{h_A^2} A_{\mathbb{C}^*}(A) = \frac{2\pi}{h_A}. \end{aligned}$$

To get an inequality in the other direction, suppose that we have an admissible weight sequence $(\rho_i)_{i \in \mathbb{N}_0}$ for the family Γ' .

For each $\varphi \in [0, 2\pi]$ the path $\alpha_\varphi: (\log r, \log R) \rightarrow A$ defined by $\alpha_\varphi(t) := te^{i\varphi}$ for $t \in (\log r, \log R)$ belongs to Γ' . There exists a family $\Gamma_0 \subseteq \Gamma'$ with $\text{mod}(\Gamma_0) = 0$ such that

$$\sum_{\alpha_\varphi \cap Q_i \neq \emptyset} \rho_i \geq 1 \quad (65)$$

for all $\varphi \in [0, 2\pi]$ with $\alpha_\varphi \notin \Gamma_0$. The set E of all $\varphi \in [0, 2\pi]$ for which this inequality fails is a Borel set (E is the preimage of $[0, 1)$ under the Borel function on $[0, 2\pi]$ given by $\sum_{i \geq 2} \rho_i \chi_{F_i}$; here $F_i = \{\varphi \in [0, 2\pi] : \alpha_\varphi \cap Q_i \neq \emptyset\}$ is a closed set for $i \geq 2$). Hence E is measurable, and it must have 1-dimensional Lebesgue measure zero, since the corresponding family of paths $\{\alpha_\varphi : \varphi \in E\}$ is contained in Γ_0 and so is a family with vanishing modulus. Thus, (65) is valid for almost every $\varphi \in [0, 2\pi]$.

By integrating this inequality over φ , and using Fubini's theorem, the Cauchy-Schwarz inequality and (63), we arrive at

$$\begin{aligned} 2\pi &\leq \sum_{i \geq 2} \ell(Q_i) \rho_i \leq \left(\sum_{i \geq 2} \ell(Q_i)^2 \right)^{1/2} \left(\sum_{i \geq 2} \rho_i^2 \right)^{1/2} \\ &= (2\pi h_A)^{1/2} \left(\sum_{i \geq 2} \rho_i^2 \right)^{1/2} \leq (2\pi h_A)^{1/2} \left(\sum_{i \in \mathbb{N}_0} \rho_i^2 \right)^{1/2}. \end{aligned} \quad (66)$$

It follows that

$$\sum_{i \in \mathbb{N}_0} \rho_i^2 \geq \frac{2\pi}{h_A}$$

for every weight sequence that is admissible for Γ' . This implies the other inequality $\mathcal{M}_{T'}(\Gamma') \geq 2\pi/h_A$, and so $\mathcal{M}_T(\Gamma) = \mathcal{M}_{T'}(\Gamma') = 2\pi/h_A$ as desired.

If we have

$$\sum_{i \in \mathbb{N}_0} \rho_i^2 = \frac{2\pi}{h_A}$$

for an admissible weight sequence, then all inequalities in (66) must be equalities. This implies that $\rho_0 = \rho_2 = 0$ and that there exists $\lambda > 0$ such that $\rho_i = \lambda \ell(Q_i)$ for $i \geq 2$. Then $\lambda = 1/h_A$, and so $\rho_i = \ell(Q_i)/h_A$ for $i \geq 2$. This shows that (62) gives the unique extremal weight sequence for $\mathcal{M}_{T'}(\Gamma')$. Since admissible weight sequences for $\mathcal{M}_{T'}(\Gamma')$ and $\mathcal{M}_T(\Gamma)$ correspond to each other by the map F (see the proof of Proposition 12.1), we see that the weight sequence (62) is also the unique extremal weight sequence for $\mathcal{M}_T(\Gamma)$. $\square$

Similarly as classical modulus and transboundary modulus are useful for proving uniqueness results for conformal maps (see Corollary 11.3), carpet modulus can be employed to establish rigidity statements for quasimetric maps on carpets. For example, using this concept (in combination with other ideas) one can show that every quasimetric self-homeomorphism of the standard Sierpiński carpet (equipped with the restriction of the Euclidean metric) is an isometry. In particular, there are precisely 8 such quasimetric symmetries (the obvious rotations and reflections). See [9] for this result and related investigations.

13 Hyperbolic groups with carpet boundary

The material in this section is independent of the rest of the paper. Its purpose is the proof of Proposition 1.4 that motivates the study of carpets whose peripheral circles are uniformly relatively separated uniform quasicircles.

We quickly review some standard facts on Gromov hyperbolic groups. See [11, 17] for general background on Gromov hyperbolic groups and Gromov hyperbolic spaces.

Let G be a finitely generated group, and S a finite set of generators of G that is *symmetric* (i.e., if $s \in S$, then $s^{-1} \in S$). The group G is called *Gromov hyperbolic* if the Cayley graph $\mathcal{G}(G, S)$ of G with respect to S is Gromov hyperbolic as a metric space. In this case, $\mathcal{G}(G, S')$ is Gromov hyperbolic for each (finite and symmetric) generating sets S' . For the basic definitions and facts here, see [17, Chap. 1].

Associated with every Gromov hyperbolic metric space X is a boundary at infinity $\partial_\infty X$ equipped with a natural class of visual metrics [11, Chap. 2]. Accordingly, one defines the boundary at infinity of a Gromov hyperbolic group G as $\partial_\infty G = \partial_\infty \mathcal{G}(G, S)$. This is well-defined, because if S' is another generating set, then there is a natural identification $\partial_\infty \mathcal{G}(G, S') = \partial_\infty \mathcal{G}(G, S)$ (the elements in both spaces can be represented by equivalence classes of sequences in G converging to infinity; moreover, equivalence of such sequences is independent of the generating sets S and S'). If d' and d are visual metrics on $\partial_\infty \mathcal{G}(G, S')$ and $\partial_\infty \mathcal{G}(G, S)$, respectively, then they are quasimetrically equivalent, i.e., the identity map between $(\partial_\infty \mathcal{G}(G, S'), d')$ and $(\partial_\infty \mathcal{G}(G, S), d)$ is a quasimetric (this follows from the fact that $\mathcal{G}(G, S)$ and $\mathcal{G}(G, S')$ are quasi-isometric, and that every quasi-isometry between geodesic Gromov hyperbolic metric spaces induces a quasimetric map between their boundaries at infinity; see [35, 5.35 Theorem] for a precise quantitative version of the last fact). So if we equip $\partial_\infty G$ with any of these visual metrics d , then we can unambiguously speak of quasimetric and quasi-Möbius maps on $\partial_\infty G$. Moreover, the space $(\partial_\infty G, d)$ is doubling (see [10, Theorem 9.2] and the remarks after this theorem; note that the proof of [10, Theorem 9.2] contains some inaccuracies; they can easily be corrected).

The natural left-action of G on $\mathcal{G}(G, S)$ by isometries induces an action of G on $\partial_\infty G$ by quasimetrics. So each $g \in G$ can be considered as a quasimetric on $\partial_\infty G$, and we write $g(x)$ for the image of a point $x \in \partial_\infty G$ under $g \in G$. In general the action of G on $\partial_\infty G$ is not *effective*, i.e., there can be elements $g \in G$ that act as the identity on $\partial_\infty G$. If G is *non-elementary* (i.e., if $\# \partial_\infty G \geq 3$), then the elements of G acting on $\partial_\infty G$ form a finite and normal subgroup of G , the *ineffective kernel* (this follows from [17, Chap. 8, 36.-Corollary]; note that every element in the ineffective kernel is *elliptic* and hence has finite order [17, Chap. 8, 28.-Proposition]).

Two properties of the action of G on $\partial_\infty G$ (equipped with a fixed visual metric d) will be used in the following. This action is *uniformly quasi-Möbius*, i.e., there exists a homeomorphism $\eta: [0, \infty] \rightarrow [0, \infty]$ such that each $g \in G$ acts as an η -quasi-Möbius map on $\partial_\infty G$ (this goes back to the remark preceding [24, Theorem 5.4]; it easily follows from [35, Theorem 5.38]).

Moreover, the action is *cocompact on triples*. This means that there exists a constant $\epsilon_0 > 0$ with the following property: whenever z_1, z_2, z_3 are three distinct points in $\partial_\infty G$, then there exists $g \in G$ such that

$$d(g(z_i), g(z_j)) \geq \epsilon_0 \quad \text{for } i, j = 1, 2, 3, i \neq j \quad (67)$$

(see the discussion in [18, pp. 215–216]).

Before we turn to the proof of Proposition 1.4, we have to explain the terminology used in its statement. Let T be a *metric carpet*, i.e., a metric space that is a carpet, and $S = \{S_i : i \in I\}$ be the collection of its peripheral circles labeled by a countable index set. Recall from Sect. 4 that we call the collection S uniformly relatively separated if there exists $s > 0$ such that $\Delta(S_i, S_j) \geq s$ whenever $i, j \in I, i \neq j$. We say that S consists of *uniform quasicircles* if any of the quantitatively equivalent conditions in Proposition 4.4 is satisfied for each peripheral circle $S_i, i \in I$, with the same parameters. If T is doubling, then the peripheral circles are *uniformly doubling*, i.e., there exists $N \in \mathbb{N}$ such that S_i is N -doubling for each $i \in I$. In this case one can establish that S consists of uniform quasicircles by showing that there exists $\delta > 0$ such that whenever $i \in I$ and x_1, x_2, x_3, x_4 are four distinct points in cyclic order on S_i , then

$$[x_1, x_2, x_3, x_4] \geq \delta.$$

Finally, we say that the peripheral circles of T *occur on all locations and scales* if there exists a constant $c > 0$ such that for each $x \in T$ and each $0 < r \leq \text{diam}(T)$ there exists a peripheral circle S of T with $S \subseteq B(x, r)$ and $\text{diam}(S) \geq cr$. Note that in this case we also have $\text{diam}(S) \leq 2r$. So the peripheral circles occur on all locations and scales if every ball in T of radius $r \leq \text{diam}(T)$ contains a peripheral circle of diameter comparable to r .

The ensuing proof of Proposition 1.4 uses a well-known idea in complex dynamics and in the theory of Kleinian groups, namely the “principle of the conformal elevator” (see [19] for more discussion): in order to establish a geometric property on all scales, one uses the dynamics to map to the “top scale”, verifies the relevant condition there, and uses suitable distortion estimates to translate between scales.

Proof of Proposition 1.4 Let G be a Gromov hyperbolic group whose boundary at infinity $\partial_\infty G$ is a carpet. We equip $\partial_\infty G$ with a fixed visual metric d . We denote the peripheral circles of $T = \partial_\infty G$ by S_i , $i \in \mathbb{N}$. Since the action of G on $\partial_\infty G$ is uniformly quasi-Möbius, there exists a distortion function η such that $g: \partial_\infty G \rightarrow \partial_\infty G$ is an η -quasi-Möbius homeomorphism for each $g \in G$. Moreover, since the action of G on $\partial_\infty G$ is cocompact on triples, there exists a constant $\epsilon_0 > 0$ as in (67).

The basic idea now is to apply the conformal elevator principle mentioned before the proof. Since the action of G on $\partial_\infty G$ is cocompact on triples, we will be able to “map every scale to the top scale” by a suitable group element. The relevant distortion estimates will be derived from the fact that the action of G on $\partial_\infty G$ is uniformly quasi-Möbius. Accordingly, we will formulate the geometric conditions in question in terms of cross-ratios.

Since $\partial_\infty G$ is doubling, there exists $N \in \mathbb{N}$ such that each circle S_i , $i \in \mathbb{N}$, is N -doubling. So for proving that the collection S_i , $i \in \mathbb{N}$, consists of uniform quasicircles it is by Proposition 4.4 enough to find $\delta > 0$ such that

$$[x_1, x_2, x_3, x_4] \geq \delta,$$

whenever x_1, x_2, x_3, x_4 are four distinct points on one of the circles S_i that are in cyclic order on S_i .

We argue by contradiction and assume that no such $\delta > 0$ exists. Then for $n \in \mathbb{N}$ we can find distinct points $x_1^n, x_2^n, x_3^n, x_4^n$ that lie in cyclic order on some peripheral circle $S'_n \in \{S_i : i \in \mathbb{N}\}$ such that

$$[x_1^n, x_2^n, x_3^n, x_4^n] \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since the action of G on $\partial_\infty G$ is cocompact on triples, for each $n \in \mathbb{N}$ there exists $g_n \in G$ such that

$$d(y_i^n, y_j^n) \geq \epsilon_0 \quad \text{for } i, j = 1, 2, 3, i \neq j. \quad (68)$$

Here we set $y_i^n = g_n(x_i^n)$ for $i = 1, 2, 3, 4$, $n \in \mathbb{N}$.

Since the action G on $\partial_\infty G$ is uniformly quasi-Möbius, we have

$$[y_1^n, y_2^n, y_3^n, y_4^n] \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Every homeomorphism on a carpet preserves the collection of peripheral circles and the cyclic order of points on peripheral circles. It follows that for each $n \in \mathbb{N}$ the set $J_n = g_n(S'_n)$ is a peripheral circle of $\partial_\infty G$ on which the points $y_1^n, y_2^n, y_3^n, y_4^n$ are in cyclic order. By (68) we have

$$\text{diam}(J_n) \geq \epsilon_0 > 0 \quad \text{for all } n \in \mathbb{N}.$$

Since every carpet has only finitely many peripheral circles whose diameter exceeds a given positive constant (this follows from the corresponding fact for the standard carpet), there are only finitely many peripheral circles among the sets $J_n, n \in \mathbb{N}$. In particular, one circle, say $J := J_{n_0}$, is repeated infinitely often in the sequence $J_1, J_2, \dots$. So by passing to a subsequence if necessary, we may assume that all points $y_1^n, y_2^n, y_3^n, y_4^n, n \in \mathbb{N}$, lie on the peripheral circle J . By passing to further subsequences if necessary, we may assume that

$$y_i^n \rightarrow y_i \in J \quad \text{as } n \rightarrow \infty \quad \text{for } i = 1, 2, 3, 4.$$

By (68) we have

$$y_i \neq y_j \quad \text{for } i = 1, 2, 3, i \neq j.$$

Moreover, since the points $y_1^n, y_2^n, y_3^n, y_4^n$ are in cyclic order on J , the point y_4 is contained in the subarc α of J with endpoints y_1 and y_3 that does not contain y_2 . Hence $y_2 \neq y_4$, and it follows that

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} [y_1^n, y_2^n, y_3^n, y_4^n] \\ &= \lim_{n \rightarrow \infty} \frac{d(y_1^n, y_3^n)d(y_2^n, y_4^n)}{d(y_1^n, y_4^n)d(y_2^n, y_3^n)} \\ &= \frac{d(y_1, y_3)d(y_2, y_4)}{d(y_1, y_4)d(y_2, y_3)} \in (0, +\infty]. \end{aligned}$$

Here the last expression is interpreted as $+\infty$ if $d(y_1, y_4) = 0$, and is a finite non-zero number if $d(y_1, y_4) \neq 0$. Note that all other terms are non-zero. In any case we get a contradiction showing that the peripheral circles of $\partial_\infty G$ are uniform quasicircles.

The argument for showing uniform relative separation of the peripheral circles uses similar ideas. Again we argue by contradiction and assume that there is a sequence of pairs S'_n and S''_n of two distinct peripheral circles of $\partial_\infty G$ such that

$$\Delta(S'_n, S''_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By Lemmas 4.6 and 4.5 we can then find distinct points $x_1^n, x_4^n \in S'_n$ and $x_2^n, x_3^n \in S''_n$ such that

$$[x_1^n, x_2^n, x_3^n, x_4^n] \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Again using that the action of G on $\partial_\infty G$ is cocompact on triples, we can find $g_n \in G$ for $n \in \mathbb{N}$ such that

$$d(y_i^n, y_j^n) \geq \epsilon_0 \quad \text{for } i, j = 1, 2, 3, i \neq j, \quad (69)$$

where $y_i^n = g_n(x_i^n)$ for $i = 1, 2, 3, 4, n \in \mathbb{N}$. Since the action of G on $\partial_\infty G$ is uniformly quasi-Möbius, we see that

$$[y_1^n, y_2^n, y_3^n, y_4^n] \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (70)$$

Let $J_n = g_n(S'_n)$ and $J'_n = g_n(S''_n)$ for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ the sets J_n and J'_n are two distinct peripheral circles of $\partial_\infty G$ with $y_1^n, y_4^n \in J_n$ and $y_2^n, y_3^n \in J'_n$. Using (70) in combination with Lemmas 4.6 and 4.5, we conclude that

$$\Delta(J_n, J'_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (71)$$

Note that

$$\text{diam}(J'_n) \geq d(y_2^n, y_3^n) \geq \epsilon_0 \quad \text{for } n \in \mathbb{N},$$

and

$$[y_1^n, y_2^n, y_3^n, y_4^n] = \frac{d(y_1^n, y_3^n)d(y_2^n, y_4^n)}{d(y_1^n, y_4^n)d(y_2^n, y_3^n)} \geq \frac{\epsilon_0 d(y_2^n, y_4^n)}{\text{diam}(\partial_\infty G)^2}.$$

This forces the relation $d(y_2^n, y_4^n) \rightarrow 0$ as $n \rightarrow \infty$, and hence

$$\text{diam}(J_n) \geq d(y_1^n, y_4^n) \geq d(y_1^n, y_2^n) - d(y_2^n, y_4^n) \geq \epsilon_0/2$$

for large n .

So all but finitely many of the peripheral circles J_n and J'_n have diameter $\geq \epsilon_0/2 > 0$. As in the first part of the proof, this shows that the collection of all peripheral circles J_n and J'_n , $n \in \mathbb{N}$, is finite, and hence there are only finitely many pairs (J_n, J'_n) . Since for each pair $\Delta(J_n, J'_n) > 0$, we must have

$$\inf_{n \in \mathbb{N}} \Delta(J_n, J'_n) > 0.$$

This contradicts (71), showing that the peripheral circles of $\partial_\infty G$ are indeed uniformly relatively separated.

To prove the final statement we start with two general remarks about arbitrary carpets. Namely, if T is a carpet, then every nonempty open set $U \subseteq T$

contains a peripheral circle. This is obviously true for the standard Sierpiński carpet, and so it holds for all carpets.

Secondly, if T is an arbitrary metric carpet, then for every $r > 0$ there exists $\delta > 0$ such that every open ball in T of radius r contains a peripheral circle J with $\text{diam}(J) > \delta$. For otherwise, there exists $r > 0$, and a sequence of balls $B_n = B(x_n, r)$ in T such that B_n does not contain any peripheral circle of diameter $\geq 1/n$. Using the compactness of T and passing to a subsequence if necessary we may assume that $x_n \rightarrow x \in T$ as $n \rightarrow \infty$. Then $B = B(x, r/2) \subseteq B(x_n, r)$ for large n and so the open and nonempty set B cannot contain any peripheral circle of T . This contradicts the first remark.

Now let G be a Gromov hyperbolic group with carpet boundary $\partial_\infty G$ as in the beginning of the proof. Let $B = B(x, r)$ with $x \in \partial_\infty G$ and $0 < r \leq \text{diam}(\partial_\infty G)$ be arbitrary. Let $\lambda \geq 2$ be a large constant whose precise value we will determine later. Define $x_1 = x$. Since $\partial_\infty G$ is connected, we can find points $x_2, x_3 \in B(x, r/\lambda)$ such that

$$d(x_i, x_j) \geq r/(4\lambda) \quad \text{for } i, j = 1, 2, 3, i \neq j.$$

Since the action of G on $\partial_\infty G$ is cocompact on triples, we can find $g \in G$ such that

$$d(y_i, y_j) \geq \epsilon_0 \quad \text{for } i, j = 1, 2, 3, i \neq j,$$

where $y_i = g(x_i)$ for $i = 1, 2, 3$.

We claim that if λ is large enough, only depending on η, ϵ_0 and $\text{diam}(\partial_\infty G)$, then

$$\text{diam}(\partial_\infty G \setminus g(B)) = \text{diam}(g(\partial_\infty G \setminus B)) < \epsilon_0/2. \quad (72)$$

To find such λ let $u, v \in \partial_\infty G \setminus B$ be arbitrary. Then using the inequalities

$$d(x_1, x_3) \leq r/\lambda \leq r/2 \leq \frac{1}{2}d(u, x_1)$$

and

$$d(u, x_3) \geq d(u, x_1) - d(x_3, x_1) \geq \frac{1}{2}d(u, x_1)$$

we obtain

$$\begin{aligned} [g(x_1), g(u), g(x_3), g(v)] &\leq \eta([x_1, u, x_3, v]) \\ &= \eta\left(\frac{d(x_1, x_3)d(u, v)}{d(x_1, v)d(u, x_3)}\right) \\ &\leq \eta\left(\frac{2r}{\lambda} \cdot \frac{d(u, x_1) + d(v, x_1)}{d(v, x_1)d(u, x_1)}\right) \end{aligned}$$

$$\begin{aligned} &\leq \eta \left(\frac{2r}{\lambda} \cdot \frac{2}{d(v, x_1) \wedge d(u, x_1)} \right) \\ &\leq \eta(4/\lambda). \end{aligned}$$

On the other hand,

$$\begin{aligned} [g(x_1), g(u), g(x_3), g(v)] &= \frac{d(y_1, y_3)d(g(u), g(v))}{d(y_1, g(v))d(g(u), y_3)} \\ &\geq \frac{\epsilon_0 d(g(u), g(v))}{\text{diam}(\partial_\infty G)^2}. \end{aligned}$$

This implies that

$$\text{diam}(\partial_\infty G \setminus g(B)) = \sup_{u, v \in \partial_\infty G \setminus B} d(g(u), g(v)) \leq \frac{1}{\epsilon_0} \text{diam}(\partial_\infty G)^2 \eta(4/\lambda).$$

As $\eta(t) \rightarrow 0$ for $t \rightarrow 0$ this shows that we can indeed find $\lambda = \lambda(\epsilon_0, \eta, \text{diam}(\partial_\infty G)) \geq 2$ independent of our initial choice of B such that (72) holds.

By the remark above we can find $\delta > 0$ such that every open ball in $\partial_\infty G$ of radius $\epsilon_0/4$ contains a peripheral circle of diameter $\geq \delta$. Hence each ball $B_i = B(y_i, \epsilon_0/4)$, $i = 1, 2, 3$, contains a peripheral circle of diameter $\geq \delta$. Note that $\text{dist}(B_i, B_j) \geq d(y_i, y_j) - \epsilon_0/2 \geq \epsilon_0/2$ for $i \neq j$. Therefore, the set $\partial_\infty G \setminus g(B)$ can meet at most one of the balls, and we can pick one of them, say $B' := B_k$, where $k \in \{1, 2, 3\}$, so that $B' \cap \partial_\infty G \setminus g(B) = \emptyset$. The ball B' contains a peripheral circle J' with $\text{diam}(J') \geq \delta$. Let $J := g^{-1}(J')$. Then J is a peripheral circle with

$$J \subseteq g^{-1}(B') \subseteq g^{-1}(g(B)) = B.$$

It remains to show that J has a diameter comparable to r . To see this pick $u, v \in J$ such that

$$d(g(u), g(v)) = \text{diam}(g(J)) = \text{diam}(J') \geq \delta.$$

Two of the points x_1, x_2, x_3 must have distance $\geq r/(8\lambda)$ to u . Of these two, one must have distance $\geq r/(8\lambda)$ to v . It follows that there exist $k, l \in \{1, 2, 3\}$, $k \neq l$, such that $d(x_k, v) \geq r/(8\lambda)$ and $d(x_l, u) \geq r/(8\lambda)$. Then

$$\begin{aligned} [g(x_k), g(u), g(x_l), g(v)] &\leq \eta([x_k, u, x_l, v]) \\ &= \eta \left(\frac{d(x_k, x_l)d(u, v)}{d(x_k, v)d(u, x_l)} \right) \\ &\leq \eta \left(128\lambda \cdot \frac{d(u, v)}{r} \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} [g(x_k), g(u), g(x_l), g(v)] &= \frac{d(y_k, y_l)d(g(u), g(v))}{d(y_k, g(v))d(g(u), y_l)} \\ &\geq \frac{\epsilon_0 \delta}{\text{diam}(\partial_\infty G)^2} =: c_1 > 0. \end{aligned}$$

Hence

$$\frac{1}{r} \text{diam}(J) \geq \frac{1}{r} d(u, v) \geq \frac{1}{128\lambda} \eta^{-1}(c_1) =: c_2 > 0.$$

Since $c_2 > 0$ is a positive constant independent of the ball B , it follows that every ball B in $\partial_\infty G$ of radius $r \leq \text{diam}(\partial_\infty G)$ contains a peripheral circle of comparable size, where the constant of comparability is independent of the ball. This proves the last statement. $\square$

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