Dimension of the SLE Light Cone, the SLE Fan, and $\text{SLE}_\kappa(\rho)$ for $\kappa \in (0, 4)$ and $\rho \in [\frac{\kappa}{2} - 4, -2)$

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Abstract: Suppose that h is a Gaussian free field (GFF) on a planar domain. Fix $\kappa \in (0, 4)$. The SLE_κ light cone $\mathbf{L}(\theta)$ of h with opening angle $\theta \in [0, \pi]$ is the set of points reachable from a given boundary point by angle-varying flow lines of the (formal) vector field $e^{ih/\chi}$, $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$, with angles in $[-\frac{\theta}{2}, \frac{\theta}{2}]$. We derive the Hausdorff dimension of $\mathbf{L}(\theta)$.

If $\theta = 0$ then $\mathbf{L}(\theta)$ is an ordinary SLE_κ curve (with $\kappa < 4$); if $\theta = \pi$ then $\mathbf{L}(\theta)$ is the range of an $\text{SLE}_{\kappa'}$ curve ($\kappa' = 16/\kappa > 4$). In these extremes, this leads to a new proof of the Hausdorff dimension formula for SLE.

We also consider $\text{SLE}_\kappa(\rho)$ processes, which were originally only defined for $\rho > -2$, but which can also be defined for $\rho \leq -2$ using Lévy compensation. The range of an $\text{SLE}_\kappa(\rho)$ is qualitatively different when $\rho \leq -2$. In particular, these curves are self-intersecting for $\kappa < 4$ and double points are dense, while ordinary SLE_κ is simple. It was previously shown (Miller and Sheffield in Gaussian free field light cones and $\text{SLE}_\kappa(\rho)$, 2016) that certain $\text{SLE}_\kappa(\rho)$ curves agree in law with certain light cones. Combining this with other known results, we obtain a general formula for the Hausdorff dimension of $\text{SLE}_\kappa(\rho)$ for all values of ρ .

Finally, we show that the Hausdorff dimension of the so-called SLE_κ fan is the same as that of ordinary SLE_κ .

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1. Introduction

Suppose that h is an instance of the Gaussian free field (GFF) on a planar domain D . Although h is not a function and does not take values at points, one can still make sense of the flow lines of the (formal) vector field $e^{ih/\chi}$ where $\chi > 0$, i.e., the (formal) solutions to the equation $\eta'(t) = e^{ih(\eta(t))/\chi}$ [She16, Dub09b, MS16b, MS17]. These paths turn out to be forms of the Schramm–Loewner evolution (SLE) [Sch00].

The purpose of this work is to compute the almost sure Hausdorff dimension of certain sets which naturally fit into the imaginary geometry (i.e., SLE/GFF coupling) framework. These sets can either be described as *light cones* associated to an imaginary geometry, or as *ranges* of $\text{SLE}_\kappa(\rho)$ processes with $\rho < -2$.

Specifically, suppose that h is a GFF on the upper half plane $\mathbf{H}$ with piecewise constant boundary conditions which change values at most a finite number of times. Fix $\kappa \in (0, 4)$ and $\theta \in [0, \pi]$. The SLE_κ **light cone** $\mathbf{L}(\theta)$ of h starting from 0 is the closure of the set of points accessible by traveling along angle-varying flow lines of the (formal) vector field $e^{ih/\chi}$, $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$, starting from 0 with angles contained in $[-\frac{\theta}{2}, \frac{\theta}{2}]$. When $\theta = 0$, the light cone is equal to the range of an SLE_κ process. It is shown in [MS16b] that when $\theta = \pi$, the light cone is equal to the range of an $\text{SLE}_{16/\kappa}$ process. By varying $\theta \in (0, \pi)$, the sets $\mathbf{L}(\theta)$ continuously interpolate between the range of an SLE_κ process ($\theta = 0$) and the range of an $\text{SLE}_{16/\kappa}$ process ($\theta = \pi$) [MS16a]. See [MS16b, Section 1] for simulations of the light cone.

Let $\dim_{\mathcal{H}}(A)$ denote the Hausdorff dimension of a set A . The purpose of this work is to compute the almost sure value of $\dim_{\mathcal{H}}(\mathbf{L}(\theta))$. Throughout, we write

$$d(\kappa, \theta) = \frac{(\kappa(1 - \bar{\theta}) + 4\bar{\theta})(\kappa + 8 + (\kappa - 4)\bar{\theta})}{8\kappa} \quad \text{where} \quad \bar{\theta} = \frac{\theta}{\pi}. \quad (1.1)$$

The main result of this article is the following:

Theorem 1.1. *Suppose that h is a GFF on $\mathbf{H}$ with piecewise constant boundary data which changes values at most a finite number of times. Let $\mathbf{L}(\theta)$ be the SLE_κ light cone ($\kappa \in (0, 4)$) of h starting from 0 with opening angle $\theta \in [0, \pi]$ and assume that the boundary data of h is such that $\mathbf{P}[\mathbf{L}(\theta) \neq \emptyset] = 1$. Almost surely,*

$$\dim_{\mathcal{H}}(\mathbf{L}(\theta)) = d(\kappa, \theta) \wedge 2. \quad (1.2)$$

The angle $\theta_c = \theta_c(\kappa)$ which solves $d(\kappa, \theta) = 2$ is given by

$$\theta_c = \frac{\pi\kappa}{4 - \kappa}. \quad (1.3)$$

We note that θ_c is equal to the so-called **critical angle** introduced in [MS16b, MS17]. Two GFF flow lines—with a common starting point and a given angle difference—intersect

each other away from the starting point if and only if the angle difference is less than or equal to this critical angle (see [MS16b, Theorem 1.5]). Note that $\theta_c \in (0, \pi)$ for $\kappa \in (0, 2)$, $\theta_c = \pi$ for $\kappa = 2$, and $\theta_c > \pi$ for $\kappa \in (2, 4)$. Since we only define light cones $\mathbf{L}(\theta)$ for $\theta \in [0, \pi]$, this implies that SLE_κ light cones can be space-filling if and only if $\kappa \in (0, 2]$. This corresponds to the fact that $\text{SLE}_{16/\kappa}$ is space-filling if and only if $\kappa \in (0, 2]$ [RS05]. We will provide additional explanation in Remark 3.2 for why θ_c naturally appears in Theorem 1.1.

The $\text{SLE}_\kappa(\rho)$ processes, first introduced in [LSW03, Section 8.3], are an important variant of SLE_κ in which one keeps track of an extra marked point V called a **force point** in addition to the Loewner driving function W . (See Sect. 2.1.) The force point can be located either in the interior of the domain or on its boundary. Throughout this article, we will primarily restrict ourselves to the case in which the force point is on the boundary of the upper half plane, so that V is a real-valued process like W .

The parameter ρ determines the strength of the “interaction” between W and V . When $\rho = 0$, $\text{SLE}_\kappa(\rho)$ is the same as ordinary SLE_κ . When $\rho > 0$ (resp. $\rho < 0$), W is pushed away from (resp. pulled towards) V . Like ordinary SLE_κ , the $\text{SLE}_\kappa(\rho)$ processes are described in terms of the Loewner evolution driven by W . However, the law of W is different, and is determined by the fact that $V - W$ is a positive multiple of a Bessel process whose dimension depends on both κ and ρ and is explicitly given by $\delta(\kappa, \rho) = 1 + \frac{2(\rho+2)}{\kappa}$, see Sect. 2.1.

Remark 1.2. There are variants of $\text{SLE}_\kappa(\rho)$ in which the *sign* of each excursion of $V - W$ away from zero is chosen independently at random with a fixed biased coin; but throughout this paper we will always assume that the sign of $V - W$ is the same for all excursions—in other words, in this paper we consider only *one-sided* $\text{SLE}_\kappa(\rho)$ and not *side-swapping* $\text{SLE}_\kappa(\rho)$.

In order to define the process for all time (as opposed to having it stop when V and W first collide), most treatments of $\text{SLE}_\kappa(\rho)$ (including [LSW03]) require that $\rho > -2$, so that $\delta = \delta(\kappa, \rho) > 1$. But the processes with $\rho \leq -2$ can also be defined (using an appropriate Lévy compensation) and are also important. As explained in [MSW17], when $\kappa \in (2, 4)$ there are certain ranges of ρ values for which $\text{SLE}_\kappa(\rho)$ can be described as the concatenation of a countable collection of SLE_κ loops, all attached to an $\text{SLE}_{\kappa'}$ “trunk” ($\kappa' = 16/\kappa > 4$) and in these cases the dimension of the whole range of the path is the dimension of the trunk, namely $(1 + \kappa'/8) \wedge 2$. As explained in [MS16a], there are other values of ρ such that the range of an $\text{SLE}_\kappa(\rho)$ process agrees in law with a light cone $\mathbf{L}(\theta)$ (defined from a GFF with particular boundary values) where the relationship between ρ and θ is given by the formula

$$\theta = \theta_\rho = \pi \left(\frac{\rho + 2}{\frac{\kappa}{2} - 2} \right). \quad (1.4)$$

Table 1 presents a phase diagram for the ρ values and the corresponding Bessel process dimensions $\delta(\kappa, \rho)$ for $\text{SLE}_\kappa(\rho)$ processes with $\kappa \in (0, 4)$, see also Fig. 1. The dimensions in the table are obtained by combining Theorem 1.1 with the main result of [MS16a]. Let us state this as a theorem:

Theorem 1.3. *Suppose $\kappa \in (0, 4)$ and that η is an $\text{SLE}_\kappa(\rho)$ process with $\rho \in ((\frac{\kappa}{2} - 4) \vee (-2 - \frac{\kappa}{2}), -2)$. (These ρ values correspond to the light cone phase described in Table 1 and its caption.) Then almost surely,*

$$\dim_{\mathcal{H}}(\eta) = \frac{(\kappa - 2(2 + \rho))(\kappa + 2(6 + \rho))}{8\kappa}. \quad (1.5)$$

Table 1. Phases of ρ values and corresponding $\delta(\kappa, \rho)$ (driving Bessel process dimension) values for $\text{SLE}_\kappa(\rho)$ processes with a single boundary force point of weight ρ , assuming $\kappa \in (2, 4)$. When $\kappa \in (0, 2]$, the phase diagram is the same except that the second two phases are replaced by a single “light cone” phase with $\rho \in (-2 - \kappa/2, -2)$ and $\delta \in (0, 1)$. When the dimension value shown in the table is greater than or equal to 2, the curve is space-filling, so the actual dimension is 2. See Fig. 1

ρ	$\delta(\kappa, \rho)$	$\dim \mathcal{H}_t(\text{Range})$	Process type	Simple
$(-\infty, -2 - \frac{\kappa}{2}]$	$(-\infty, 0]$	—	Not defined	—
$(-2 - \frac{\kappa}{2}, \frac{\kappa}{2} - 4]$	$(0, 2 - \frac{4}{\kappa}]$	$1 + \frac{2}{\kappa} = 1 + \frac{\kappa'}{8}$	Trunk plus loops	X
$(\frac{\kappa}{2} - 4, -2)$	$(2 - \frac{4}{\kappa}, 1)$	$\frac{(\kappa - 2)(2 + \rho)(\kappa + 2)(6 + \rho))}{8\kappa}$	Light cone	X
-2	1	1	Boundary tracing	✓
$(-2, \frac{\kappa}{2} - 2)$	$(1, 2)$	$1 + \frac{\kappa}{8}$	Boundary hitting	✓
$[\frac{\kappa}{2} - 2, \infty)$	$[2, \infty)$	$1 + \frac{\kappa}{8}$	Boundary avoiding	✓

Fig. 1. $\text{SLE}_\kappa(\rho)$ phases from Table 1. This paper computes dimensions within the yellow “light cone” region described in more detail in [MS16a]. As one approaches the $\rho = -2$ upper boundary from below, the curve converges in law to a boundary-tracing curve, but the dimension converges to $1 + \kappa/8$ (which is also the dimension above the $\rho = -2$ line). As one approaches the lower right edge of the yellow triangle from above, θ tends to π , the range of the path (though not the path itself) converges in law to that of an $\text{SLE}_{\kappa'}$ type curve, and the dimension converges to $1 + \kappa'/8$, which is also the dimension throughout the blue triangle. As one approaches the lower left edge of the yellow triangle from above, θ tends to θ_c , the corresponding $\text{SLE}_\kappa(\rho)$ curves becomes space-filling, and the dimension tends to 2. In short, as one approaches the yellow three triangle edges (clockwise from the top, respectively) the dimensions converge to $1 + \kappa/8$ and $1 + \kappa'/8$ and 2

The almost sure Hausdorff dimension for ordinary SLE_κ is given by $1 + \frac{\kappa}{8}$ for $\kappa \in (0, 8)$ and by 2 for $\kappa \geq 8$. The upper bound for this result was first obtained by Rohde and Schramm [RS05] and the lower bound was established by Beffara [Bef08]. Now suppose that η is the trace of an $\text{SLE}_\kappa(\rho)$ process η with driving function W and force point process V . By the Girsanov theorem [RY99], the evolution of an $\text{SLE}_\kappa(\rho)$ process η —started at a time when $V_t \neq W_t$ and stopped at a time before V and W collide—has a law that is absolutely continuous with respect to that of an ordinary SLE_κ process restricted to the same time interval. From this it is easy to show that the dimension of $\eta(\{t : V_t \neq W_t\})$ is a.s. the same as the dimension of ordinary SLE_κ .

But what about the set $\eta(\{t : V_t = W_t\})$? When $\rho > -2$, the times when $V_t = W_t$ correspond to times when η is hitting the boundary, so this set is a subset of $\mathbf{R}$.

Consequently, $\dim_{\mathcal{H}}(\eta)$ in this case is the same as the dimension of ordinary SLE_κ because we trivially have that $\dim_{\mathcal{H}}(\eta \cap \mathbf{R}) \leq 1 \leq 1 + \frac{\kappa}{8}$. (The almost sure value of $\dim_{\mathcal{H}}(\eta \cap \mathbf{R})$ as a function of ρ and κ is given in [MW17, Theorem 1.6].) For $\rho \in [\frac{\kappa}{2} - 4, -2)$, the problem is more interesting because the set $\eta(\{t : V_t = W_t\})$ includes points in the interior of the domain. In fact, Theorem 1.3 implies that the dimension of this set is strictly larger than the dimension of $\eta(\{t : V_t \neq W_t\})$.

In [MSW17], it is shown that the same is true for $\rho \in (-2 - \frac{\kappa}{2}, \frac{\kappa}{2} - 4]$. In this case, the dimension of the range turns out to be $1 + \frac{2}{\kappa}$, the same as the dimension of an $\text{SLE}_{\kappa'}$ process, for all $\rho \in (-2 - \frac{\kappa}{2}, \frac{\kappa}{2} - 4]$. Together with this work, this covers the entire range of possible ρ values.

Theorem 1.3 also implies that the dimension of an $\text{SLE}_\kappa(\rho)$ process, $\kappa \in (0, 4)$, continuously interpolates between that of ordinary SLE_κ and that of $\text{SLE}_{\kappa'}$.

The method we use to derive the so-called one point estimate (the exponent for the probability that the path gets within distance $\epsilon > 0$ of a given point as $\epsilon \rightarrow 0$) which, in turn, leads to the upper bounds in Theorems 1.1 and 1.3, is rather different in spirit from the method used by Rohde and Schramm [RS05] to derive the corresponding one point estimate for ordinary SLE_κ . The strategy employed in [RS05] is to try to find a martingale which becomes large on the event that an SLE_κ process gets close to a given point. This leads one to derive and solve a certain PDE. In the setting of an $\text{SLE}_\kappa(\rho)$ process with $\rho < -2$, extending this method seems to be technically challenging because the presence of the force point introduces a second spatial variable into the corresponding PDE and, as we remarked earlier, one cannot use absolute continuity to compare to ordinary SLE_κ . To circumvent this difficulty, in the present article we will relate the event that $\mathbf{L}(\theta)$ (or an $\text{SLE}_\kappa(\rho)$ process with $\rho \in [\frac{\kappa}{2} - 4, -2)$) gets close to a given point to the local structure of the flow lines of the GFF starting from that point. One of the highlights of this approach is that it is conceptual in nature rather than computational. The basic idea is illustrated in more detail in Figs. 5, 6 and 7. We will then use the martingales from [SW05] to estimate the probability that the local structure of the flow lines at a given point exhibits the necessary behavior for $\mathbf{L}(\theta)$ to hit.

The lower bound is proved by relating the correlation structure of the points in $\mathbf{L}(\theta)$ to the correlation structure of the values of h . Roughly speaking, the approximate “tree structure” used in the lower bound arises because the collection of flow lines of h with a common angle themselves form a tree (see [MS16b, Theorem 1.5 and Figure 1.7] as well as [MS17, Figures 1.4–1.6]). Since $\mathbf{L}(0)$ is equal to the range of an SLE_κ process for $\kappa \in (0, 4)$ and $\mathbf{L}(\pi)$ is equal to the range of an $\text{SLE}_{\kappa'}$ process, we obtain as a special case of Theorem 1.1 the almost sure dimension of ordinary SLE. We remark that this is not the first article in which the imaginary geometry framework is used to compute dimensions related to SLE: it is used in [MW17] to derive the cut point, double point, and other dimensions associated with intersection sets of SLE paths; in [GMS14] to derive the almost sure multifractal spectrum of SLE; and in [GHM15, GHM16] to derive certain KPZ-type formulas (using also the tools of [DMS14]).

Fix $\theta \in [0, \pi]$. The SLE_κ fan $\mathbf{F}(\theta)$ is the set of points accessible by flow lines of h starting from 0 with fixed angles in $[-\frac{\theta}{2}, \frac{\theta}{2}]$. (This is in contrast to the paths which generate $\mathbf{L}(\theta)$, since they are allowed to change angles.) See Fig. 2 for a numerical simulation of $\mathbf{F}(\pi)$ for $\kappa = \frac{1}{4}$. Obviously, $\dim_{\mathcal{H}}(\mathbf{F}(\theta)) \geq 1 + \frac{\kappa}{8}$ because $\mathbf{F}(\theta)$ contains the range of the 0 angle flow line of h which is itself an SLE_κ process (or possibly an $\text{SLE}_\kappa(\rho)$ process depending on the boundary data of h). It was shown in [MS16b] that the Lebesgue measure of $\mathbf{F}(\theta)$ is almost surely zero. Our final result gives that $\dim_{\mathcal{H}}(\mathbf{F}(\theta)) = 1 + \frac{\kappa}{8}$:

Fig. 2. Numerically generated flow lines, started at $-i$ of $e^{i(h/\chi+\theta)}$ where h is the projection of a GFF on $[-1, 1]^2$ onto the space of functions piecewise linear on the triangles of a 300×300 grid; $\kappa = 1/4$. Different colors indicate different values of $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. The boundary data for h is chosen so that the central (“north-going”) curve shown should approximate an $\text{SLE}_{1/4}$ process. The other paths should approximate $\text{SLE}_{1/4}(\rho_1; \rho_2)$ processes where the values of $\rho_1, \rho_2 > -2$ are a function of θ . We prove in Theorem 1.4 that the almost sure Hausdorff dimension of the entire set shown is equal to the dimension of each of the individual paths

Theorem 1.4. *For each $\kappa \in (0, 4)$ and $\theta \in [0, \pi]$, the almost sure Hausdorff dimension of the SLE_κ fan $\mathbf{F}(\theta)$ is $1 + \frac{\kappa}{8}$ (assuming that the boundary data of h is such that $\mathbf{P}[\mathbf{F}(\theta) \neq \emptyset] = 1$), the same as that of ordinary SLE_κ .*

The reader might find Theorem 1.4 surprising because $\mathbf{F}(\theta)$ consists of *many* SLE_κ paths and one might suspect that the limit points of these paths would make the dimension strictly larger than that of a single SLE_κ path. Theorem 1.4, however, implies that this is not the case. Another reason that the reader may find the result to be surprising is that the numerical simulations [MS16b, Figures 1.3–1.5] suggest that as $\kappa \downarrow 0$ (but for a fixed value of $\theta \in (0, \pi)$), $\mathbf{F}(\theta)$ converges almost surely in the Hausdorff topology to a *two-dimensional* set. However, Theorem 1.4 implies that $\dim_{\mathcal{H}}(\mathbf{F}(\theta))$ almost surely converges to 1 as $\kappa \downarrow 0$.

In the proofs of this work, we will assume that the reader has some familiarity with imaginary geometry as presented in [MS16b, MS16c, MS16d, MS17] (though we will provide a reminder of the basic facts in Sect. 2.4). We will in particular make use of the notation introduced in [MS16b, Figure 1.10]. Throughout, we assume that $\kappa \in (0, 4)$, $\kappa' = 16/\kappa \in (4, \infty)$, and let

$$\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}, \quad \lambda = \frac{\pi}{\sqrt{\kappa}}, \quad \text{and} \quad \lambda' = \frac{\pi}{\sqrt{\kappa'}} = \lambda - \frac{\pi}{2}\chi. \quad (1.6)$$

We will also use η to refer to an SLE_κ process and η' to refer to an $\text{SLE}_{\kappa'}$ process.

Outline. The remainder of this article is structured as follows. In Sect. 2, we will collect several estimates which are used throughout this article as well as give a brief review of the results from [MS16b, MS16c, MS16d, MS17] which will be used in this article. Next, in Sect. 3 we will prove the upper bound for Theorem 1.1, hence also Theorem 1.3, and complete the proof of Theorem 1.4. Finally, in Sect. 4 we will complete the proof of the lower bound for Theorem 1.1 hence also Theorem 1.3.

2. Preliminaries

2.1. $\text{SLE}_\kappa(\rho)$ processes. We will now give a very brief introduction to SLE. More detailed introductions can be found in many excellent surveys of the subject, e.g., [Wer04, Law05]. Chordal SLE_κ in $\mathbf{H}$ from 0 to ∞ is defined by the random family of conformal maps (g_t) obtained by solving the Loewner ODE

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z \quad (2.1)$$

with $W = \sqrt{\kappa}B$ and B a standard Brownian motion. Write $K_t := \{z \in \mathbf{H} : \tau(z) \leq t\}$ where $\tau(z)$ is the swallowing time of z defined by $\sup\{t \geq 0 : \inf_{s \in [0, t]} |g_s(z) - W_s| > 0\}$. Then g_t is the unique conformal map from $\mathbf{H}_t := \mathbf{H} \setminus K_t$ to $\mathbf{H}$ satisfying $\lim_{|z| \rightarrow \infty} |g_t(z) - z| = 0$.

Rohde and Schramm [RS05] showed that there almost surely exists a curve η (the so-called **SLE trace**) such that for each $t \geq 0$ the domain $\mathbf{H}_t$ of g_t is the unbounded connected component of $\mathbf{H} \setminus \eta([0, t])$, in which case the (necessarily simply connected and closed) set K_t is called the “filling” of $\eta([0, t])$ [RS05]. An SLE_κ connecting boundary points x and y of an arbitrary simply connected Jordan domain can be constructed as the image of an SLE_κ on $\mathbf{H}$ under a conformal transformation $\varphi: \mathbf{H} \rightarrow D$ sending 0 to x and ∞ to y . (The choice of φ does not affect the law of this image path, since the law of SLE_κ on $\mathbf{H}$ is scale invariant.) For $\kappa \in [0, 4]$, SLE_κ is simple and, for $\kappa > 4$, SLE_κ is self-intersecting [RS05]. The dimension of the path is $1 + \frac{\kappa}{8}$ for $\kappa \in [0, 8]$ and 2 for $\kappa > 8$ [Bef08].

An $\text{SLE}_\kappa(\underline{\rho}_L; \underline{\rho}_R)$ process is a generalization of SLE_κ in which one keeps track of additional marked points which are called **force points**. These processes were first introduced in [LSW03, Section 8.3]. Fix $\underline{x}_L = (x_{\ell,L} < \cdots < x_{1,L} \leq 0)$ and $\underline{x}_R = (0 \leq x_{1,R} < \cdots < x_{r,R})$. We associate with each $x_{i,q}$ for $q \in \{L, R\}$ a weight $\rho_{i,q} \in \mathbf{R}$. An $\text{SLE}_\kappa(\underline{\rho}_L; \underline{\rho}_R)$ process with force points $(\underline{x}_L; \underline{x}_R)$ is the measure on continuously growing compact hulls K_t generated by the Loewner chain with W_t replaced by the solution to the system of SDEs:

$$\begin{aligned} dW_t &= \sum_{i=1}^{\ell} \frac{\rho_{i,L}}{W_t - V_t^{i,L}} dt + \sum_{i=1}^r \frac{\rho_{i,R}}{W_t - V_t^{i,R}} dt + \sqrt{\kappa} dB_t, \\ dV_t^{i,q} &= \frac{2}{V_t^{i,q} - W_t} dt, \quad V_0^{i,q} = x_{i,q}, \quad i \in \mathbf{N}, \quad q \in \{L, R\}. \end{aligned} \quad (2.2)$$

It is explained in [MS16b, Section 2] that for all $\kappa > 0$, there is a unique solution to (2.2) up until the **continuation threshold** is hit—the first time t for which either

$$\sum_{i: V_t^{i,L} = W_t} \rho_{i,L} \leq -2 \quad \text{or} \quad \sum_{i: V_t^{i,R} = W_t} \rho_{i,R} \leq -2.$$

In the case of a single boundary force point, the existence of a unique solution to (2.2) can be derived by relating $W_t - V_t$ to a Bessel process; see [She09]. The almost sure continuity of the $\text{SLE}_\kappa(\rho)$ processes, up until the continuation threshold is reached, is proved in [MS16b, Theorem 1.3]. It is possible to make sense of the solution to (2.2) even after the continuation threshold is reached. These processes are analyzed and shown to be continuous in [MS16a, MSW17] (see also [She09]).

2.2. Radon–Nikodym derivatives. Let $c = (D, z_0, \underline{x}_L, \underline{x}_R, z_\infty)$ be a configuration consisting of a Jordan domain D in $\mathbf{C}$ with $\ell + r + 2$ marked points on ∂D . An $\text{SLE}_\kappa(\underline{\rho}_L; \underline{\rho}_R)$ process η with configuration c is given by the image of an $\text{SLE}_\kappa(\underline{\rho}_L; \underline{\rho}_R)$ process $\tilde{\eta}$ in $\mathbf{H}$ which takes the force points of $\tilde{\eta}$ to those of η . Suppose that $\tilde{c} = (\tilde{D}, z_0, \tilde{\underline{x}}_L, \tilde{\underline{x}}_R, z_\infty)$ and $c = (D, z_0, \underline{x}_L, \underline{x}_R, z_\infty)$ are two configurations such that $\tilde{D}$ agrees with D in a neighborhood U of z_0 . Let μ_c^U denote the law of an $\text{SLE}_\kappa(\underline{\rho}_L; \underline{\rho}_R)$ process in c stopped at the first time τ that it exits U and define $\mu_{\tilde{c}}^U$ analogously. The following estimate is a restatement of [MW17, Lemma 2.8] which, in turn, is based on extending [Dub09a, Lemma 13] to the setting of boundary-intersecting $\text{SLE}_\kappa(\underline{\rho})$ processes using the SLE/GFF coupling.

Lemma 2.1. *Assume that we have the setup described just above where $D = \mathbf{H}$, $\tilde{D} \subseteq \mathbf{H}$, $U \subseteq \mathbf{H}$ is bounded, and $z_0 = 0$. Fix $\zeta > 0$ and suppose that the distance between U and $\mathbf{H} \setminus \tilde{D}$ is at least ζ , the force points of $c, \tilde{c}$ in $\bar{U}$ are identical, the corresponding weights are also equal, and the force points which are outside of U are at distance at least ζ from U . There exists a constant $C \geq 1$ depending on U, ζ, κ , and the weights of the force points such that*

$$\frac{1}{C} \leq \frac{d\mu_c^U}{d\mu_{\tilde{c}}^U} \leq C.$$

Proof. See [MW17, Lemmas 2.7 and 2.8] as well as [Dub09a, Lemma 13]. $\square$

2.3. Estimates for conformal maps. Throughout, we will make frequent use of the following three estimates for conformal maps. The first is [Law05, Corollary 3.18]:

Lemma 2.2. *Suppose that $f: D \rightarrow D'$ is a conformal transformation with $f(z) = z'$. Then*

$$\frac{d'}{4d} \leq |f'(z)| \leq \frac{4d'}{d}$$

where $d = \text{dist}(z, \partial D)$ and $d' = \text{dist}(z', \partial D')$.

The second is [Law05, Corollary 3.23]:

Lemma 2.3. *Suppose that $f: D \rightarrow D'$ is a conformal transformation with $f(z) = z'$. For all $r \in (0, 1)$ and all $|w - z| \leq r \times \text{dist}(z, \partial D)$, we have that*

$$|f(w) - z'| \leq \frac{4|w - z|}{1 - r^2} \times \frac{\text{dist}(z', \partial D')}{\text{dist}(z, \partial D)}.$$

Finally, we state the Beurling estimate [Law05, Theorem 3.76] which we will frequently use in conjunction with the conformal invariance of Brownian motion.

Theorem 2.4. (Beurling Estimate) *Suppose that B is a Brownian motion in $\mathbf{C}$ and $\tau_{\mathbf{D}} = \inf\{t \geq 0 : B(t) \in \partial \mathbf{D}\}$. There exists a constant $c < \infty$ such that if $\gamma: [0, 1] \rightarrow \mathbf{C}$ is a curve with $\gamma(0) = 0$ and $|\gamma(1)| = 1$, $z \in \mathbf{D}$, and $\mathbf{P}^z$ is the law of B when started at z , then*

$$\mathbf{P}^z[B([0, \tau_{\mathbf{D}}]) \cap \gamma([0, 1]) = \emptyset] \leq c|z|^{1/2}.$$

2.4. Imaginary geometry review. Throughout this work, we assume that the reader is familiar with the GFF as well as with imaginary geometry. We refer the reader to [She07] for a more in-depth introduction to the former and to [MS16b] for more on the latter. For the convenience of the reader, we will review some of the results from [MS16b] which will be used repeatedly throughout the present work.

We begin by describing the coupling of the $\text{SLE}_\kappa(\rho)$ processes as flow lines of the GFF. Fix $\kappa \in (0, 4)$ and let $\kappa' = \frac{16}{\kappa} \in (4, \infty)$. Recall the constants $\lambda = \frac{\pi}{\sqrt{\kappa}}$, $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$, and $\lambda' = \frac{\pi}{\sqrt{\kappa'}} = \lambda - \frac{\pi}{2}\chi$ as defined in (1.6).

Suppose that η is an $\text{SLE}_\kappa(\rho)$ process in $\mathbf{H}$ from 0 to ∞ . Then η can be coupled with a GFF h on $\mathbf{H}$ so that it may be interpreted as a flow line of the (formal) vector field $e^{ih/\chi}$. The boundary data for h is given by

$$-\lambda \left(1 + \sum_{i=0}^j \rho_{i,L} \right) \quad \text{on } (x_{j+1,L}, x_{j,L}] \quad \text{and} \quad \lambda \left(1 + \sum_{i=0}^j \rho_{i,R} \right) \quad \text{on } (x_{j,R}, x_{j+1,R}].$$

Here, we have taken $\rho_{0,L} = \rho_{0,R} = 0$, $x_{\ell+1,L} = -\infty$, and $x_{r+1,R} = +\infty$. On the left (resp. right) hand side, j varies between 0 and ℓ (resp. r). If (g_t) denotes the Loewner evolution associated with η and $V_t^{i,q}$ denotes the evolution of the force points, then we have for each stopping time τ for η which is almost surely finite and before the continuation threshold is hit that $h \circ g_\tau^{-1} - \chi \arg(g_\tau^{-1})'$ is a GFF on $\mathbf{H}$ with boundary conditions given by

$$-\lambda \left(1 + \sum_{i=0}^j \rho_{i,L} \right) \quad \text{on } (V_\tau^{j+1,L}, V_\tau^{j,L}] \quad \text{and} \quad \lambda \left(1 + \sum_{i=0}^j \rho_{i,R} \right) \quad \text{on } (V_\tau^{j,R}, V_\tau^{j+1,R}].$$

Here, we take $V_\tau^{0,L} = V_\tau^{0,R} = W_\tau$, $V_\tau^{\ell+1,L} = -\infty$, and $V_\tau^{r+1,R} = +\infty$. As before, on the left (resp. right) hand side, j varies between 0 and ℓ (resp. r).

See Fig. 3 for an illustration in the case that $\ell = r = 1$.

If D is a simply connected domain and $x, y \in \partial D$ are distinct, then one can also realize the flow line of a GFF h on D as an $\text{SLE}_\kappa(\rho)$ process from x to y provided one chooses the boundary data for h appropriately. Namely, one needs to take

Fig. 3. An illustration of the coupling of an $\text{SLE}_\kappa(\rho)$ process as the flow line of a GFF h on $\mathbf{H}$ from 0 to ∞ . In this case, $\ell = r = 1$ (i.e., the process has one force point on each side of the seed). The conformal map $f_\tau : \mathbf{H} \setminus \eta([0, \tau]) \rightarrow \mathbf{H}$ shown is given by $f_\tau = g_\tau - W_\tau$

Fig. 4. Illustration of a counterflow line η' and a flow line η with angle θ on the strip $\mathbf{R} \times [0, \pi]$. In this case, η' is an $\text{SLE}_{\kappa'}(\rho'_L; \rho'_R)$ process with $\rho'_L = \frac{b}{\lambda'} - 1$ and $\rho'_R = \frac{a}{\lambda'} - 1$ and η is an $\text{SLE}_{\kappa}(\rho_L; \rho_R)$ with $\rho_L = \frac{a-\theta\chi}{\lambda} + \frac{\kappa}{2} - 3$ and $\rho_R = \frac{b+\theta\chi}{\lambda} + \frac{\kappa}{2} - 3$. If $\theta = -\frac{\pi}{2}$, then η is equal to the right boundary of η' and if $\theta = \frac{\pi}{2}$, then η is equal to the left boundary of η' . For all intermediate values of $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, η is contained in the range of η' and η' visits the range of η in the opposite order in which the points are drawn by η . If $\theta \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$, then η is either to the left or right of η'

$$h = \tilde{h} \circ \varphi^{-1} - \chi \arg(\varphi^{-1})' \quad (2.3)$$

where $\varphi: \mathbf{H} \rightarrow D$ is a conformal transformation with $\varphi(0) = x$ and $\varphi(x) = \infty$ and $\tilde{h}$ is a GFF on $\mathbf{H}$ whose flow line from 0 to ∞ is an $\text{SLE}_{\kappa}(\rho)$ process.

The formula (2.3) is the change of coordinates formula for imaginary geometry.

One can also consider flow lines of a GFF with different angles. More specifically, the flow line of a GFF h with angle θ is the $\text{SLE}_{\kappa}(\rho)$ process coupled with the field $h + \theta\chi$. It has the interpretation as being the flow line of the (formal) vector field $e^{i(h/\chi + \theta)}$ (i.e., where all of the arrows have been rotated by the angle θ) (Fig. 4).

In [MS16b], it is described how flow lines with different angles and starting points interact with each other. In particular, if η_1, η_2 are flow lines of a GFF h on $\mathbf{H}$ starting from $x_1 \leq x_2$ with angles θ_1, θ_2 , then [MS16b, Theorem 1.5] implies that:

- η_1 stays to the left of η_2 if $\theta_1 > \theta_2$. If $\theta_1 \in (\theta_2, \theta_2 + 2\lambda'/\chi)$, then η_1 can intersect and bounce off η_2 . If $\theta_1 \geq \theta_2 + 2\lambda'/\chi$, then η_1 and η_2 do not intersect. (Note that $2\lambda'/\chi = \pi\kappa/(4 - \kappa)$.)
- η_1 merges with η_2 (and the paths do not subsequently separate) upon their first intersection if $\theta_1 = \theta_2$.
- η_1 crosses (and does not cross back but may bounce off) η_2 from left to right upon intersecting if $\theta_1 \in (\theta_2 - \pi, \theta_2)$.

Thus to determine the manner in which flow lines interact with each other, one needs to compute the difference between their angles and then check in which of the aforementioned three ranges the difference falls into.

An $\text{SLE}_{\kappa'}(\rho')$ process η' can similarly be coupled with a GFF h on $\mathbf{H}$. In this case, the boundary data is given by

$$\lambda' \left(1 + \sum_{i=0}^j \rho'_{i,L} \right) \quad \text{on } (x_{j+1,L}, x_{j,L}] \quad \text{and} \quad -\lambda' \left(1 + \sum_{i=0}^j \rho'_{i,R} \right) \quad \text{on } (x_{j,R}, x_{j+1,R}].$$

Such an $\text{SLE}_{\kappa'}(\rho')$ process is referred to as a **counterflow line** of h , the reason being that it can be realized as a tree of flow lines. One makes sense of $\text{SLE}_{\kappa'}(\rho')$ processes coupled

with GFFs on other domains D as counterflow lines using the change of coordinates formula (2.3). Here, $\chi = \frac{\sqrt{\kappa'}}{2} - \frac{2}{\sqrt{\kappa'}}$. Note that this is the same as the value of χ associated with $\kappa = \frac{16}{\kappa'}$. It is often convenient to apply the change of coordinates $z \mapsto -1/z$ so that the counterflow line grows from ∞ to 0. In this case, the left (resp. right) boundary of η' is given by the flow line of h from 0 to ∞ with angle $\frac{\pi}{2}$ (resp. $-\frac{\pi}{2}$) [MS16b, Theorem 1.4]. More generally, it follows from [MS16b, Theorem 1.4] that the entire range of η' can be realized as the light cone of flow lines starting from 0 which are allowed to change angles but with angle always constrained to be in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

When illustrating GFF flow lines, it is often convenient to use the notation $\mathfrak{X}$ to indicate the boundary data for the GFF. It is used to indicate the boundary data along a flat segment of the domain boundary and means that the boundary data changes according to χ times its winding relative to the flat part. This notation is described in detail in [MS16b, Figure 1.10].

3. Upper Bound

In this section, we will prove the upper bound of Theorems 1.1 and 1.3. We will then explain how to extract Theorem 1.4 (in its entirety) from the upper bound. We will begin in Sect. 3.1 by recording an estimate of the moments of the derivative of the Loewner map when an SLE_κ process gets close to a given point (Proposition 3.3). Next, in Sect. 3.2 we will derive the exponent for the probability that two flow lines of the GFF (with a particular choice of boundary data) starting from $\pm\frac{1}{2}\epsilon$ do not intersect before hitting $\partial\mathbf{D}$ as $\epsilon \rightarrow 0$ (Lemma 3.5). We will then combine these results to establish the upper bounds for the dimensions in Sect. 3.3.

The main result of this section is the following proposition.

Proposition 3.1. *Suppose that h is a GFF on $\mathbf{H}$ with piecewise constant boundary data which changes values at most a finite number of times. Let $\mathbf{L}(\theta)$ be the SLE_κ light cone ($\kappa \in (0, 4)$) of h starting from 0 with opening angle $\theta \in [0, \pi]$. Almost surely,*

$$\dim_{\mathcal{H}}(\mathbf{L}(\theta)) \leq d(\kappa, \theta) \wedge 2$$

where $d(\kappa, \theta)$ is as in (1.1). In particular, the dimension of an $\text{SLE}_\kappa(\rho)$ process with $\rho \in ((\frac{\kappa}{2} - 4) \vee (-2 - \frac{\kappa}{2}), -2)$ is bounded from above by the expression in (1.5).

We are now going to give an overview of how the estimates proved in this section will be used to establish Proposition 3.1. Fix $\kappa \in (0, 4)$ and $\theta \in [0, \pi]$. If $d(\kappa, \theta) \geq 2$, then the upper bound given is trivially true. Consequently, we may assume without loss of generality that $\theta \in [0, \pi]$ is such that $d(\kappa, \theta) < 2$. We are going to prove the result by combining Proposition 3.3 with Lemma 3.5. For the proof of Proposition 3.1, it will be more convenient to perform a change of coordinates which swaps 0 and ∞ so that $\mathbf{L}(\theta)$ grows from ∞ towards 0 rather than from 0 towards ∞ . By the absolute continuity properties of the GFF [MS16b, Proposition 3.2], we may assume without loss of generality that the boundary data for h is as described in the left side of Fig. 7. Let η' be the counterflow line of h from 0 to ∞ , (g_t) its chordal Loewner evolution, W its Loewner driving function, and let $f_t = g_t - W_t$ be its centered Loewner evolution. It is explained in Figs. 5, 6 and 7 that in order for $\mathbf{L}(\theta)$ to get within distance ϵ of a given point $z \in \mathbf{H}$, it must be that η' gets within distance ϵ of z and the flow lines η^L and η^R with angles $\frac{\theta}{2}$ and $-\frac{\theta}{2}$, respectively, starting near the tip of η' do not intersect each other. As explained in Fig. 7, the exponent for this probability can be estimated

Fig. 5. Suppose that h is a GFF on the strip $\mathcal{S} = \mathbf{R} \times [0, 1]$ with the illustrated boundary data. Then the counterflow line η' of h starting from i is an $\text{SLE}_{\kappa'}$ process and the flow line starting from 0 is an $\text{SLE}_{\kappa}(\frac{3\kappa}{4} - 3; \frac{3\kappa}{4} - 3)$ process. (Note that η hits both $\mathbf{R}_-$ and $\mathbf{R}_+$ since $\frac{3\kappa}{4} - 3 < \frac{\kappa}{2} - 2$ for all $\kappa \in (0, 4)$.) Since the range of η' almost surely contains the range of η , in order for η to hit a given point z , it must be that η' hits z . Moreover, on the event that η hits z , the flow lines η^L and η^R starting from z with angles π and $-\pi$, respectively, stay to the left and right of η , respectively, and the left side of η^L does not intersect the right side of η^R . The converse statement also holds: if η' hits a given point z and the left side of the flow line η^L starting from z with angle π does not intersect the right side of the flow line η^R starting from z with angle $-\pi$, then η must hit z because η cannot hit the left (resp. right) side of η^L (resp. η^R)

Fig. 6. (Continuation of Fig. 5.) In order for the light cone $\mathbf{L}(\theta)$ of h starting from 0 with opening angle $\theta \in [0, \pi]$ to hit point z , it must be that η' hits z and that the left side of the flow line η^L starting from z with angle $-\frac{\theta}{2} + \pi$ does not hit the right side of the flow line η^R starting from z with angle $\frac{\theta}{2} - \pi$. Note that the angles $-\frac{\theta}{2} + \pi$ and $\frac{\theta}{2} - \pi$ are dual to the angles of the left and right sides of $\mathbf{L}(\theta)$, respectively. The converse statement also holds: if η' hits a given point z and the left side of the flow line η^L with angle $-\frac{\theta}{2} + \pi$ starting from z does not intersect the right side of the flow line η^R with angle $\frac{\theta}{2} - \pi$ starting from z then $\mathbf{L}(\theta)$ must contain z . The reason is that a flow line with angle $-\frac{\theta}{2}$ cannot hit the left side of η^L and a flow line with angle $\frac{\theta}{2}$ cannot hit the right side of η^R . Hence, an angle-varying flow line which gets arbitrarily close to z can be generated by taking a path which starts off with angle $\frac{\theta}{2}$ until it gets very close to η^L , then travels with angle $-\frac{\theta}{2}$ until getting very close to η^R , etc. Indeed, such a path will travel back and forth between η^L and η^R and get progressively closer to z with each pass. The figure shows such an angle-varying flow line in red where the angle changes from $\frac{\theta}{2}$ to $-\frac{\theta}{2}$ at the black dot

by computing the moments of $|f'_{\tau_\epsilon}(z)|$ and computing the exponent for the probability of the event that two GFF flow lines starting close to each other do not intersect before reaching a macroscopic distance from their starting points.

Remark 3.2. Recall that the critical angle θ_c from (1.3) is the angle at or above which it is claimed in Theorem 1.1 that $\dim_{\mathcal{H}}(\mathbf{L}(\theta))$ is almost surely equal to 2. The reason that

Fig. 7. This is a rotation of the picture in Figs. 5 and 6 by 180° and conformally mapped so that we are working in $\mathbf{H}$. Here, h is a GFF on $\mathbf{H}$ with the boundary data shown, η' is its counterflow line starting from 0, and η^L (resp. η^R) is the flow line with angle $\frac{\theta}{2}$ (resp. $-\frac{\theta}{2}$) starting from a point on the left (resp. right) side of η' . To estimate the probability that $\mathbf{L}(\theta)$ hits $\partial B(z, \epsilon)$ for a given point z , we first estimate the probability that η' hits $\partial B(z, \epsilon)$, say for the first time at time τ_ϵ . Given this, we estimate the probability that η^L and η^R both reach a macroscopic distance before hitting each other. Let (g_t) denote the chordal Loewner evolution of η' , W its Loewner driving function, and let $f_t = g_t - W_t$ be its centered Loewner evolution. We can estimate the latter probability by first conformally mapping away $\eta'|_{[0, \tau_\epsilon]}$ using f_{τ_ϵ} and then estimating the probability that the images of η^L and η^R avoid each other until reaching a macroscopic size using Lemma 3.5

this is the case is that when $\theta \geq \theta_c$, the left side of the flow line η^L cannot intersect the right side of the flow line η^R from Figs. 5, 6 and 7.

3.1. Derivative estimate. Fix $\kappa > 0$ and suppose that η is an SLE_κ process in $\mathbf{H}$ from 0 to ∞ . For each $r \in \mathbf{R}$, we let

$$v = v(r) = \frac{r^2}{4}\kappa + r \left(1 - \frac{\kappa}{4}\right) \quad \text{and} \quad \xi = \xi(r) = \frac{r^2}{8}\kappa.$$

Let (g_t) be the Loewner evolution associated with η , let W be its Loewner driving function, and let $f_t = g_t - W_t$ be its centered Loewner evolution. For each $z \in \mathbf{H}$, we let

$$Z_t = Z_t(z) = X_t + iY_t = f_t(z)$$

and

$$\Delta_t = |g'_t(z)|, \quad \Upsilon_t = \frac{Y_t}{|g'_t(z)|}, \quad \Theta_t = \arg Z_t, \quad \text{and} \quad S_t = \sin \Theta_t. \quad (3.1)$$

Proposition 3.3. For each $\epsilon > 0$, let $\tau_\epsilon = \inf\{t \geq 0 : |\eta(t) - z| \leq \epsilon\}$. We have that

$$\mathbf{E}[|g'_{\tau_\epsilon}(z)|^{v+r} \mathbf{1}_{\{\tau_\epsilon < \infty\}}] \asymp \epsilon^{-\xi-r} \quad (3.2)$$

where the constants in $\asymp$ depend only on κ and r . For each $R \geq 2|z|$, we also let $\sigma_R = \inf\{t \geq 0 : |\eta(t)| = R\}$. Fix $\delta \in (0, \frac{\pi}{2})$ and assume that $\arg(z) \in (\delta, \pi - \delta)$. We also have that

$$\mathbf{E}[|g'_{\tau_\epsilon}(z)|^{v+r} \mathbf{1}_{\{\tau_\epsilon \leq \sigma_R < \infty\}}] \asymp \epsilon^{-\xi-r} \quad (3.3)$$

where the constants in $\asymp$ depend only on κ , r , and δ .

Fig. 8. Suppose that h is a GFF on $\mathbf{H}$ with the boundary data depicted above. For angles θ_1, θ_2 , we let η_i for $i = 1, 2$ be the flow line of h starting at x_i with angle θ_i and let τ_i (resp. $\hat{\tau}_i$) be the first time that η_i hits $\partial\mathbf{D}$ (resp. $\partial(\frac{1}{2}\mathbf{D})$). Assume that θ_1, θ_2 are chosen so that η_1 can hit η_2 , i.e. $\theta_1 \in (\theta_2 - \pi, \theta_2 + 2\lambda'/\chi)$. In Proposition 3.4 we compute the exponent for the probability of the event that $\eta_1([0, \tau_1]) \cap \eta_2([0, \tau_2]) \neq \emptyset$ as $x_1, x_2 \rightarrow 0$. The main step of the proof is Lemma 3.5, which gives the exponent in the special case that $\text{dist}(\eta_1([\hat{\tau}_1, \tau_1]), \eta_2([\hat{\tau}_2, \tau_2]))$ is bounded from below by a positive constant and $\eta_i|_{[0, \tau_i]}$ for $i = 1, 2$ both do not get too close to 0

Proof. See [JVL12, Section 6.3] for a proof of (3.2) and [MW17, Lemma 4.2] for a proof of (3.3). $\square$

This result can be derived from [JVL12, Proposition 6.1], which gives that

$$M_t = |Z_t|^r Y_t^{\xi} \Delta_t^v = S_t^{-r} \Upsilon_t^{\xi+r} \Delta_t^{v+r}$$

is a local martingale. This martingale also appears in [SW05, Theorem 3 and Theorem 6] and is part of the same family of martingales that we will use in Sect. 3.2 to get the exponent for the probability that two GFF flow lines do not intersect each other before making it to a macroscopic distance from their starting points.

3.2. Non-intersection exponent. In this section, we are going to derive the exponent for the probability that two flow lines of the GFF starting from $\pm\frac{1}{2}\epsilon$ do not intersect before hitting $\partial\mathbf{D}$ as $\epsilon \rightarrow 0$ (see Fig. 8 for an illustration of the setup). The main result is:

Proposition 3.4. Fix $\epsilon > 0$ and let $x_1 = -\frac{1}{2}\epsilon$ and $x_2 = \frac{1}{2}\epsilon$. Let θ_1, θ_2 be angles with $\theta_1 \in (\theta_2 - \pi, \theta_2 + 2\lambda'/\chi)$. Suppose that h is a GFF on $\mathbf{H}$ with the boundary data illustrated in Fig. 8 where $a, b \in \mathbf{R}$ are constants so that η_1, η_2 do not hit the continuation threshold immediately almost surely. For $i = 1, 2$, let τ_i be the first time that η_i hits $\partial\mathbf{D}$ and let $F = \{\eta_1([0, \tau_1]) \cap \eta_2([0, \tau_2]) = \emptyset\}$. Let

$$\alpha = \frac{1}{2\kappa} (\kappa - 4 - 2\rho) \left(\frac{b-a}{\lambda} - \rho \right) \quad \text{and} \quad \rho = \frac{(\theta_1 - \theta_2)\chi}{\lambda} - 2. \quad (3.4)$$

Then we have that

$$\mathbf{P}[F] = \epsilon^{\alpha+o(1)}$$

where the $o(1)$ term tends to zero as $\epsilon \rightarrow 0$ at a rate depending only on $\theta_1, \theta_2, \kappa$, and a, b .

We will not use Proposition 3.4 as stated in the proof of Proposition 3.1 and have included it just for completeness. The main ingredient in its proof is Lemma 3.5 which gives the corresponding estimate in the special case that the segments of the paths between first hitting $\partial(\frac{1}{2}\mathbf{D})$ and $\partial\mathbf{D}$ have positive distance from each other and neither path gets too close to 0 before exiting $\partial\mathbf{D}$. From Lemma 3.5, we will prove Lemma 3.7 which is a version which holds with more general boundary data and is the estimate that we will actually make use of in this article.

Lemma 3.5. *Suppose that we have the same setup as in Proposition 3.4. Let $\widehat{\tau}_i$ be the first time that η_i hits $\partial(\frac{1}{2}\mathbf{D})$ for $i = 1, 2$. For each $\delta \in (0, \frac{1}{2})$, let E_δ be the event that*

- (i) $\eta_1([0, \tau_1]) \cap \eta_2([0, \tau_2]) = \emptyset$,
- (ii) $\text{dist}(\eta_1([\widehat{\tau}_1, \tau_1]), \eta_2([\widehat{\tau}_2, \tau_2])) \geq \delta$, and
- (iii) $\eta_i([0, \tau_i]) \cap (\delta\epsilon\mathbf{D}) = \emptyset$ for $i = 1, 2$.

Then

$$\mathbf{P}[E_\delta] \asymp \epsilon^\alpha \quad (3.5)$$

where α is as in (3.4) and the constants in $\asymp$ depend only on $\delta, \theta_1, \theta_2, \kappa$, and a, b .

Before we prove Lemma 3.5, we need to collect the following lemma.

Lemma 3.6. *Suppose that we have the same setup as in Lemma 3.5 and let $V_{s,t}^j$ for $s, t \geq 0$ and $j \in \{1, \dots, 5\}$ be as in Fig. 9. There exists a universal constant $A > 0$ such that*

$$V_{\tau_1, \tau_2}^{j+1} - V_{\tau_1, \tau_2}^j \leq A \quad \text{for } j \in \{1, \dots, 4\}. \quad (3.6)$$

Moreover, there exists a constant $B_\delta > 0$ depending only on $\delta \in (0, \frac{1}{2})$ such that

$$|V_{\tau_1, \tau_2}^j - V_{\tau_1, \tau_2}^3| \geq B_\delta \quad \text{for } j \in \{1, 2, 4, 5\} \quad \text{on } E_\delta. \quad (3.7)$$

Fig. 9. (Continuation of Fig. 8.) For each $s, t \geq 0$, let $g_{s,t}$ be the unique conformal map which takes the unbounded connected component of $\mathbf{H} \setminus (\eta_1([0, s]) \cup \eta_2([0, t]))$ back to $\mathbf{H}$ satisfying $g_{s,t}(z) = z + o(1)$ as $z \rightarrow \infty$. Let $V_{s,t}^1 = g_{s,t}(\inf \eta_1([0, s]) \cap \mathbf{R})$, $V_{s,t}^2 = g_{s,t}(\eta_1(s))$, $V_{s,t}^3$ be the image of the most recent intersection of $\eta_1|_{[0, s]}$ and $\eta_2|_{[0, t]}$ if they intersect, or $\eta_1|_{[0, s]}$ and $[0, x_2]$ if they intersect, or $\eta_1|_{[0, s]}$ and $[x_1, 0]$ if they intersect, and otherwise be equal to $g_{s,t}(0)$, $V_{s,t}^4 = g_{s,t}(\eta_2(t))$, and $V_{s,t}^5 = g_{s,t}(\sup \eta_2([0, t]) \cap \mathbf{R})$.

Let ρ_i be such that as one traces $\mathbf{R}$ from left to right, the heights in the right side jump by $\rho_i \lambda$ at the points $V_{s,t}^i$.

The product $M_{s,t} := \prod_{j \neq 3} |V_{s,t}^j - V_{s,t}^3|^{\tilde{\rho}_3 - \rho_3} \rho_j / (2\kappa)$ evolves as a martingale in each of s and t separately where $\tilde{\rho}_3 = \kappa - 4 - \rho_3$ [SW05]. Reweighting the law of the pair of paths (η_1, η_2) by $M_{s,t}$ yields the law of a new pair of paths which almost surely do not intersect each other. When one path is fixed the evolution of the other in the weighted law is the same as in the unweighted law except with ρ_3 replaced by $\tilde{\rho}_3$. As explained in more detail in the proof of Lemma 3.5, this new pair of paths can be constructed as flow lines of a GFF with modified boundary data and angles

Proof. For each $z \in \mathbf{C}$, let $\mathbf{P}^z$ denote the law of a standard planar Brownian motion B starting from z which is independent of η_1 and η_2 . For each $s, t \geq 0$, let $\mathbf{H}_{s,t}$ be the unbounded connected component of $\mathbf{H} \setminus (\eta_1([0, s]) \cup \eta_2([0, t]))$ and, for each $1 \leq j \leq 4$, let $A_{s,t}^{j+1}$ be the segment of $\partial \mathbf{H}_{s,t}$ which connects $g_{s,t}^{-1}(V_{s,t}^j)$ to $g_{s,t}^{-1}(V_{s,t}^{j+1})$ in the clockwise direction. By [Law05, Remark 3.50], we have that

$$V_{s,t}^{j+1} - V_{s,t}^j = \lim_{y \rightarrow \infty} \pi y \mathbf{P}^{iy} [B \text{ exits } \mathbf{H}_{s,t} \text{ in } A_{s,t}^{j+1}].$$

Since $A_{s,t}^{j+1} \subseteq \overline{\mathbf{D}}$, this, in turn, implies for $0 \leq s \leq \tau_1$ and $0 \leq t \leq \tau_2$ that

$$V_{s,t}^{j+1} - V_{s,t}^j \leq \lim_{y \rightarrow \infty} \pi y \mathbf{P}^{iy} [B \text{ exits } \mathbf{H} \setminus \mathbf{D} \text{ in } \partial \mathbf{D}] < \infty.$$

This proves (3.6).

Let $\phi_i = \arg(\eta_i(\tau_i))$ for $i = 1, 2$ and let $\psi_1 = \phi_1 - \frac{\delta}{4}$ and $\psi_2 = \phi_2 + \frac{\delta}{4}$. We note that the probability that a Brownian motion starting from iy exits $\mathbf{H} \setminus \mathbf{D}$ in $\partial \mathbf{D}$ with argument contained in $[\psi_2, \psi_1]$ is at least a δ -dependent constant times $1/y$. From this and Condition (ii) in the definition of E_δ , (3.7) follows. $\square$

Proof of Lemma 3.5. Let $\rho_1, \dots, \rho_5$ be such that the jumps in the heights from left to right in the right side of Fig. 9 are equal to $\rho_i \lambda$. Explicitly, the values of the ρ_i are given by

$$\begin{aligned} \rho_1 &= \frac{-\theta_1 \chi - a}{\lambda} - 1, \quad \rho_2 = 2, \quad \rho_3 = \frac{(\theta_1 - \theta_2) \chi}{\lambda} - 2, \\ \rho_4 &= 2 \quad \text{and} \quad \rho_5 = \frac{b + \theta_2 \chi}{\lambda} - 1. \end{aligned}$$

Let

$$\tilde{\rho}_3 = \kappa - 4 - \rho_3 = \kappa - 2 + \frac{(\theta_2 - \theta_1) \chi}{\lambda}$$

be the reflection of ρ_3 about the value $\frac{\kappa}{2} - 2$. By [SW05, Theorem 6], reweighting the law of (η_1, η_2) by the local martingale

$$M_{s,t} = \prod_{j \neq 3} |V_{s,t}^j - V_{s,t}^3|^{(\tilde{\rho}_3 - \rho_3) \rho_j / (2\kappa)} \quad (3.8)$$

corresponds to changing ρ_3 to $\tilde{\rho}_3$. This yields a pair of paths $(\widehat{\eta}_1, \widehat{\eta}_2)$ which are flow lines of the GFF as shown in the left side of Fig. 9 where the values θ_1 and θ_2 (both as angles and as indicated in the boundary conditions) are replaced by

$$\widehat{\theta}_1 = \theta_1 \quad \text{and} \quad \widehat{\theta}_2 = 2\theta_1 - \theta_2 - \frac{4\lambda'}{\chi},$$

respectively, and b is replaced by $\widehat{b} = b + \lambda(\tilde{\rho}_3 - \rho_3)$. In particular, the angle gap between $(\widehat{\eta}_1, \widehat{\eta}_2)$ is given by

$$\widehat{\theta}_1 - \widehat{\theta}_2 = \theta_2 - \theta_1 + \frac{4\lambda'}{\chi} > \frac{2\lambda'}{\chi}$$

since we assumed that $\theta_2 - \theta_1 > -2\lambda'/\chi$. Thus, $(\widehat{\eta}_1, \widehat{\eta}_2)$ almost surely do not intersect each other [MS16b, Theorem 1.5]. Observe that α is equal to the sum of the exponents in the definition of $M_{s,t}$ from (3.8):

$$\alpha = \left(\frac{\widetilde{\rho}_3 - \rho_3}{2\kappa} \right) \sum_{j \neq 3} \rho_j.$$

Let

$$\widetilde{M}_t = M_{t \wedge \tau_1, t \wedge \tau_2}$$

Since $M_{s,t}$ is a local martingale, it follows that $\widetilde{M}_t$ is also a local martingale. Lemma 3.6 implies that

$$M_{\tau_1, \tau_2} \asymp 1 \quad \text{on } E_\delta \tag{3.9}$$

where the constants in $\asymp$ depend only on $\delta, \kappa, \theta_1, \theta_2$, and a, b . For each $u \geq 0$, let $T_u = \inf\{t \geq 0 : \widetilde{M}_t = u\}$. It follows from (3.9) that there exists a constant $u_1 > 0$ depending on $\delta, \kappa, \theta_1, \theta_2$, and a, b such that $E_\delta \subseteq \{T_{u_1} < T_0\}$. Consequently,

$$\mathbf{P}[E_\delta] \leq \mathbf{P}[T_{u_1} < T_0] = \frac{1}{u_1} \mathbf{E}[\widetilde{M}_{T_{u_1} \wedge T_0}] \asymp \epsilon^\alpha$$

where the constants in $\asymp$ depend on $\delta, \theta_1, \theta_2, \kappa$, and a, b . This proves the upper bound in (3.5).

We will now give the lower bound for $\mathbf{P}[E_\delta]$. Lemma 3.6 implies that there exists a constant $u_2 > 0$ depending only on $\delta, \theta_1, \theta_2, \kappa$, and a, b such that, on E_δ , we have that $T_{u_2} \leq \tau_1 \wedge \tau_2 \wedge T_0$. We have,

$$\mathbf{P}[E_\delta] \geq \mathbf{P}[E_\delta \mid T_{u_2} < T_0] \mathbf{P}[T_{u_2} < T_0].$$

It is easy to see that $\mathbf{P}[E_\delta \mid T_{u_2} < T_0]$ is bounded from below by universal positive constant depending only on $\delta, \kappa, \theta_1, \theta_2$, and a, b using the results of [MW17, Section 2]. This gives the lower bound since, arguing as in the proof of the upper bound, we know that $\mathbf{P}[T_{u_2} < T_0] \asymp \epsilon^\alpha$ where the constants in $\asymp$ depend only on $\delta, \theta_1, \theta_2, \kappa$, and a, b . This proves the desired result for E_δ . $\square$

In Lemma 3.5, we computed the exponent for the probability that two GFF flow lines starting from $\pm \frac{1}{2}\epsilon$ hit $\partial \mathbf{D}$ before intersecting each other or hitting $\partial(\delta\epsilon \mathbf{D})$ as $\epsilon \rightarrow 0$ when the field has the boundary data illustrated in Fig. 8. We are now going to deduce from this and the Radon–Nikodym derivative estimate Lemma 2.1 that the same is true if we consider a field which has the same boundary data as illustrated in Fig. 8 outside of the interval $(-\frac{1}{2}\delta\epsilon, \frac{1}{2}\delta\epsilon)$ and has general, piecewise constant boundary data in $(-\frac{1}{2}\delta\epsilon, \frac{1}{2}\delta\epsilon)$.

Lemma 3.7. *Suppose that we have the same setup as Lemma 3.5 except we take h to be a GFF on $\mathbf{H}$ whose boundary conditions are piecewise constant, change values at most a finite number of times, are at most δ^{-1} in magnitude, and take the form illustrated in Fig. 8 outside of the interval $(-\frac{1}{2}\delta\epsilon, \frac{1}{2}\delta\epsilon)$. Then*

$$\mathbf{P}[E_\delta] \asymp \epsilon^\alpha \tag{3.10}$$

where the constants in $\asymp$ depend only on $\delta, \theta_1, \theta_2, \kappa$, and a, b .

Proof. Suppose that $\tilde{h}$ is a GFF whose boundary conditions are as in the statement of Lemma 3.5, let $\tilde{\eta}_i$ for $i = 1, 2$ be the flow line of $\tilde{h}$ starting from x_i and let $\tilde{\tau}_i$ be the first time that $\tilde{\eta}_i$ hits $\partial\mathbf{D}$. We also let σ_i (resp. $\tilde{\sigma}_i$) for $i = 1, 2$ be the first time that η_i (resp. $\tilde{\eta}_i$) hits $\partial(\delta\epsilon\mathbf{D})$. Let μ denote the law of $(\eta_1|_{[0, \tau_1 \wedge \sigma_1]}, \eta_2|_{[0, \tau_2 \wedge \sigma_2]})$ and let $\tilde{\mu}$ denote the law of $(\tilde{\eta}_1|_{[0, \tilde{\tau}_1 \wedge \tilde{\sigma}_1]}, \tilde{\eta}_2|_{[0, \tilde{\tau}_2 \wedge \tilde{\sigma}_2]})$. It follows from Lemma 2.1 that μ and $\tilde{\mu}$ are mutually absolutely continuous with

$$\frac{1}{C} \leq \frac{d\mu}{d\tilde{\mu}} \leq C \quad \text{on } E_\delta$$

where $C \geq 1$ is a constant depending only on $\delta, \kappa, \theta_1, \theta_2$, and a, b . The desired result follows since $\tau_i \leq \sigma_i$ for $i = 1, 2$ on E_δ . $\square$

Proof of Proposition 3.4. We are going to establish the upper bound by iteratively applying Lemma 3.5. Fix $\beta \in (0, 1)$ and let $n_\beta = \lfloor \beta \log \frac{1}{\epsilon} \rfloor$. For each $i = 1, 2$ and $1 \leq j \leq n_\beta$, we let $\tau_{i,j}$ (resp. $\widehat{\tau}_{i,j}$) be the first time that η_i hits $\partial(\epsilon^\beta e^j \mathbf{D})$ (resp. $\partial(\frac{1}{2}\epsilon^\beta e^j \mathbf{D})$) and let $G_{j,\delta}$ be the event that

1. $\eta_1([0, \tau_{1,j}]) \cap \eta_2([0, \tau_{2,j}]) = \emptyset$ and
2. either $\text{dist}(\eta_1([\widehat{\tau}_{1,j}, \tau_{1,j}]), \eta_2([\widehat{\tau}_{2,j}, \tau_{2,j}])) < \delta\epsilon^\beta e^j$ or $(\cup_{i=1}^2 \eta_i([0, \tau_{i,j}])) \cap (\delta\epsilon\mathbf{D}) \neq \emptyset$.

Let $\mathcal{F}_j$ be the σ -algebra generated by $\eta_i|_{[0, \tau_{i,j}]}$ for $i = 1, 2$. It is easy to see that there exists a function $p: [0, 1] \rightarrow [0, 1]$ with $p(\delta) \downarrow 0$ as $\delta \downarrow 0$ such that

$$\mathbf{P}[G_{j+1,\delta} | \mathcal{F}_j] \mathbf{1}_{G_{j,\delta}} \leq p(\delta) \quad \text{for } 1 \leq j \leq n_\beta.$$

Consequently, it follows that $\mathbf{P}[\cap_{j=1}^{n_\beta} G_{j,\delta}] \leq (p(\delta))^{n_\beta}$. Choose $\delta > 0$ sufficiently small so that $p(\delta)^{n_\beta} \leq \epsilon^{(1-\beta)\alpha}$. For each $1 \leq j \leq n_\beta$, let $E_{j,\delta}$ be the event that $\eta_1([0, \tau_{1,j}]) \cap \eta_2([0, \tau_{2,j}]) = \emptyset$, $\text{dist}(\eta_1([\widehat{\tau}_{1,j}, \tau_{1,j}]), \eta_2([\widehat{\tau}_{2,j}, \tau_{2,j}])) \geq \delta\epsilon^\beta e^j$, and $\eta_i([0, \tau_{i,j}]) \cap (\delta\epsilon\mathbf{D}) \neq \emptyset$ for $i = 1, 2$. We have that

$$\begin{aligned} \mathbf{P}[F] &\leq \mathbf{P}[F \cap (\cup_j G_{j,\delta}^c)] + \mathbf{P}[\cap_j G_{j,\delta}] \\ &\leq \sum_{j=1}^{n_\beta} \mathbf{P}[E_{j,\delta}] + \epsilon^{(1-\beta)\alpha} \\ &\lesssim (\log \frac{1}{\epsilon}) \epsilon^{(1-\beta)\alpha} + \epsilon^{(1-\beta)\alpha} \quad (\text{Lemma 3.5}). \end{aligned}$$

The upper bound follows because this holds for every $\beta \in (0, 1)$. The lower bound follows because we have that $\mathbf{P}[F] \geq \mathbf{P}[E_\delta] \asymp \epsilon^\alpha$ by Lemma 3.5. $\square$

3.3. Proof of the upper bound. We are now going to combine the estimates of Sects. 3.1 and 3.2 to complete the proof of the upper bound. Throughout, we suppose that h is a GFF on $\mathbf{H}$ with the boundary data illustrated in Fig. 7 and let h' be the counterflow line of h starting from 0. For each $n \in \mathbf{N}$, we let $\mathcal{D}_n$ be the set of squares with side length 2^{-n} and with corners in $2^{-n}\mathbf{Z}^2$ which are contained in $\bar{\mathbf{H}}$. For each $Q \in \mathcal{D}_n$, let $z(Q)$ be the center of Q and let $\tilde{Q}_n(Q) = B(z(Q), 2^{1-n})$. Note that $Q \subseteq \tilde{Q}_n(Q)$. For each $z \in \mathbf{H}$, let $Q_n(z)$ be the element of $\mathcal{D}_n$ which contains z and let $\tilde{Q}_n(z) = \tilde{Q}_n(Q_n(z))$. See Fig. 10 for an illustration of these definitions.

Fig. 10. Shown in the illustration are $Q_n(z)$ and $\tilde{Q}_n(z)$ for a given point $z \in \mathbf{H}$

For each $n \in \mathbf{N}$, we let $\zeta_{z,n} = \inf\{t \geq 0 : \eta'(t) \in \partial \tilde{Q}_n(z)\}$ and

$$r_{z,n} = \frac{2^{6-n} |f'_{\zeta_{z,n}}(z)|}{\sin(\Theta_{\zeta_{z,n}}^z)} \quad \text{on } \{\zeta_{z,n} < \infty\}. \quad (3.11)$$

We also let Θ_t^z be the process of (3.1) with respect to z .

Lemma 3.8. Fix $z \in \mathbf{H}$ and $n \in \mathbf{N}$. On $\{\zeta_{z,n} < \infty\}$, the following hold:

- (i) $f_{\zeta_{z,n}}(\tilde{Q}_{n+3}(z)) \subseteq r_{z,n} \mathbf{D} \cap \mathbf{H}$.
- (ii) There exists constants $c_1, c_2 > 0$ such that

$$c_1 r_{z,n} (2^{k/2} \sin(\Theta_{\zeta_{z,n}}^z) - c_2) \mathbf{D} \cap \mathbf{H} \subseteq f_{\zeta_{z,n}}(\tilde{Q}_{n-k}(z)) \quad \text{for each } 1 \leq k \leq n.$$

Proof. Throughout the proof, we shall assume that we are working on $\{\zeta_{z,n} < \infty\}$. We first note that

$$\text{dist}(z, \partial \tilde{Q}_n(z)) \geq \text{dist}(Q_n(z), \partial \tilde{Q}_n(z)) = 2^{1-n} - 2^{-n-1/2} \geq 2^{-n}.$$

Consequently, we have that

$$2^{-n} \leq \text{dist}(z, \partial \tilde{Q}_n(z)) \leq 2^{1-n} \quad \text{for } z \in \mathbf{H} \text{ and } n \in \mathbf{N}. \quad (3.12)$$

Hence applying Lemma 2.3 with $r = \frac{1}{2}$, we have that

$$\begin{aligned} |f_{\zeta_{z,n}}(z) - f_{\zeta_{z,n}}(w)| &\leq 4 \times \frac{2^{-n-1}}{1 - (1/2)^2} \times \frac{\text{Im}(f_{\zeta_{z,n}}(z))}{2^{-n}} \\ &\leq 4 |f_{\zeta_{z,n}}(z)| \quad \text{for } w \in \tilde{Q}_{n+3}(z). \end{aligned} \quad (3.13)$$

This leaves us to bound $|f_{\zeta_{z,n}}(z)|$. Applying Lemma 2.2 and (3.12), we have that

$$\frac{\text{Im}(f_{\zeta_{z,n}}(z))}{8} \leq \frac{|f'_{\zeta_{z,n}}(z)|}{2^n} \leq 4 \text{Im}(f_{\zeta_{z,n}}(z)). \quad (3.14)$$

Applying the lower bound of (3.14) in the inequality, we thus have that

$$|f_{\zeta_{z,n}}(z)| = \frac{\text{Im}(f_{\zeta_{z,n}}(z))}{\sin(\Theta_{\zeta_{z,n}}^z)} \leq \frac{2^{3-n} |f'_{\zeta_{z,n}}(z)|}{\sin(\Theta_{\zeta_{z,n}}^z)}. \quad (3.15)$$

Since

$$|f_{\zeta_{z,n}}(w)| \leq |f_{\zeta_{z,n}}(z) - f_{\zeta_{z,n}}(w)| + |f_{\zeta_{z,n}}(z)|,$$

combining (3.13) with (3.15) gives (i).

To prove (ii), we first note by the [Beurling estimate](#) that there exists a constant $c_3 > 0$ such that the probability that a Brownian motion starting from z hits $\partial\tilde{Q}_{n-k}(z)$ before hitting $\eta'([0, \zeta_{z,n}])$ is at most $c_3 2^{-k/2}$. Consequently, by the conformal invariance of Brownian motion, the probability that a Brownian motion starting from $f_{\zeta_{z,n}}(z)$ hits $f_{\zeta_{z,n}}(\partial\tilde{Q}_{n-k}(z))$ before hitting $f_{\zeta_{z,n}}(\eta'([0, \zeta_{z,n}]))$ is also at most $c_3 2^{-k/2}$. By standard estimates for Brownian motion, it follows that there exists a constant $c_4 > 0$ such that

$$\text{dist}(f_{\zeta_{z,n}}(z), f_{\zeta_{z,n}}(\partial\tilde{Q}_{n-k}(z))) \geq c_4 \text{Im}(f_{\zeta_{z,n}}(z)) 2^{k/2}. \quad (3.16)$$

Consequently, we have that

$$\begin{aligned} \text{dist}(0, f_{\zeta_{z,n}}(\partial\tilde{Q}_{n-k}(z))) &\geq \text{dist}(f_{\zeta_{z,n}}(z), f_{\zeta_{z,n}}(\partial\tilde{Q}_{n-k}(z))) - |f_{\zeta_{z,n}}(z)| \\ &\geq c_4 \text{Im}(f_{\zeta_{z,n}}(z)) 2^{k/2} - |f_{\zeta_{z,n}}(z)| \quad (\text{by (3.16)}) \\ &\geq c_4 \text{Im}(f_{\zeta_{z,n}}(z)) 2^{k/2} - \frac{2^{3-n} |f'_{\zeta_{z,n}}(z)|}{\sin(\Theta_{\zeta_{z,n}}^z)} \quad (\text{by (3.15)}). \end{aligned} \quad (3.17)$$

In analogy with (3.15), the upper bound of (3.14) implies that

$$|f_{\zeta_{z,n}}(z)| = \frac{\text{Im}(f_{\zeta_{z,n}}(z))}{\sin(\Theta_{\zeta_{z,n}}^z)} \geq \frac{2^{-2-n} |f'_{\zeta_{z,n}}(z)|}{\sin(\Theta_{\zeta_{z,n}}^z)}. \quad (3.18)$$

By the definition of $r_{z,n}$ and (3.18), it follows that there exists constants $c_1, c_2 > 0$ such that the expression in (3.17) is bounded from below by

$$c_1 r_{z,n} (2^{k/2} \sin(\Theta_{\zeta_{z,n}}^z) - c_2).$$

This proves the desired result. $\square$

On $\{\zeta_{z,n} < \infty\}$, we let $\tilde{h}_n = h \circ f_{\zeta_{z,n}}^{-1} - \chi \arg(f_{\zeta_{z,n}}^{-1})'$ be the GFF which arises after conformally mapping away $\eta'([0, \zeta_{z,n}])$. We then let $\tilde{\eta}_{z,n}^L$ (resp. $\tilde{\eta}_{z,n}^R$) be the flow line of $\tilde{h}_n$ starting from $-r_{z,n}$ (resp. $r_{z,n}$) with angle $\frac{\theta}{2}$ (resp. $-\frac{\theta}{2}$). Let $\tilde{\tau}_{z,n}^L$ (resp. $\tilde{\tau}_{z,n}^R$) be the first time that $\tilde{\eta}_{z,n}^L$ (resp. $\tilde{\eta}_{z,n}^R$) hits $\partial\mathbf{D}$. For $q \in \{L, R\}$, let $\eta_{z,n}^q = f_{\zeta_{z,n}}^{-1}(\tilde{\eta}_{z,n}^q)$. See Fig. 11 for an illustration of the construction. We are now going to show that the set of points z at which the paths η' , $\eta_{z,n}^L$, $\eta_{z,n}^R$ behave in a consistently pathological manner as η' approaches z is almost surely empty. In particular, we will prove in Lemma 3.9 that the set $S^{1,\delta}$ of points z that η' approaches at an angle which is consistently outside of $(\delta, \pi - \delta)$ is almost surely empty for a sufficiently small choice of $\delta > 0$. Then we will show in Lemma 3.10 that the set $S^{2,k}$ of points z that η' approaches and for consistently large values of n either $\tilde{\eta}_{z,n-k}^L$ or $\tilde{\eta}_{z,n-k}^R$ hits $\partial(r_{z,n}\mathbf{D})$ is almost surely empty for a sufficiently large choice of k . These results, in turn, will be used in the proof of Proposition 3.1 to generate a cover of $\mathbf{L}(\theta)$ in the manner described in Figs. 5, 6 and 7.

Fig. 11. An illustration of the construction of the paths $\eta_{z,n}^L$ and $\eta_{z,n}^R$

Lemma 3.9. For each $z \in \mathbf{H}$, $n \in \mathbf{N}$, and $\delta > 0$, we let $E_{z,n}^{1,\delta} = \{\zeta_{z,n} < \infty, \Theta_{\zeta_{z,n}}^z \notin (\delta, \pi - \delta)\}$. Let $\mathcal{S}_n^{1,\delta}$ be the set of points $z \in \mathbf{H}$ such that $E_{z,n}^{1,\delta}$ occurs and let $\mathcal{S}^{1,\delta} = \bigcup_{n=1}^\infty \bigcap_{m=n}^\infty \mathcal{S}_m^{1,\delta}$. There exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ we have that $\mathcal{S}^{1,\delta} = \emptyset$ almost surely.

We remark that a version of Lemma 3.9 is proved in [MW17] for $\kappa \in (0, 4)$ using a different argument.

Proof of Lemma 3.9. By [SW05, Theorem 3], we can view η' as a radial $\text{SLE}_{\kappa'}(\kappa' - 6)$ process targeted at z . After reparameterizing the path by log conformal radius, Θ^z solves the SDE

$$d\Theta_t^z = (\kappa' - 4) \cot(\Theta_t^z) dt + \sqrt{\kappa'} dB_t$$

where B is a standard Brownian motion (see [She09, Equation (4.1)]). When $\kappa' \in (4, 8)$, $\kappa' - 6 < \frac{\kappa'}{2} - 2$ which means that Θ^z almost surely hits either 0 or π in finite time [Law05, Lemma 1.26]. In particular, if $\Theta_t^z = \theta \notin (\delta, \pi - \delta)$ for some fixed t then the probability that $\Theta_s^z \in \{0, \pi\}$ for some $s \in [t, t + 1]$ tends to 1 as $\delta \rightarrow 0$ uniformly in $\theta \notin (\delta, \pi - \delta)$. It follows that there exists a function $p_1 : [0, \frac{\pi}{2}] \rightarrow [0, 1]$ with $p_1(\delta) \rightarrow 0$ as $\delta \rightarrow 0$ such that

$$\mathbf{P}[\zeta_{z,n+8} < \infty \mid \Theta_{\zeta_{z,n}}^z = \theta, \zeta_{z,n} < \infty] \leq p_1(\delta) \quad \text{for all } \theta \notin (\delta, \pi - \delta) \quad (3.19)$$

(by standard distortion estimates for conformal maps, it takes at least 1 unit of log conformal radius time for the path to travel from $\partial \tilde{Q}_n(z)$ to $\partial \tilde{Q}_{n+8}(z)$). Iterating (3.19) implies that with $p(\delta) = p_1^{1/8}(\delta)$ we have that

$$\mathbf{P}[\cap_{m=r}^n E_{z,m}^{1,\delta}] \leq p(\delta)^{n-r} \quad \text{for all } n \geq r \geq -\log_2 \text{Im}(z) + 2 \quad (3.20)$$

($r \geq -\log_2 \text{Im}(z) + 2$ implies $\tilde{Q}_r(z) \subseteq \mathbf{H}$.)

Note that for $Q \in \mathcal{D}_j$, the function $Q \rightarrow \mathbf{R}$ given by $w \mapsto \Theta_{\zeta_{w,j}}^w$ is positive and harmonic. Consequently, the Harnack inequality [Law05, Proposition 2.22] implies that

there exists a constant $K \geq 1$ such that for all $z, w \in Q$ we have that $\Theta_{\zeta w, j}^w \leq K \Theta_{\zeta z, j}^z$. Thus letting $E_{Q, m}^{1, \delta} = \cup_{z \in Q} E_{z, m}^{1, \delta}$ for $m \leq j$, it follows from (3.20) that

$$\mathbf{P}[\cap_{m=r}^n E_{Q, m}^{1, \delta}] \leq p(K\delta)^{n-r} \quad \text{for all } j \geq n \geq r. \quad (3.21)$$

Fix $\varsigma \in (0, 1)$ and let $r = -\log_2 \varsigma + 2$, $U_\varsigma = (\varsigma^{-1} \mathbf{D}) \cap (\mathbf{H} + \varsigma i)$, and $\mathcal{V}_j^{\varsigma, \delta}$ for $j \geq r$ consist of those $Q \in \mathcal{D}_j$ with $Q \subseteq U_\varsigma$ such that $\cap_{m=r}^j E_{Q, m}^\delta$ occurs. It is easy to see that there exists a constant $C > 0$ such that

$$\sum_{m=r}^{\infty} \mathbf{E}[|\mathcal{V}_m^{\varsigma, \delta}|] \leq \frac{C}{\varsigma^2} \sum_{m=r}^{\infty} 2^{2m} p(K\delta)^{m-r}. \quad (3.22)$$

Consequently, choosing $\delta > 0$ sufficiently small so that $4p(K\delta) < 1$, we see that the summations in (3.22) are finite. This implies that the set of squares in $\mathcal{V}_m^{\varsigma, \delta}$ is non-empty for finitely many m almost surely, from which the claimed result follows for $\kappa' \in (4, 8)$.

For $\kappa' \geq 8$, we have that $\kappa' - 6 \geq \frac{\kappa'}{2} - 2$ which means that Θ^z almost surely does not hit 0 or π [Law05, Lemma 1.26]. In this case, it is easy to see from the form of the SDE that there exists a function $q_1 : [0, \frac{\pi}{2}] \rightarrow [0, 1]$ such that

$$\mathbf{P}[E_{z, n+8}^{1, \delta} \mid \Theta_{\zeta z, n}^z = \theta, \zeta_{z, n} < \infty] \leq q_1(\delta) \quad \text{for all } \theta \notin (\delta, \pi - \delta).$$

Therefore the same argument we used to complete the proof for $\kappa' \in (4, 8)$ also applies here, which proves the claimed result for $\kappa' \geq 8$. $\square$

Lemma 3.10. *For each $z \in \mathbf{H}$ and $k, n \in \mathbf{N}$ with $1 \leq k \leq n$, we let*

$$E_{z, n}^{2, k} = \left\{ \zeta_{z, n} < \infty, \left(\cup_{q \in \{L, R\}} \tilde{\eta}_q^{n-k}([0, \tilde{\tau}_q^{n-k}]) \right) \cap (r_{z, n} \mathbf{D}) \neq \emptyset \right\}.$$

Let $\mathcal{S}_n^{2, k}$ be the set of points $z \in \mathbf{H}$ such that $E_{z, n}^{2, k}$ occurs and let $\mathcal{S}^{2, k} = \cup_{n=1}^{\infty} \cap_{m=n}^{\infty} \mathcal{S}_m^{2, k}$. There exists k_0 such that for every $k \geq k_0$ we have that $\mathcal{S}^{2, k} = \emptyset$ almost surely.

See Fig. 12 for an illustration of the setup of Lemma 3.10.

Proof of Lemma 3.10. Fix $\delta > 0$ sufficiently small so that the statement of Lemma 3.9 holds. Let $\tilde{\mathcal{S}}_n^{2, k}$ be the set of points such that $E_{z, n}^{2, k} \setminus E_{z, n}^{1, \delta}$ occurs and let $\tilde{\mathcal{S}}^{2, k} = \cup_{n=1}^{\infty} \cap_{m=n}^{\infty} \tilde{\mathcal{S}}_m^{2, k}$. We are going to prove the lemma by showing that there exists $k_0 \in \mathbf{N}$ such that $k \geq k_0$ implies that $\tilde{\mathcal{S}}^{2, k} = \emptyset$ almost surely. Let $\tilde{\theta}_L \leq \frac{\theta}{2}$ be an angle such that a flow line of $\tilde{h}_j$ starting from $\mathbf{R}_-$ can hit $\mathbf{R}_+$ and let $\tilde{\theta}_R = -\tilde{\theta}_L$ so that a flow line of $\tilde{h}_j$ starting from $\mathbf{R}_+$ with angle $\tilde{\theta}_R$ can hit $\mathbf{R}_-$. For each $n \in \mathbf{N}$, we let $\tilde{\gamma}_n^L$ (resp. $\tilde{\gamma}_n^R$) be the flow line of $\tilde{h}_n$ starting from $c_1 2^{k/4} r_{z, n}$ (resp. $-c_1 2^{k/4} r_{z, n}$) with angle $\tilde{\theta}_L$ (resp. $\tilde{\theta}_R$) where c_1 is the constant from Lemma 3.8. Let $\tilde{\tau}_{z, n}^L$ (resp. $\tilde{\tau}_{z, n}^R$) be the first time that $\tilde{\gamma}_n^L$ (resp. $\tilde{\gamma}_n^R$) hits either $\partial B(0, r_{z, n})$ or $\partial B(0, c_1 2^{k/2} r_{z, n})$ and let $\tilde{F}_{z, n}^k$ be the event that both $\tilde{\gamma}_n^L([0, \tilde{\tau}_{z, n}^L])$ and $\tilde{\gamma}_n^R([0, \tilde{\tau}_{z, n}^R])$ separate $B(0, r_{z, n})$ from $\partial B(0, c_1 2^{k/2} r_{z, n})$. Note that $(E_{z, n}^{2, k})^c \subseteq \tilde{F}_{z, n}^k$ because $\tilde{\eta}_q^n$ cannot cross $\tilde{\gamma}_n^q$ for $q \in \{L, R\}$.

Let $\mathcal{F}_n$ be the σ -algebra generated by $\eta^j|_{[0, \zeta_{z, n}]}$ as well as the paths $\tilde{\eta}_j^q|_{[0, \tilde{\tau}_j^q]}$ for $1 \leq j \leq n$. We next claim that there exists a function $p : \mathbf{N} \rightarrow [0, 1]$ with $p(k) \downarrow 0$ as $k \rightarrow \infty$ such that

$$\mathbf{P}[\tilde{F}_{z, n+k}^k \mid \mathcal{F}_n] \mathbf{1}_{\{\zeta_{z, n} < \infty\}} \leq p(k). \quad (3.23)$$

Fig. 12. In Lemma 3.10, we show that there exists k_0 such that for $k \geq k_0$ the set of points $z \in \mathbf{H}$ such that $\{\zeta_{z,n} < \infty\}$ and either $\tilde{\eta}_{z,n-k}^L$ or $\tilde{\eta}_{z,n-k}^R$ hits $\partial(r_{z,n}\mathbf{D})$ before hitting $\partial\mathbf{D}$ for arbitrarily large values of n is almost surely empty. On the event that $\mathbf{L}(\theta)$ hits $f_{\zeta_{z,n-k}}^{-1}(\partial(r_{z,n}\mathbf{D}))$ (equivalently, $\tilde{\mathbf{L}}(\theta) = f_{\zeta_{z,n-k}}(\mathbf{L}(\theta))$ hits $\partial(r_{z,n}\mathbf{D})$) and both $\tilde{\eta}_{z,n-k}^L$ and $\tilde{\eta}_{z,n-k}^R$ do not hit $\partial(r_{z,n}\mathbf{D})$ before hitting $\partial\mathbf{D}$, it follows that $\tilde{\eta}_{z,n-k}^L$ and $\tilde{\eta}_{z,n-k}^R$ do not intersect each other before hitting $\partial\mathbf{D}$. Indeed, $\tilde{\mathbf{L}}(\theta)$ cannot enter into the region surrounded by $\tilde{\eta}_{z,n-k}^L$ and $\tilde{\eta}_{z,n-k}^R$, so if $\tilde{\eta}_{z,n-k}^L$ and $\tilde{\eta}_{z,n-k}^R$ did intersect then $\tilde{\mathbf{L}}(\theta)$ could not intersect $\partial(r_{z,n}\mathbf{D})$.

We are first going to explain why there exists a function $p_1 : \mathbf{N} \rightarrow [0, 1]$ with $p_1(k) \downarrow 0$ as $k \rightarrow \infty$ such that

$$\mathbf{P}[\tilde{F}_{z,n}^k] \leq p_1(k) \quad \text{for all } n \in \mathbf{N}. \quad (3.24)$$

We will then explain using the Radon–Nikodym derivative estimate Lemma 2.1 why (3.23) follows once we establish (3.24). First of all, we note that the probability that $\tilde{\gamma}_n^L$ hits $\mathbf{R}_+$ before hitting $\partial B(0, c_1 2^{k/2} r_{z,n})$ tends to 1 as $k \rightarrow \infty$, the probability that it hits $B(0, r_{z,n})$ tends to 0 as $k \rightarrow \infty$, and the analogous statements are likewise true with $\tilde{\gamma}_n^R$ in place of $\tilde{\gamma}_n^L$. Indeed, this follows since the law of $\tilde{\gamma}_n^L$ rescaled by $(c_1 2^{k/4} r_{z,n})^{-1}$ stopped upon hitting $\partial\mathbf{D}$ is that of an $\text{SLE}_\kappa(\rho_{1,L}; \rho_{1,R}, \rho_{2,R})$ process starting from 1 with $\rho_{1,L}, \rho_{1,R} > -2$, $\rho_{1,R} + \rho_{2,R} \in (\frac{\kappa}{2} - 4, \frac{\kappa}{2} - 2)$, and with the force points located at 1^- and $0, 1^+$, respectively. This proves (3.24). To extract (3.23) from (3.24), we note that part (ii) of Lemma 3.8 implies that the paths involved in the definition of $\tilde{F}_{z,n+k}^k$ are disjoint and at a positive distance from those involved in the definition of $\tilde{F}_{z,j}^k$ for all $j \leq n$. Consequently, the claimed bound follows from Lemma 2.1. That there exists $k_0 \in \mathbf{N}$ such that $\mathcal{S}^{2,k} = \emptyset$ almost surely for $k \geq k_0$ then follows from the same argument used to establish the corresponding result for $\mathcal{S}^{1,\delta}$ in Lemma 3.9. $\square$

Proof of Proposition 3.1. We begin by partitioning $\mathbf{L}(\theta)$ as follows. For each $\delta > 0$, let $\mathbf{L}^\delta(\theta)$ consist of those $z \in \mathbf{L}(\theta)$ such that for every $n \in \mathbf{N}$ there exists $m \geq n$ such that the event E_δ of Lemma 3.5 occurs for the pair of paths $(\tilde{\eta}_{z,m}^L, \tilde{\eta}_{z,m}^R)$. It follows from Lemma 3.10 and the argument described in Fig. 12 that $\mathbf{L}(\theta) = \bigcup_{\delta > 0} \mathbf{L}^\delta(\theta)$. Moreover, note that $0 < \delta_1 < \delta_2$ implies that $\mathbf{L}^{\delta_2}(\theta) \subseteq \mathbf{L}^{\delta_1}(\theta)$. Consequently, it suffices to show that there exists $\bar{\delta} > 0$ such that the desired upper bound for $\dim_{\mathcal{H}}(\mathbf{L}^{\bar{\delta}}(\theta))$ holds for each $\delta \in (0, \bar{\delta})$. We are going to set the value of $\bar{\delta} > 0$ in the proof. We begin by assuming that $\bar{\delta} \in (0, \delta_0)$ where $\delta_0 > 0$ is the constant from Lemma 3.9.

Fix $\delta \in (0, \bar{\delta})$, $\varsigma \in (0, 1)$, and let $U_\varsigma = (\varsigma^{-1}\mathbf{D}) \cap (\mathbf{H} + \varsigma i)$. For each $n \in \mathbf{N}$, we are now going to construct a cover of $\mathbf{L}^\delta(\theta) \cap U_\varsigma$ consisting of squares in $\cup_{m \geq n} \mathcal{D}_m$. Let $\mathcal{U}_m^{\varsigma, \delta}$ be the set of squares in $\mathcal{Q} \in \mathcal{D}_m$ which are contained in U_ς such that the following hold:

- (i) η' hits $\tilde{Q} = \tilde{Q}_m(Q)$, say for the first time at time $\tau_{\tilde{Q}}$,
- (ii) $\Theta_{\tau_{\tilde{Q}}}^{z(Q)} \in (\delta, \pi - \delta)$,
- (iii) The event E_δ of Lemma 3.5 defined in terms of the paths $\tilde{\eta}_{z,m}^L$ and $\tilde{\eta}_{z,m}^R$ occurs.

For each $n \in \mathbf{N}$, we let $\mathcal{V}_n^{\varsigma, \delta} = \cup_{m \geq n} \mathcal{U}_m^{\varsigma, \delta}$. To complete the proof, we need to show that $\mathcal{V}_n^{\varsigma, \delta}$ is a cover of $\mathbf{L}^\delta(\theta) \cap U_\varsigma$ and then get a bound on the expected number of squares in $\mathcal{V}_n^{\varsigma, \delta}$.

Fix $z \in \mathbf{L}^\delta(\theta) \cap U_\varsigma$. Since $\mathbf{L}^\delta(\theta)$ is contained in the range of η' , it follows that $\zeta_{z,n} < \infty$ for all $n \in \mathbf{N}$. For each $j \in \mathbf{N}$, let $Q_j \in \mathcal{D}_j$ be the square which contains z and let $\tilde{Q}_j = \tilde{Q}(Q_j)$. It follows from Lemmas 3.9 and 3.10, possibly by decreasing the value of $\bar{\delta} > 0$, that there exists a sequence (j_ℓ) in $\mathbf{N}$ tending to ∞ such that $\Theta_{\tau_{\tilde{Q}_{j_\ell}}}^z \in (\delta, \pi - \delta)$, $\tilde{\eta}_{j_\ell}^L$ and $\tilde{\eta}_{j_\ell}^R$ do not hit $B(0, \delta r_{j_\ell})$ for all $\ell \in \mathbf{N}$. Therefore $Q_{j_\ell} \in \mathcal{V}_n^{\varsigma, \delta}$ for all $\ell \in \mathbf{N}$ so that $j_\ell \geq n$, hence $\mathcal{V}_n^{\varsigma, \delta}$ is a cover of $\mathbf{L}^\delta(\theta) \cap U_\varsigma$, as desired.

We are now going to estimate $\mathbf{P}[Q \in \mathcal{U}_n^{\varsigma, \delta}]$ for a given square $Q \in \mathcal{D}_n$ which is contained in U_ς . Take $\theta_1 = \frac{\theta}{2}$, $\theta_2 = -\frac{\theta}{2}$, $a = \lambda'$, and $b = -\lambda'$. The exponent α from (3.4) of Lemma 3.7 corresponding to these parameters is given by

$$\alpha = \frac{1}{\kappa} \left(1 - \frac{\kappa}{4}\right) (1 - \bar{\theta}) (\kappa(1 + \bar{\theta}) - 4\bar{\theta}) \quad \text{where} \quad \bar{\theta} = \frac{\theta}{\pi}.$$

Therefore

$$\mathbf{P}[Q \in \mathcal{U}_n^{\varsigma, \delta}] \asymp \mathbf{E}[|f'_{\zeta_{z,n}}(z)| r_{z,n}^\alpha \mathbf{1}_{\{\zeta_{z,n} < \infty\}}] = r_{z,n}^\alpha \mathbf{E}[|f'_{\zeta_{z,n}}(z)|^\alpha \mathbf{1}_{\{\zeta_{z,n} < \infty\}}] \quad (3.25)$$

where the constants in $\asymp$ depend only on δ , κ , and θ . Recall that $\kappa' = \frac{16}{\kappa}$. Set

$$r = - \left(\frac{4 + 4\bar{\theta}}{\kappa'} \right) + \bar{\theta} < \frac{1}{2} - \frac{4}{\kappa'} \quad (3.26)$$

so that

$$v(r) + r = \alpha \quad \text{and} \quad -\xi(r) - r = \frac{2}{\kappa'} - \frac{8}{\kappa'} ((\kappa' - 4)\bar{\theta})^2. \quad (3.27)$$

With this choice of r , we can apply Proposition 3.3 and this leads to an exponent for $\mathbf{P}[Q \in \mathcal{U}_n^{\varsigma, \delta}]$ given by

$$\begin{aligned} \beta &= \beta(\kappa, \theta) = (v(r) + r) - (\xi(r) + r) = v(r) - \xi(r) \\ &= \frac{(4 + (4 - \kappa')\bar{\theta})(2(\kappa' - 2) + (4 - \kappa')\bar{\theta})}{8\kappa'} \end{aligned}$$

That is,

$$\mathbf{P}[Q \in \mathcal{U}_n^{\varsigma, \delta}] \asymp 2^{-n\beta}$$

where the constants in $\asymp$ depend only on θ , δ , and κ . By making the substitution $\kappa' = \frac{16}{\kappa}$, we note that $2 - \beta = d(\kappa, \theta)$. Fix $\xi > 0$. By performing a union bound over $\{Q \in \mathcal{U}_m^{\zeta, \delta}\}$ for $m \geq n$, we consequently have that

$$\begin{aligned} \mathbf{E}[\mathcal{H}_{d(\kappa, \theta) + \xi}(\mathbf{L}^\delta(\theta) \cap U_\zeta)] &\lesssim \sum_{m=n}^{\infty} 2^{2m} \times 2^{-m(d(\kappa, \theta) + \xi)} \times 2^{-m\beta} \\ &\asymp \sum_{m=n}^{\infty} 2^{-\xi m} < \infty \end{aligned}$$

where the constants in $\lesssim$ and $\asymp$ depend only on ζ , δ , κ , and θ . Taking a limit as $n \rightarrow \infty$ implies that $\mathcal{H}_{d(\kappa, \theta) + \xi}(\mathbf{L}^\delta(\theta) \cap U_\zeta) = 0$ almost surely. Since $\zeta \in (0, 1)$ was arbitrary, we therefore have that $\dim_{\mathcal{H}}(\mathbf{L}^\delta(\theta)) \leq d(\kappa, \theta) + \xi$ almost surely. The result follows since $\delta, \xi > 0$ were arbitrary. $\square$

Now that we have proved Proposition 3.1, hence the upper bounds of Theorems 1.1 and 1.3, we turn to complete the proof of Theorem 1.4.

Proof of Theorem 1.4. Fix $\theta \in [0, \pi]$. Suppose that h is a GFF on $\mathbf{H}$ with piecewise constant boundary data which changes values at most a finite number of times and let $\mathbf{F}(\theta)$ be the fan of h with opening angle θ starting from 0. For each $\theta_1 \leq \theta_2$, we let $\mathbf{L}(\theta_1, \theta_2)$ be the closure of the set of points accessible by angle-varying flow lines starting from 0 with rational angles contained in $[\theta_1, \theta_2]$ and which change angles a finite number of times and only at positive rational times. Using this notation, $\mathbf{L}(\theta) = \mathbf{L}(-\frac{\theta}{2}, \frac{\theta}{2})$. By Proposition 3.1, the dimension $\mathbf{L}(\theta_1, \theta_2)$ is at most $d(\kappa, \theta_2 - \theta_1)$. Note that, as $\theta_2 - \theta_1$ decreases to 0, $d(\kappa, \theta_2 - \theta_1)$ decreases to $1 + \frac{\kappa}{8}$, the dimension of ordinary SLE_κ . For each $\epsilon > 0$, we have that $\mathbf{F}(\theta)$ is contained in the finite union $\cup_{j=0}^{\lceil \theta/\epsilon \rceil} \mathbf{L}(-\frac{\theta}{2} + j\epsilon, -\frac{\theta}{2} + (j+1)\epsilon)$ of light cones. By Proposition 3.1, the Hausdorff dimension of each of these light cones is almost surely at most $d(\kappa, \epsilon)$. Therefore $\dim_{\mathcal{H}}(\mathbf{F}(\theta)) \leq d(\kappa, \epsilon)$ almost surely. Since this holds for each $\epsilon > 0$, $\dim_{\mathcal{H}}(\mathbf{F}(\theta)) \leq d(\kappa, 0) = 1 + \frac{\kappa}{8}$ almost surely. We also have that $\dim_{\mathcal{H}}(\mathbf{F}(\theta)) \geq 1 + \frac{\kappa}{8}$ almost surely since $\mathbf{F}(\theta)$ contains the 0 angle flow line of h starting from 0 which itself has dimension $1 + \frac{\kappa}{8}$ almost surely by [RS05, Bef08]. $\square$

4. Lower Bound

We are now going to finish the proof of Theorem 1.3 by establishing the lower bound. We will make use of a multi-scale refinement of the second moment method (see [DPRZ01, HMP10, MSW14, MW17, GMS14, MWW16] for similar applications of this technique). In particular, we will introduce a special class of points—so called “perfect points”—which are contained in $\mathbf{L}(\theta)$ whose correlation structure is easier to control than for general points in $\mathbf{L}(\theta)$ and then get a lower bound for the dimension of this set of points.

4.1. Definition of events. We will now work towards defining the perfect points. See Fig. 13 for an illustration of the event which is used to define a perfect point and which we will now describe. Fix $u \in \partial\mathbf{H} \setminus \{0\}$ and $\beta > \Delta^2 > \Delta > 1$; we will eventually take a limit first as $\beta \rightarrow \infty$ and then as $\Delta \rightarrow \infty$. Suppose that we have five non-crossing paths $\gamma', \gamma^L, \widehat{\gamma}^L, \gamma^R, \widehat{\gamma}^R$ in $\overline{\mathbf{H}}$. We assume that γ' starts from 0 and let τ be the first time that γ' hits $\partial B(i, e^{-\beta})$. We assume that γ' admits a (chordal) Loewner

Fig. 13. Illustration of the event $E_u^{\beta, \Delta}(\gamma', \gamma^L, \gamma^R, \widehat{\gamma}^L, \widehat{\gamma}^R)$ used to define a perfect point to establish the lower bound. Here, $f_t = g_t - W_t$ where (g_t) and W are, respectively, the Loewner evolution and driving function of γ' and τ is the first time that γ' hits $\partial B(i, e^{-\beta})$. In our particular application, γ' will be a counterflow line hence self-intersecting and the other paths will be flow lines hence simple, as illustrated

evolution (g_t) with continuous Loewner driving function W and let $f_t = g_t - W_t$ be its centered Loewner evolution. We also assume that γ^L (resp. γ^R) starts on the left (resp. right) side of $\gamma'([0, \tau])$ and $\widehat{\gamma}^L$ (resp. $\widehat{\gamma}^R$) starts on $\mathbf{R}_-$ (resp. $\mathbf{R}_+$). We then let $E_u^{\beta, \Delta}(\gamma', \gamma^L, \gamma^R, \widehat{\gamma}^L, \widehat{\gamma}^R)$ be the event that the following hold:

- (i) γ' hits $\partial B(i, e^{-\beta})$ and does so before hitting $\partial B(i, e^{\Delta})$.
- (ii) The harmonic measure of the left (resp. right) side of $\gamma'([0, \tau])$ as seen from i in $\mathbf{H} \setminus \gamma'([0, \tau])$ is at least $\frac{1}{2} - e^{-4\Delta}$.
- (iii) For $q \in \{L, R\}$, let ζ^q (resp. $\widehat{\zeta}^q$) be the first time that $f_\tau(\gamma^q)$ hits $\partial \mathbf{D}$ (resp. $\partial(\frac{1}{2}\mathbf{D})$). Then $\text{dist}(f_\tau(\gamma^L([\widehat{\zeta}^L, \zeta^L])), f_\tau(\gamma^R([\widehat{\zeta}^R, \zeta^R]))) \geq e^{-\Delta}$.
- (iv) γ^L and γ^R intersect $\widehat{\gamma}^L$ and $\widehat{\gamma}^R$, respectively, before intersecting each other and also before leaving $B(i, e^{\Delta})$. Moreover, $e^{\Delta-\beta} |f'_\tau(i)| \mathbf{D}$ is (completely) contained in the connected component of $\mathbf{H} \setminus f_\tau(\gamma^L \cup \gamma^R \cup \widehat{\gamma}^L \cup \widehat{\gamma}^R)$ which has 0 on its boundary.

Before we finish defining the perfect points of $\mathbf{L}(\theta)$, we first record the following lemma.

Lemma 4.1. *Suppose that we have the setup described just above. There exists a constant $C_1 > 0$ such that the following is true. On the event $\{\tau < \infty\}$, let $\varphi: \mathbf{H} \setminus \gamma'([0, \tau]) \rightarrow \mathbf{H}$ be the unique conformal transformation with $\varphi(\gamma'(\tau)) = 0$ and $\varphi(i) = i$. For each $\varsigma \in (0, 1)$ we have that $B(i, C_1 e^{(1-\varsigma)\beta}) \subseteq \varphi(B(i, e^{-\varsigma\beta/2}))$.*

Proof. Throughout, we shall suppose that $\{\tau < \infty\}$ occurs. Fix $\varsigma \in (0, 1)$. The probability that a Brownian motion starting from i hits $\partial B(i, e^{-\varsigma\beta})$ before hitting $\partial \mathbf{H} \cup \gamma'([0, \tau])$ is $O(e^{-(1-\varsigma)\beta/2})$ by the [Beurling estimate](#). By the conformal invariance of Brownian motion, the probability of the event X that a Brownian motion starting from i exits $\varphi(B(i, e^{-\varsigma\beta/2}))$ in $\varphi(\partial B(i, e^{-\varsigma\beta/2}))$ is also $O(e^{-(1-\varsigma)\beta/2})$. Let

$$d = \text{dist}(\varphi(\partial B(i, e^{-\varsigma\beta/2})), i).$$

We claim $\mathbf{P}[X] \gtrsim d^{-1}$. Indeed, $X_1 \cap X_2 \subseteq X$ where X_1 is the event that the Brownian motion exits $\partial B(0, d)$ before hitting $\partial \mathbf{H}$ at a point with argument in $[\frac{\pi}{4}, \frac{3\pi}{4}]$ and X_2 is the event that it hits $\varphi(\partial B(i, e^{-\varepsilon\beta}))$ after hitting $\partial B(0, d)$ before hitting $\partial \mathbf{H}$. It is easy to see that $\mathbf{P}[X_1] \gtrsim d^{-1}$ and $\mathbf{P}[X_2 | X_1] \gtrsim 1$. Consequently, $e^{-(1-\varepsilon)\beta/2} \gtrsim d^{-1}$ hence $d \gtrsim e^{(1-\varepsilon)\beta/2}$, as desired. $\square$

We now define the perfect points of $\mathbf{L}(\theta)$ using these events as follows. We suppose that $u_1 \in \partial \mathbf{H} \setminus \{0\}$ and that h_1 is a GFF on $\mathbf{H}$ with boundary data given by

$$h_1|_{(-\infty, u_1]} \equiv -\lambda' + 2\pi\chi, \quad h_1|_{(u_1, 0]} \equiv \lambda', \quad \text{and} \quad h_1|_{(0, \infty)} \equiv -\lambda' \quad \text{if} \quad u_1 < 0$$

and

$$h_1|_{(-\infty, 0]} \equiv \lambda', \quad h_1|_{(0, u_1]} \equiv -\lambda', \quad \text{and} \quad h_1|_{(u_1, \infty)} \equiv \lambda' - 2\pi\chi \quad \text{if} \quad u_1 > 0.$$

These two possibilities correspond to the type of boundary data which arises by starting with a GFF on $\mathbf{H}$ with boundary data as in Fig. 7 and then applying a conformal change of coordinates which takes a given point $z \in \mathbf{H}$ to i and leaves 0 fixed; u_1 should be thought of as the image of ∞ under such a map.

Let η'_1 be the counterflow line of h_1 starting from 0 with associated Loewner evolution (g_t^1) , Loewner driving function W^1 , and let $f_t^1 = g_t^1 - W_t^1$ be its centered Loewner evolution. Let τ_1 be the first time that η'_1 hits $\partial B(i, e^{-\beta})$. On $\{\tau_1 < \infty\}$, we let $\tilde{h}_1 = h_1 \circ (f_{\tau_1}^1)^{-1} - \chi \arg((f_{\tau_1}^1)^{-1})'$ and let $\tilde{\eta}_1^L$ (resp. $\tilde{\eta}_1^R$) be the flow line of $\tilde{h}_1$ starting from $-e^{2\Delta-\beta}|f_{\tau_1}^1(i)|$ (resp. $e^{2\Delta-\beta}|f_{\tau_1}^1(i)|$) with angle $\frac{\theta}{2}$ (resp. $-\frac{\theta}{2}$). We take $\eta_1^L = (f_{\tau_1}^1)^{-1}(\tilde{\eta}_1^L)$ (resp. $\eta_1^R = (f_{\tau_1}^1)^{-1}(\tilde{\eta}_1^R)$) and let $E_1 = E_{u_1}^{\beta, \Delta}(\eta_1^L, \eta_1^L, \eta_1^R, \mathbf{R}_-, \mathbf{R}_+)$. Let τ_1^L (resp. τ_1^R) be the first time that η_1^L (resp. η_1^R) hits $\mathbf{R}_-$, $\partial f_{\tau_1}^{-1}(e^{\Delta-\beta}|f_{\tau_1}^1(i)|\mathbf{D})$, or $\partial B(i, e^\Delta)$ (resp. $\mathbf{R}_+$, $\partial f_{\tau_1}^{-1}(e^{\Delta-\beta}|f_{\tau_1}^1(i)|\mathbf{D})$, or $\partial B(i, e^\Delta)$). Finally, we let φ_1 be the unique conformal transformation from the connected component of $\mathbf{H} \setminus \eta'_1([0, \tau_1])$ which contains i to $\mathbf{H}$ with $\varphi_1(i) = i$ and $\varphi_1(\eta'_1(\tau_1)) = 0$.

Suppose that $k \in \mathbf{N}$ and that, for each $1 \leq j \leq k$, paths η'_j , η_j^L , η_j^R , $\tilde{\eta}_j^L$, and $\tilde{\eta}_j^R$, Loewner evolutions (g_t^j) with driving functions W^j , centered Loewner evolutions $f_t^j = g_t^j - W_t^j$, conformal maps φ_j , stopping times τ_j , τ_j^L , τ_j^R , GFFs h_j , points $u_j \in \partial \mathbf{H} \setminus \{0\}$, and events E_j have been defined. We then take $h_{k+1} = h_k \circ \varphi_k^{-1} - \chi \arg(\varphi_k^{-1})'$, $u_{k+1} = \varphi_k(u_k)$, let η'_{k+1} be the counterflow line of h_{k+1} starting from 0, (g_t^{k+1}) its Loewner evolution, W^{k+1} its Loewner driving function, $f_t^{k+1} = g_t^{k+1} - W_t^{k+1}$ its centered Loewner evolution, and let τ_{k+1} be the first time that η'_{k+1} hits $\partial B(i, e^{-\beta})$. We define η_{k+1}^L , η_{k+1}^R , and φ_{k+1} analogously to η_1^L , η_1^R , and φ_1 , respectively, and we take $\tilde{\eta}_{k+1}^L = \varphi_k(\eta_{k+1}^L)$ and $\tilde{\eta}_{k+1}^R = \varphi_k(\eta_{k+1}^R)$. We let τ_{k+1}^L (resp. τ_{k+1}^R) be the first time that η_{k+1}^L (resp. η_{k+1}^R) hits $\tilde{\eta}_{k+1}^L$, $f_{\tau_{k+1}}^{-1}(e^{\Delta-\beta}|f_{\tau_{k+1}}^1(i)|\mathbf{D})$, or $\partial B(i, e^\Delta)$ (resp. $\tilde{\eta}_{k+1}^R$, $f_{\tau_{k+1}}^{-1}(e^{\Delta-\beta}|f_{\tau_{k+1}}^1(i)|\mathbf{D})$, or $\partial B(i, e^\Delta)$). We let

$$E_{k+1} = \begin{cases} E_{u_{k+1}}^{\beta, \Delta}(\eta'_{k+1}, \eta_{k+1}^L, \eta_{k+1}^R, \tilde{\eta}_{k+1}^L, \tilde{\eta}_{k+1}^R) & \text{on } E_k, \\ E_{u_{k+1}}^{\beta, \Delta}(\eta'_{k+1}, \eta_{k+1}^L, \eta_{k+1}^R, \mathbf{R}_-, \mathbf{R}_+) & \text{on } E_k^c. \end{cases}$$

We also let

$$E^{m,n} = \cap_{j=m+1}^n E_j \quad \text{and} \quad E^n = E^{0,n}.$$

Remark 4.2. We note that:

- (i) $E^{m,n}$ can occur even if only some of or perhaps none of $E_1, \dots, E_m$ occur and
- (ii) the conformal maps φ_j and starting points of the paths η_j^L and η_j^R are measurable with respect to η_1' .

Remark 4.3. The reason for assumption (ii) in the definition of the events E_j is that it implies that $\widehat{\eta}_{j+1}^L$ (resp. $\widehat{\eta}_{j+1}^R$) starts in $\mathbf{R}_-$ (resp. $\mathbf{R}_+$) for large enough values of $\Delta > 1$. Indeed, recall that $\widehat{\eta}_{j+1}^L = \varphi_j(\eta_j^L) = \varphi_j((f_{\tau_j}^j)^{-1}(\widetilde{\eta}_j^L))$ and the starting point of $\widetilde{\eta}_j^L$ is given by $-e^{2\Delta-\beta}|f_{\tau_j}'(i)|$. Assumption (ii) implies that there exists a constant $c_1 > 0$ such that $\varphi_j \circ (f_{\tau_j}^j)^{-1}$ maps $[-ce^{4\Delta}\text{Im}(f_{\tau_j}(i)), 0]$ into $\mathbf{R}_-$. The claim for $\widehat{\eta}_{j+1}^L$ follows since Lemma 2.2 implies that $\text{Im}(f_{\tau_j}(i)) \asymp |f_{\tau_j}'(i)|e^{-\beta}$ and the claim for $\widehat{\eta}_{j+1}^R$ is proved analogously.

For $z \in \mathbf{H}$, we let $\psi_z: \mathbf{H} \rightarrow \mathbf{H}$ be the unique conformal map with $\psi_z(0) = 0$ and $\psi_z(z) = i$. We define the events $E_n(z)$, $E^{m,n}(z)$, and $E^n(z)$ exactly in the same manner as E_n , $E^{m,n}$, and E^n except in terms of the paths which arise after applying the change of coordinates ψ_z . We similarly define paths $\eta_{z,k}'$, $\eta_{z,k}^L$, $\eta_{z,k}^R$, $\widehat{\eta}_{z,k}^L$, $\widehat{\eta}_{z,k}^R$, stopping times $\sigma_{z,k}$, $\tau_{z,k}$, $\tau_{z,k}^L$, $\tau_{z,k}^R$, and conformal maps $\varphi_{z,j}$. (In other words, everything defined as above except starting with the GFF $h_1 \circ \psi_z^{-1} - \chi \arg(\psi_z^{-1})'$ in place of h_1 .)

4.2. Estimates of probabilities. We are now going to give the one and two point estimates for the perfect points and then complete the proof of the lower bound for Theorems 1.1 and 1.3. Throughout, we let

$$\alpha = 2 - d(\kappa, \theta)$$

where $d(\kappa, \theta)$ is as in (1.1). Recall from the proof of Proposition 3.1 that this is the value of α from (3.4) with the choice of parameters $a = \lambda'$, $b = -\lambda'$, $\theta_1 = \frac{\theta}{2}$, and $\theta_2 = -\frac{\theta}{2}$. Since $\mathbf{L}(\theta)$ is increasing in θ , to prove the theorem we may assume without loss of generality that $\theta \in [0, \pi]$ is such that $d(\kappa, \theta) < 2$.

Proposition 4.4. *We have that*

$$\mathbf{P}[E^n] \asymp e^{-\beta(1+o_\beta(1))\alpha n}$$

where the $o_\beta(1)$ term tends to 0 as $\beta \rightarrow \infty$. Moreover, the rate at which the $o_\beta(1)$ term tends to 0 as $\beta \rightarrow \infty$ and the constants in $\asymp$ depend only on κ , θ , Δ , and u_1 .

The proof of Proposition 4.4 has two inputs. The first is the following lemma and the second is Lemma 4.6.

Lemma 4.5. *There exists $\Delta_0 > 1$ such that for all $\beta > \Delta^2 > \Delta > \Delta_0$ we have that*

$$\mathbf{P}[E_1] \asymp e^{-\beta(1+o_\beta(1))\alpha} \quad (4.1)$$

where the $o_\beta(1)$ term tends to 0 as $\beta \rightarrow \infty$. Moreover, the rate at which the $o_\beta(1)$ term tends to 0 as $\beta \rightarrow \infty$ and the constants in $\asymp$ depend only on κ , θ , Δ , and u_1 .

Proof. Let F_1 be the event that $\{\tau_1 < \infty\}$, $\eta'_1([0, \tau_1]) \subseteq B(i, e^\Delta)$, and the harmonic measure of the left (resp. right) side of $\eta'_1([0, \tau_1])$ as seen from i is at least $\frac{1}{2} - e^{-4\Delta}$. By [MW17, Equation (3.8) of Lemma 3.4], there exists a universal constant $C > 0$ such that

$$|f_{\tau_1}(0^-)| + |f_{\tau_1}(0^+)| \leq Ce^\Delta \quad \text{on } F_1. \quad (4.2)$$

For $q \in \{L, R\}$, let $\tilde{\zeta}_1^q$ (resp. $\hat{\zeta}_1^q$) be the first time that $\tilde{\eta}_1^q$ hits $\partial\mathbf{D}$ (resp. $\partial(\frac{1}{2}\mathbf{D})$). Let F_2 be the event that all of the following hold:

- (i) $\tilde{\eta}_1^L([0, \tilde{\zeta}_1^L]) \cap \tilde{\eta}_1^R([0, \tilde{\zeta}_1^R]) = \emptyset$,
- (ii) $\text{dist}(\tilde{\eta}_1^L([\tilde{\zeta}_1^L, \tilde{\zeta}_1^L]), \tilde{\eta}_1^R([\hat{\zeta}_1^R, \tilde{\zeta}_1^R])) \geq e^{-\Delta}$, and
- (iii) $\tilde{\eta}_1^q([0, \tilde{\zeta}_1^q]) \cap (e^{\Delta-\beta}|f'_{\tau_1}(i)|\mathbf{D}) = \emptyset$ for $q \in \{L, R\}$.

Lemma 3.7 together with Lemma 2.1 implies that

$$\mathbf{P}[F_2 \mid F_1, |f'_{\tau_1}(i)|] \asymp (e^{-\beta}|f'_{\tau_1}(i)|)^\alpha$$

where the constants in $\asymp$ depend only on κ, θ, Δ , and u_1 . Combining, we have that

$$\mathbf{P}[F_1 \cap F_2] \asymp e^{-\beta\alpha} \mathbf{E}[|f'_{\tau_1}(i)|^\alpha \mathbf{1}_{F_1}]. \quad (4.3)$$

Applying Proposition 3.3 as in the proof of Proposition 3.1 with the value of r as in (3.26) (except we use (3.3) in place of (3.2)) we see that

$$\mathbf{P}[F_1 \cap F_2] \asymp e^{-\beta(1+o_\beta(1))\alpha} \quad (4.4)$$

where the rate at which the $o_\beta(1)$ term tends to zero as $\beta \rightarrow \infty$ and the constants in $\asymp$ depend only on κ, θ, Δ , and u_1 . Let F_3 be the event that $\tilde{\eta}_1^L|_{[\tilde{\zeta}_1^L, \infty)}$ (resp. $\tilde{\eta}_1^R|_{[\tilde{\zeta}_1^R, \infty)}$) hits the left (resp. right) component of $\mathbf{R} \setminus f_{\tau_1}(\eta'([0, \tau_1]))$ before intersecting $\tilde{\eta}_1^R$ (resp. $\tilde{\eta}_1^L$) and before intersecting $e^{\Delta-\beta}|f'_{\tau_1}(i)|\mathbf{D}$. Then (4.2), [MW17, Lemma 2.3], and [MW17, Lemma 2.5] together imply that there exists a constant $p_1 > 0$ depending only on κ, θ, Δ , and u_1 such that

$$\mathbf{P}[F_3 \mid F_1, F_2] \geq p_1.$$

This proves the lemma since $E_1 = \cap_{i=1}^3 F_i$. $\square$

Let

$$\varphi^{j,k} = \varphi_{j+1} \circ \cdots \circ \varphi_k \quad \text{for } j < k \quad \text{and} \quad \varphi^k = \varphi^{0,k}. \quad (4.5)$$

Lemma 4.6. *There exists $\Delta_0 > 1$ such that for all $\beta > \Delta^2 > \Delta > \Delta_0$ we have that*

$$\mathbf{P}[E^{m,n} \mid E^k, E_m] \asymp \mathbf{P}[E^{n-m}] \quad \text{for } 1 \leq k \leq m \leq n-1$$

where the constants in $\asymp$ depend only on κ, θ, Δ , and u_1 .

Proof. See Fig. 14 for an illustration of the setup of the proof. Let T_Δ be the $e^{-\Delta}$ -neighborhood of $[0, i]$. Throughout, unless explicitly stated otherwise, we shall assume that the paths in the proof are stopped upon exiting T_Δ . By the definition of the stopping times τ_j^q , we know that $\eta_j^q([0, \tau_j^q])$ does not intersect $(f_{\tau_j}^j)^{-1}(e^{\Delta-\beta}|(f_{\tau_j}^j)'(i)|\mathbf{D})$ for $q \in \{L, R\}$ and $1 \leq j \leq m$. By Lemma 2.2, we know that $\text{Im}(f_{\tau_j}^j(i)) \asymp |(f_{\tau_j}^j)'(i)|e^{-\beta}$. It follows that the probability that a Brownian motion starting from

Fig. 14. Setup for the proof of Lemma 4.6, which gives the approximate independence of the events E^m and $E^{m,n}$. Recall that $\varphi^{j,k} = \varphi^{j+1} \circ \dots \circ \varphi^k$. Let T_Δ be the $e^{-\Delta}$ -neighborhood of $[0, i]$. On the event that η_{m+1}^L and $(\varphi^{m,n-1})^{-1}(\eta_n^L)$ merge as shown, the paths $(\varphi^{m,m+1})^{-1}(\eta_{m+2}^L), \dots, (\varphi^{m,n-2})^{-1}(\eta_{n-1}^L)$ stopped upon merging with either η_{m+1}^L or $(\varphi^{m,n-1})^{-1}(\eta_n^L)$ are contained in the light green region which is to the left of η'_{m+1} . The analogous statement holds for the paths $(\varphi^{m,m+1})^{-1}(\eta_{m+2}^R), \dots, (\varphi^{m,n-2})^{-1}(\eta_{n-1}^R)$

$f_{\tau_j^i}^j(i)$ hits $\partial(e^{\Delta-\beta}|(f_{\tau_j^i}^j)'(i)|\mathbf{D})$ before hitting $\partial\mathbf{H}$ is $\asymp e^{-\Delta}$. Consequently, it follows from the conformal invariance of Brownian motion that there exists a constant $c > 0$ such that $\varphi_j((f_{\tau_j^i}^j)^{-1}(e^{\Delta-\beta}|(f_{\tau_j^i}^j)'(i)|\mathbf{D})) \cap B(i, ce^\Delta) = \emptyset$. This, in turn, implies that $\varphi^{j-1,m}(\eta_j^q([0, \tau_j^q]))$ does not intersect $B(i, ce^\Delta)$ for $q \in \{L, R\}$ and $1 \leq j \leq m$. Let $\mathcal{G}_m = \sigma(\varphi^{j-1,m}(\eta_j^q([0, \tau_j^q])) : q \in \{L, R\}, 1 \leq j \leq m)$. By Lemma 2.1, we thus have that the law of η'_{m+1} given $\mathcal{G}_m$ and the law of η'_1 are mutually absolutely continuous with Radon–Nikodym derivative which is bounded from above and below by a finite and positive constant provided we make $\Delta_0 > 1$ sufficiently large. Fix a path γ' which is contained in the support of the law of η'_1 . Again applying Lemma 2.1, we also have that the Radon–Nikodym derivative between the conditional law of (η_1^L, η_1^R) given $\eta'_1 = \gamma'$ and $(\eta_{m+1}^L, \eta_{m+1}^R)$ given $\mathcal{G}_m$ and $\eta'_{m+1} = \gamma'$ is bounded from above and below by finite and positive constants. Fix a pair of paths (γ_1^L, γ_1^R) which are contained in the support of the law of (η_1^L, η_1^R) given $\eta'_1 = \gamma'$. Applying Lemma 2.1 a final time, we have that the Radon–Nikodym derivative between the conditional law of $(\eta_{n-m}^L, \eta_{n-m}^R)$ given $\eta'_1 = \gamma'$ and $(\eta_1^L, \eta_1^R) = (\gamma_1^L, \gamma_1^R)$ and the conditional law of (η_n^L, η_n^R) given $\mathcal{G}_m$, $\eta'_{m+1} = \gamma'$, and $(\eta_{m+1}^L, \eta_{m+1}^R) = (\gamma_1^L, \gamma_1^R)$ is bounded from above and below by finite and positive constants. If γ_1^L and $(\varphi^{n-m-1})^{-1}(\gamma_{n-m}^L)$ intersect each other and γ_1^R and $(\varphi^{n-m-1})^{-1}(\gamma_{n-m}^R)$ intersect each other, then the conditional law of $(\varphi^{1,2})^{-1}(\eta_2^q), \dots, (\varphi^{1,n-m-2})^{-1}(\eta_{n-m-1}^q)$ for $q \in \{L, R\}$ given $\eta'_1 = \gamma'$, $(\eta_1^L, \eta_1^R) = (\gamma_1^L, \gamma_1^R)$, and $(\eta_{n-m}^L, \eta_{n-m}^R) = (\gamma_{n-m}^L, \gamma_{n-m}^R)$ is equal to the conditional law of $(\varphi^{m,m+1})^{-1}(\eta_{m+2}^q), \dots, (\varphi^{m,n-2})^{-1}(\eta_{n-1}^q)$ for $q \in \{L, R\}$ given $\mathcal{G}_m$, $\eta'_{m+1} = \gamma'$, $(\eta_{m+1}^L, \eta_{m+1}^R) = (\gamma_1^L, \gamma_1^R)$, and $(\eta_n^L, \eta_n^R) = (\gamma_{n-m}^L, \gamma_{n-m}^R)$.

By [MW17, Lemma 2.5], we know that the conditional probability that η_1^L and η_1^R hit $\mathbf{R}_-$ and $\mathbf{R}_+$, respectively, before leaving $B(i, e^\Delta)$ and intersecting each other given their realization up until exiting T_Δ and the other paths is uniformly positive. Similarly,

[MW17, Lemma 2.5] implies that, on E^m , the conditional probability that η_{m+1}^L and η_{m+1}^R merge into $\hat{\eta}_{m+1}^L$ and $\hat{\eta}_{m+1}^R$, respectively, before leaving $B(i, e^\Delta)$ given their realization up until exiting T_Δ and the other paths is uniformly positive. Combining everything completes the proof. $\square$

Proof of Proposition 4.4. This follows by using Lemma 4.6 to iterate the estimate from Lemma 4.5. $\square$

Now that we have proved the one point estimate for the perfect points, we turn to establish the two point estimate. We let

$$\varphi_z^{j,k} = \varphi_{z,j+1} \circ \cdots \circ \varphi_{z,k} \quad \text{for } j < k \quad \text{and} \quad \varphi_z^k = \varphi_z^{0,k}. \quad (4.6)$$

For each $n \in \mathbf{N}$ and $z \in \mathbf{H}$, we also let

$$\underline{V}_n(z) = B(z, 2^{-5n-5} \text{Im}(z) e^{-n\beta}) \quad \text{and} \quad \overline{V}_n(z) = B(z, 2^{5n+5} \text{Im}(z) e^{-n\beta}).$$

Lemma 4.7. *There exists $\Delta_0 > 1$ such that for all $\beta > \Delta^2 > \Delta \geq \Delta_0$, the following are true.*

- (i) *For each $m, n \in \mathbf{N}$ with $m \geq n+2$, on $\{\tau_{z,m} < \infty\}$ we have that $\psi_z^{-1} \circ (\varphi_z^{m-1})^{-1}(\gamma) \subseteq \overline{V}_n(z)$ for $\gamma = \eta_{z,m}^q|_{[0, \tau_{z,m}^q]}$ for $q \in \{L, R\}$ and for $\gamma = \eta'_{z,m}|_{[0, \tau_{z,m}]}$.*
- (ii) *For each $m, n \in \mathbf{N}$ with $m+2 \leq n$, on $\{\tau_{z,m} < \infty\}$ we have that $\psi_z^{-1} \circ (\varphi_z^{m-1})^{-1}(\gamma) \cap \underline{V}_n(z) = \emptyset$ for $\gamma = \eta_{z,m}^q|_{[0, \tau_{z,m}^q]}$ for $q \in \{L, R\}$ and for $\gamma = \eta'_{z,m}|_{[0, \tau_{z,m}]}$.*

Proof. We are first going to give the proof in the case that $z = i$ and we will first establish part (i). Fix $m, n \in \mathbf{N}$ with $m \geq n+2$. Throughout, we shall assume that we are working on E^m . It follows from Lemma 2.3 that if $r \in (0, \frac{1}{2}]$ then

$$B(i, \frac{1}{16} r e^{-\beta}) \subseteq \varphi_k^{-1}(B(i, r)) \subseteq B(i, 16 r e^{-\beta}) \quad \text{for } 1 \leq k \leq m. \quad (4.7)$$

Iterating (4.7) implies that

$$B(i, 2^{-5k} e^{-k\beta}) \subseteq (\varphi^k)^{-1}(B(i, \frac{1}{2})) \subseteq B(i, 2^{5k} e^{-k\beta}) \quad \text{for } 1 \leq k \leq m \quad (4.8)$$

(provided we take $\Delta_0 > 1$ large enough).

Note that $\eta_m^q([0, \tau_m^q]) \subseteq B(i, e^\Delta)$ for $q \in \{L, R\}$ by the definition of the events. Consequently, it follows from Lemma 4.1 that $\varphi_{m-1}^{-1}(\eta_m^q([0, \tau_m^q])) \subseteq B(i, e^{-\beta/4})$ for $q \in \{L, R\}$ provided $\Delta_0 > 1$ is large enough. We also assume that $\Delta_0 > 1$ is sufficiently large so that $e^{-\Delta_0/4} < \frac{1}{2}$. Applying (4.8) proves part (i) for $\eta_m^q|_{[0, \tau_m^q]}$ for $q \in \{L, R\}$; the proof for $\eta'_m|_{[0, \tau_m]}$ is analogous. This proves part (i) for $z = i$. For the case that $z \neq i$, we note that applying Lemma 2.3 with $r \in (0, \frac{1}{2}]$ again yields,

$$B(i, \frac{1}{16} r \text{Im}(z)) \subseteq \psi_z^{-1}(B(i, r)) \subseteq B(i, 16 r \text{Im}(z)). \quad (4.9)$$

Combining (4.8) with (4.9) gives part (i). The proof of part (ii) is the same. $\square$

Proposition 4.8. *Fix $\delta \in (0, \frac{\pi}{2})$. Suppose that $z, w \in \mathbf{H} \cap \mathbf{D}$ are distinct with $\arg(z), \arg(w) \in (\delta, \pi - \delta)$. Let m be the smallest integer such that $\overline{V}_{m-1}(z) \cap \overline{V}_{m-1}(w) = \emptyset$. Then we have that*

$$\mathbf{P}[E^n(z), E^n(w)] \lesssim e^{O(\beta) + \beta(1+o_\beta(1))\alpha m} \mathbf{P}[E^n(z)] \mathbf{P}[E^n(w)]$$

where the $o_\beta(1)$ term tends to zero as $\beta \rightarrow \infty$. Moreover, the rate at which the $o_\beta(1)$ term tends to zero and the constants in the $O(\beta)$ term and $\asymp$ depend only on κ, θ, u_1 , and δ .

Fig. 15. Illustration of the setup for the two-point estimate proved in Lemma 4.9. Shown is η'_1 on the event that it hits $\bar{V}_k(w)$ (not shown) before hitting $\bar{V}_{m-1}(z)$ as well as the auxiliary paths involved in the definition of the events $E^{m-5}(w)$, $E^{m,k}(w)$, and $E^{m,n}(z)$. By Lemma 4.7, these paths are contained in $\mathbf{H} \setminus \bar{V}_{m-2}(w)$, $\bar{V}_{m-1}(w)$, and $\bar{V}_{m-1}(z)$ respectively. Since these regions are disjoint, we can use the Radon–Nikodym estimate Lemma 2.1 to show that the events $E^{m-5}(w)$, $E^{m,k}(w)$, and $E^{m,n}(z)$ are approximately independent

Before we prove Proposition 4.8, we will need to collect the following lemma which is the analog of Lemma 4.6 in the setting of two points. For each $z \in \mathbf{H}$ and $k \in \mathbf{N}$, we let $\mathcal{F}_m^k(z)$ be the σ -algebra generated by $\eta'_{z,j}|_{[0,\tau'_{z,j}]}$ for $1 \leq j \leq k$ and $\eta^q_{z,j}|_{[0,\tau^q_{z,j}]}$ for $q \in \{L, R\}$ and $1 \leq j \leq k$, $j \neq m-4, \dots, m$. See Fig. 15 for an illustration of the setup as well as the proof.

Lemma 4.9. *There exists $\Delta_0 > 1$ such that $\beta > \Delta^2 > \Delta > \Delta_0$ implies that the following is true. Fix $\delta \in (0, \frac{\pi}{2})$ and suppose that $z, w \in \mathbf{H} \cap \mathbf{D}$ are distinct with $\arg(z), \arg(w) \in (\delta, \pi - \delta)$. Let m be the smallest integer such that $\bar{V}_{m-1}(z) \cap \bar{V}_{m-1}(w) = \emptyset$. Fix $n \geq m$ and let P_w be the event that η'_1 hits $\bar{V}_k(w)$ before hitting $\bar{V}_{m-1}(z)$. Let $E_m^k(w) = E^{m-5}(w) \cap E^{m,k}(w)$. For all $k \geq m$, we have that*

$$\mathbf{P}[E^{m+1,n}(z) \mid \mathcal{F}_m^k(w), E_{m+1}] \mathbf{1}_{E_m^k(w), P_w} \asymp \mathbf{P}[E^{n-m-1}] \mathbf{1}_{E_m^k(w), P_w} \quad (4.10)$$

where the constants in $\asymp$ depend only on $\kappa, \theta, \Delta, u_1$, and δ .

Proof. We assume that $\Delta_0 > 1$ is sufficiently large so that $\beta > \Delta_0$ implies that $\bar{V}_{n+1}(u) \subseteq \bar{V}_n(u)$ for all $n \in \mathbf{N}$ and $u \in \mathbf{H}$ and also so that Lemma 4.7 holds. The proof is analogous to that of Lemma 4.6. By applying ψ_z , we may assume without generality that $z = i$. Note that the event $E^{m,n}$ is defined in terms of the paths $(\varphi^{j-1})^{-1}(\eta'_j|_{[0,\tau_j]})$ and $(\varphi^{j-1})^{-1}(\eta^q_j|_{[0,\tau^q_j]})$ for $q \in \{L, R\}$ and $m+1 \leq j \leq n$ and that the event $E_m^k(z)$ is defined in terms of the paths $(\varphi^{j-1})^{-1}(\eta'_{w,j}|_{[0,\tau_{w,j}]})$ and $(\varphi^{j-1})^{-1}(\eta^q_{w,j}|_{[0,\tau^q_{w,j}]})$ for $q \in \{L, R\}$ and $1 \leq j \leq k$, $j \neq m-4, \dots, m$. Lemma 4.7 implies that the paths involved in the definition of $E^{m,n}$ (resp. $E^{m,k}(w)$) are contained in $\bar{V}_{m-1}(i)$ (resp. $\bar{V}_{m-1}(w)$). Lemma 4.7 also implies that the paths involved in the definition of $E^{m-5}(w)$ do not intersect $\bar{V}_{m-3}(w) \subseteq \bar{V}_{m-2}(w)$. By the choice of Δ_0 , we have that $\bar{V}_{m-2}(w)$ contains $\bar{V}_{m-1}(i)$ and $\bar{V}_{m-1}(w)$. That is, the paths involved in the definition of $E^{m,n}$ and those

involved in the definition of $E_m^k(w)$ are disjoint on the event P_w . Thus by conformally mapping back and using the Radon–Nikodym derivative estimate Lemma 2.1 as in the proof of Lemma 4.6 it is not hard to see that (4.10) holds, as desired. $\square$

Proof of Proposition 4.8. We are going to extract the result from Lemma 4.9. Let m , $E_m^n(z)$, P_z , $E_m^n(w)$, and P_w be as in the statement of Lemma 4.9 and assume that $\Delta_0 > 1$ is large enough so that Lemmas 4.5, 4.6 and 4.9 hold. We have that,

$$\begin{aligned} \mathbf{P}[E^n(z), E^n(w)] &\leq \mathbf{E} \left[\mathbf{P}[E^{m+1,n}(z) \mid \mathcal{F}_m^n(w), E_{m+1}(z)] \mathbf{1}_{E_m^n(w), P_w} \right] \\ &\quad + \mathbf{E} \left[\mathbf{P}[E^{m+1,n}(w) \mid \mathcal{F}_m^n(z), E_{m+1}(w)] \mathbf{1}_{E_m^n(z), P_z} \right]. \end{aligned}$$

We are now going to explain how to bound the first summand above. The second summand is bounded similarly, so this will complete the proof. We have that,

$$\begin{aligned} &\mathbf{E} \left[\mathbf{P}[E^{m+1,n}(z) \mid \mathcal{F}_m^n(w), E_{m+1}(z)] \mathbf{1}_{E_m^n(w), P_w} \right] \\ &\asymp \mathbf{P}[E^{n-m-1}] \mathbf{P}[E_m^n(w)] \quad (\text{Lemma 4.9}) \\ &\lesssim e^{O(\beta)} \mathbf{P}[E^{n-m}] \mathbf{P}[E^n(w)] \quad (\text{Lemmas 4.5 and 4.6}) \\ &\asymp \frac{e^{O(\beta)}}{\mathbf{P}[E^m]} \mathbf{P}[E^n(z)] \mathbf{P}[E^n(w)] \quad (\text{Lemma 4.6}) \\ &\leq e^{O(\beta) + \beta(1+o_\beta(1))\alpha m} \mathbf{P}[E^n(z)] \mathbf{P}[E^n(w)] \quad (\text{Proposition 4.4}). \end{aligned}$$

$\square$

See the left side of Fig. 16 for an illustration of the of the setup of the following lemma, which we will use to show that the perfect points are almost surely contained in $\mathbf{L}(\theta)$.

Lemma 4.10. *Suppose that τ is an almost surely finite stopping time for η' . Fix x^L (resp. x^R) on the left (resp. right) side of $\eta'([0, \tau])$ and let η^L (resp. η^R) be the flow line with angle $\frac{\theta}{2}$ (resp. $-\frac{\theta}{2}$) starting from x^L (resp. x^R) stopped upon hitting $\mathbf{R}_-$ (resp. $\mathbf{R}_+$). Let A be the segment on the outer boundary of $\eta'([0, \tau])$ which runs from x^L to x^R with a clockwise orientation. On the event that η^L and η^R do not intersect each other, we have that $\mathbf{L}(\theta) \cap A \neq \emptyset$ almost surely.*

Proof. It follows from the flow line interaction rules [MS16b, Theorem 1.5] that the left side of $\mathbf{L}(\theta)$ cannot cross η^L from right to left (otherwise it would intersect with a height difference of $-\pi\chi$) and it cannot hit $\mathbf{R}_-$. Similarly, the right side of $\mathbf{L}(\theta)$ cannot cross η^R from left to right and it cannot hit $\mathbf{R}_+$. It thus follows that either the left or right side of $\mathbf{L}(\theta)$ hits A or A is contained inside the region surrounded by the outer boundary of $\mathbf{L}(\theta)$. In the former case, there is nothing to prove so we shall assume that we are in the latter case. Assume for contradiction that $\mathbf{L}(\theta)$ does not intersect A . Then x^L and x^R are both contained in a common complementary pocket P of $\mathbf{L}(\theta)$ as shown in the right panel of Fig. 16. It follows from the flow line interaction rules that η^L cannot intersect the right side of ∂P (otherwise it would intersect with a height difference of $-\pi\chi$, counted from right to left) and η^R cannot intersect the left side of ∂P (otherwise it would intersect with a height difference of $-\pi\chi$, counted from right to left). Moreover, η^L (resp. η^R) is prevented from intersecting the left (resp. right) side of ∂P because doing so would force η^L (resp. η^R) either to cross η^R (resp. η^L) or to cross η' (not shown

Fig. 16. Suppose that τ is an almost surely finite stopping time for η' . Fix a point x^L (resp. x^R) on the left (resp. right) side of $\eta'([0, \tau])$ and let η^L (resp. η^R) be the flow line of h starting from x^L (resp. x^R) with angle $\frac{\theta}{2}$ (resp. $-\frac{\theta}{2}$) stopped upon hitting $\mathbf{R}_-$ (resp. $\mathbf{R}_+$). Let A be the clockwise segment of the outer boundary of $\eta'([0, \tau])$ which runs from x^L to x^R (purple in the illustration). We show in Lemma 4.10 that on the event that η^L and η^R do not intersect each other we have that $\mathbf{L}(\theta) \cap A \neq \emptyset$ almost surely; this is the converse of the statement explained in Fig. 12. The reason that the “right” and “left” sides of $\mathbf{L}(\theta)$ appear to be flipped above is because $\mathbf{L}(\theta)$ is growing from ∞ . Note that the angle of the flow line which makes up the left (resp. right) side of $\mathbf{L}(\theta)$ is the same as that which makes up the right (resp. left) side of P . In the right panel, we have only drawn part of η' which is in P ; the initial part of η' has been omitted. Note that the initial part of η' in particular contains the closing point of P because counterflow lines visit the ranges of flow lines in reverse chronological order. This is what prevents η^L, η^R from exiting P at the closing point

in the right panel of Fig. 16). Therefore the only possibility is that both η^L and η^R exit P from its opening point as shown in the right panel of Fig. 16. This is a contradiction because then η^L and η^R are forced to intersect at the pocket opening point. $\square$

For each $\beta > 1$, let $\mathcal{D}_n^\beta$ be the set of squares with corners in $e^{-\beta n} \mathbf{Z}^2$ which are contained in $[-1, 1] \times [1, 2]$. As before, we let $z(Q)$ denote the center of a given square Q and, for each $z \in [-1, 1] \times [1, 2]$ and $n \in \mathbf{N}$, we let $Q_n(z)$ denote the element of $\mathcal{D}_n^\beta$ which contains z . Let $\mathcal{E}_n^\beta = \{z(Q) : Q \in \mathcal{D}_n^\beta\}$. For each $\Delta > 1$ such that $\beta > \Delta^2 > \Delta$, let $\mathcal{C}_n^{\beta, \Delta}$ consist of those $Q \in \mathcal{D}_n^\beta$ with $Q \subseteq [-1, 1] \times [1, 2]$ for which $E^n(z(Q))$ occurs. Let

$$\mathcal{P}^{\beta, \Delta} = \overline{\bigcap_{n \in \mathbf{N}} \bigcup_{Q \in \mathcal{C}_n^{\beta, \Delta}} Q}.$$

Lemma 4.11. *There exists $\Delta_0 > 1$ such that for each $\beta, \Delta > 1$ with $\beta > \Delta^2 > \Delta > \Delta_0$ we have that $\mathcal{P}^{\beta, \Delta} \subseteq \mathbf{L}(\theta)$ almost surely.*

Proof of Lemma 4.11. This follows from Lemma 4.10 and the definition of the events. $\square$

We can now complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Standard arguments for computing the Hausdorff dimension of a random fractal imply that an estimate of the form given in Proposition 4.8 combined with Lemma 4.11 gives that, for each $\xi > 0$, the probability of the event that $\dim_{\mathcal{H}}(\mathbf{L}(\theta)) \geq d(\kappa, \theta) - 2\xi$ is positive (see, for example, the arguments in [DPRZ01, HMP10, MSW14, MW17, GMS14, MWW16]). For completeness, we will include the entire argument. For each $n \in \mathbb{N}$, let μ_n be the measure on $X = [-1, 1] \times [1, 2]$ defined by

$$\mu_n(A) = \int_A \sum_{z \in \mathcal{E}_n^\beta} \frac{\mathbf{1}_{E^n}(z)}{\mathbf{P}[E^n(z)]} \mathbf{1}_{Q_n(z)}(z') dz' \quad \text{for } A \subseteq X \text{ Borel.}$$

Then $\mathbf{E}[\mu_n(X)] = 1$. Recall that $\alpha = 2 - d(\kappa, \theta)$. Moreover, we have that

$$\begin{aligned} \mathbf{E}[\mu_n(X)^2] &= e^{-4\beta n} \sum_{\substack{z, w \in \mathcal{E}_n^\beta \\ z \neq w}} \frac{\mathbf{P}[E^n(z) \cap E^n(w)]}{\mathbf{P}[E^n(z)]\mathbf{P}[E^n(w)]} \\ &= e^{-4\beta n} \sum_{\substack{z, w \in \mathcal{E}_n^\beta \\ z \neq w}} \frac{\mathbf{P}[E^n(z) \cap E^n(w)]}{\mathbf{P}[E^n(z)]\mathbf{P}[E^n(w)]} + e^{-4\beta n} \sum_{z \in \mathcal{E}_n^\beta} \frac{1}{\mathbf{P}[E^n(z)]}. \end{aligned}$$

If we choose n, β , and Δ large enough, then applying Proposition 4.8 to the first summand and Proposition 4.4 to the second summand yields that the above is bounded by

$$\lesssim e^{-4\beta n} \sum_{\substack{z, w \in \mathcal{E}_n^\beta \\ z \neq w}} |z - w|^{-\alpha - \xi} + e^{-4\beta n} \sum_{z \in \mathcal{E}_n^\beta} e^{(\alpha + \xi)\beta n} \lesssim 1$$

Set $d_\xi = d(\kappa, \theta) - 2\xi$. Let $I_{d_\xi}(\mu)$ denote the d_ξ -energy of μ . We also have that

$$\begin{aligned} \mathbf{E}[I_{d_\xi}(\mu_n)] &= \sum_{\substack{z, w \in \mathcal{E}_n^\beta \\ z \neq w}} \frac{\mathbf{P}[E^n(z) \cap E^n(w)]}{\mathbf{P}[E^n(z)]\mathbf{P}[E^n(w)]} \iint_{Q_n(z) \times Q_n(w)} \frac{dz' dw'}{|z' - w'|^{d_\xi}} \\ &= \sum_{\substack{z, w \in \mathcal{E}_n^\beta \\ z \neq w}} \frac{\mathbf{P}[E^n(z) \cap E^n(w)]}{\mathbf{P}[E^n(z)]\mathbf{P}[E^n(w)]} \iint_{Q_n(z) \times Q_n(w)} \frac{dz' dw'}{|z' - w'|^{d_\xi}} \\ &\quad + \sum_{z \in \mathcal{E}_n^\beta} \frac{1}{\mathbf{P}[E^n(z)]} \iint_{Q_n(z) \times Q_n(z)} \frac{dz' dw'}{|z' - w'|^{d_\xi}} \\ &\lesssim \sum_{\substack{z, w \in \mathcal{E}_n^\beta \\ z \neq w}} \frac{\mathbf{P}[E^n(z) \cap E^n(w)]}{\mathbf{P}[E^n(z)]\mathbf{P}[E^n(w)]} \left(\frac{e^{-4\beta n}}{|z - w|^{d_\xi}} \right) + \sum_{z \in \mathcal{E}_n^\beta} \frac{1}{\mathbf{P}[E^n(z)]} \times e^{(d_\xi - 4)\beta n} \\ &\lesssim \sum_{\substack{z, w \in \mathcal{E}_n^\beta \\ z \neq w}} |z - w|^{-\alpha - \xi} \times e^{-4\beta n} \times |z - w|^{-d_\xi} \\ &\quad + \sum_{z \in \mathcal{E}_n^\beta} e^{(\alpha + \xi)\beta n} \times e^{(d_\xi - 4)\beta n} \lesssim 1. \end{aligned}$$

Consequently, the sequence (μ_n) has a subsequence (μ_{n_k}) that converges weakly to some measure μ which is non-zero with positive probability. It is clear that μ is supported on $\mathcal{P}^{\beta, \Delta}$ and has finite d_ξ -energy. From [MP10, Theorem 4.27], we know that

$$\mathbf{P}[\dim_{\mathcal{H}}(\mathbf{L}(\theta)) \geq d_\xi] > 0.$$

It is left to explain the 0-1 law: that for each $d \in [0, 2]$, $\mathbf{P}[\dim_{\mathcal{H}}(\mathbf{L}(\theta)) = d] \in \{0, 1\}$. We will use the same argument used in the proof of [MW17, Theorem 1.5]. By swapping the roles of 0 and ∞ using the conformal transformation $z \mapsto -1/z$, we now assume that $\mathbf{L}(\theta)$ grows from 0 towards ∞ rather than from ∞ towards 0. For each $r > 0$, we let $D_r = \dim_{\mathcal{H}}(\mathbf{L}(\theta) \cap B(0, r) \cap \mathbf{H})$. It is clear that $0 < r_1 < r_2$ implies $D_{r_1} \leq D_{r_2}$. By the scale invariance of the setup, we have that D_{r_1} has the same law as D_{r_2} . Thus $D_{r_1} = D_{r_2}$ almost surely for all $0 < r_1 < r_2$. In particular, $\mathbf{P}[D_\infty = D_r] = 1$ for all $r > 0$. Thus the events $\{D_\infty = d\}$ and $\{D_r = d\}$ are the same up to a set of probability zero. The latter is measurable with respect to the h restricted to $B(0, r)$. Letting $r \downarrow 0$, we see that this implies that the event $\{D_\infty = d\}$ is trivial, which completes the proof. $\square$

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