Quaternionic Kähler Detour Complexes and $\mathcal{N} = 2$ Supersymmetric Black Holes

D. Cherney¹, E. Latini^{1,2}, A. Waldron¹

¹ Department of Mathematics, University of California, Davis, CA 95616, USA.

E-mail: cherney@math.ucdavis.edu; emanuele@math.ucdavis.edu; wally@math.ucdavis.edu

² INFN, Laboratori Nazionali di Frascati, CP 13, I-00044 Frascati, Italy. E-mail: latini@lnf.infn.it

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Abstract: We study a class of supersymmetric spinning particle models derived from the radial quantization of stationary, spherically symmetric black holes of four dimensional $\mathcal{N} = 2$ supergravities. By virtue of the c -map, these spinning particles move in quaternionic Kähler manifolds. Their spinning degrees of freedom describe mini-superspace-reduced supergravity fermions. We quantize these models using BRST detour complex technology. The construction of a nilpotent BRST charge is achieved by using local (worldline) supersymmetry ghosts to generate special holonomy transformations. (An interesting byproduct of the construction is a novel Dirac operator on the superghost extended Hilbert space.) The resulting quantized models are gauge invariant field theories with fields equaling sections of special quaternionic vector bundles. They underly and generalize the quaternionic version of Dolbeault cohomology discovered by Baston. In fact, Baston’s complex is related to the BPS sector of the models we write down. Our results rely on a calculus of operators on quaternionic Kähler manifolds that follows from BRST machinery, and although directly motivated by black hole physics, can be broadly applied to any model relying on quaternionic geometry.

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1. Introduction

The main result of this paper is a detour complex for quaternionic Kähler manifolds. In physics language, this amounts to a gauge theory of higher (quaternionic) “forms” on these manifolds. To be precise, we utilize special holonomy to split the tangent bundle of a $4n$ -dimensional quaternionic Kähler manifold M into a product of rank 2 and $2n$ vector bundles H and E [1],

$$TM \cong E \otimes H,$$

and present an equation of motion and gauge invariances for sections of $\wedge E$ (or, more generally, $\wedge E \otimes \odot H$).

The results of the paper will appeal to multiple audiences including: (i) Those readers interested in the differential geometry of quaternionic Kähler spaces. (ii) Readers studying various supersymmetric quantum mechanical and spinning particle models in quaternionic Kähler and hyperKähler backgrounds (such as gravitational instanton moduli spaces [2], Hitchin’s moduli space of stable Higgs bundles [3,4], geometric Langlands theory [5] and hypermultiplet moduli spaces [6], to name a few). (iii) Readers looking for applications of the BRST detour quantization of orthosymplectic constraint algebras developed for applications to higher spin systems in [7], on which these results heavily rely. (iv) Readers wanting to apply our results to supergravity (SUGRA) black hole quantization since, remarkably, the mathematical structure presented above is exactly what is called for when studying the minisuperspace quantization of $\mathcal{N} = 2$ SUGRA black holes [8–11]. (In particular, wavefunctions valued in $\wedge E$ describe the fermionic degrees of freedom of these models.) Therefore the paper is structured so that any of these readerships can easily extract the information they need.

In Sect. 2, we introduce the notion of a detour complex, beginning with simple examples. We then generalize our previous results on Kähler detour complexes to hyperKähler manifolds. This result follows immediately from an isomorphism between super Lie algebras of geometric operators mapping Dolbeault and Lefschetz operators on Kähler forms to their hyperKähler analogues acting on sections of $\wedge E$. We then explain a main difficulty solved in this paper: the construction of a geometric detour complex for quaternionic Kähler manifolds is seemingly obstructed by the higher rank of the analogous geometric super algebra. This problem is overcome in later sections by understanding the key *rôle* played by the BRST superghosts in the description of quaternionic geometry. The main requisite geometric data is presented in Sect. 3 together with our notations and conventions.

In Sect. 4 we review the relationship between quaternionic Kähler spinning particles and four dimensional black holes; the original motivation for this work. The latter can be described by a spinning particle model coming from the minisuperspace reduction of $\mathcal{N} = 2$ supergravities [8,9]. The “BPS” conditions of this spinning particle model (*i.e.*, requiring solutions for which the local fermion supersymmetry transformations vanish) equal the reduction of the analogous conditions in the four dimensional SUGRA.

Since those conditions amount to the attractor mechanism [12–14] for four dimensional supersymmetric black holes, the quantized spinning particle model is an excellent laboratory for studying these objects.¹ In particular, it allows a minisuperspace analysis of the Ooguri–Strominger–Vafa conjecture [19] and the relationship between black hole wave functions and vacuum selection in string theory [20]. This equivalence between the attractor flow equation and supersymmetric geodesic motion was observed in [8, 9, 21].

The introduction of BRST techniques to solve what could be stated as a purely geometrical problem suggests the presence of an underlying gauge invariant physical model. This is indeed the case. The first of the relevant models is a hyperKähler supersymmetric quantum mechanics. This model can be enhanced to include quaternionic Kähler backgrounds once its four worldline supersymmetries are gauged. This yields a supersymmetric spinning particle model consistent in any quaternionic Kähler manifold. We describe these models in Sects. 5 and 6, respectively.

Sections 3, 8 and 9 can in principle be read by geometers in isolation from the other more physical sections. In Sect. 8, we give a calculus of geometric operators acting on sections of $\wedge E$. Although, we were motivated to write these operators for quantum mechanical BRST reasons, the results themselves are purely geometric. They form the basic building blocks of our quaternionic detour complex. They also place in a much more general setting the Dirac, Dirac–Fueter and detour operator employed some time ago by Baston [24].

Finally our main result is given in Sect. 9, orchestrating all the previous results to build a gauge invariant, higher “form” quantum field theory on quaternionic Kähler manifolds. It relies on the construction of a nilpotent BRST charge given in Sect. 7 achieved by utilizing the supersymmetry ghosts to generate special holonomy transformations. An interesting byproduct of this computation is a novel Dirac operator on the BRST superghost Hilbert space.

Aside from providing an explicit quantization of the fermion modes of minisuperspace $\mathcal{N} = 2$ supersymmetric black holes, our quaternionic detour complex has many potential further applications and generalizations. In particular, it is closely related to the twistor methods of [25]. Also, in some sense, the model is a higher spin theory, so the methods of Vasiliev may be applicable to writing interactions for infinite towers of these quantum fields (see [26, 27] for an excellent review of these methods). Given the existence of the underlying SUGRA theory, this is a very tantalizing possibility. These and other directions for future work are discussed in the conclusions.

2. Detour Complexes

The simplest example of a geometrical detour complex is given by the superalgebra, on any Riemannian manifold M , generated by the exterior derivative $\mathbf{d}$ and the codifferential δ :

$$\{\delta, \mathbf{d}\} = \Delta. \quad (1)$$

Here, the right hand side is the form Laplacian which is a central element of this algebra. These operators act on differential forms $\Psi \in \Gamma(\wedge M)$, which may be viewed as wavefunctions of an $\mathcal{N} = 2$ supersymmetric quantum mechanical model [28], with Δ the Hamiltonian and $(\delta, \mathbf{d})$ the two supercharges. Gauging the corresponding worldline

¹ A very useful introduction to BPS black holes and the attractor mechanism is [10, 11] (the formulation in [16–18] also fits our viewpoint well).

translation and supersymmetries yields a spinning particle (or 1-dimensional SUGRA) model which can be quantized using BRST machinery. In mathematical terms this amounts to computing the Lie algebra cohomology of the superalgebra (1).

However, when defining Lie algebra cohomology for superalgebras, some care is needed [29]. In physics terms this amounts to choices of vacua/polarizations for commuting superghosts [30,31]. It turns out that a distinguished choice exists such that the cohomology is neatly arranged in terms of gauge invariances, Bianchi identities and the equations of motion of a gauge invariant field theory. In a higher spin setting this was first observed in the context of an unfolded formulation and what is called the “twisted adjoint representation” [37,38]. (Very recently the unfolding technique has been shown to be equivalent to the BRST one [32]. The idea of studying worldline descriptions of higher spin systems, via detour and path integral quantization has also been analyzed in [33] and [34,35].) In [36] we used a split choice of ghost polarization² to construct detour complexes from constraint algebras. (For systems with anti-commuting ghosts, this method reproduces known results [44,45] for totally symmetric higher spin fields). The term “detour complex” was chosen because the BRST technology produced complexes of the type studied recently by conformal geometers, the main idea being to connect standard complexes and their duals by (typically higher order in derivatives) detour operators [46–49]. For the simplest case of the de Rham complex, the detour machinery yields a cohomology neatly encapsulated by the complex.

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\mathbf{d}} & \Lambda M & \xrightarrow{\mathbf{d}} & \Lambda M & \xrightarrow{\mathbf{d}} & \Lambda M \rightarrow \cdots \\ & & & & & & \uparrow \delta \mathbf{d} \\ & & & & & & \Lambda M \xrightarrow{\delta} \Lambda M \xrightarrow{\delta} \Lambda M \xrightarrow{\delta} \cdots \end{array}$$

The self-adjoint detour operator $\delta \mathbf{d}$ encodes the equations of motion $\delta \mathbf{d}A = 0$ of a p -form gauge field A and connects the standard de Rham complex to its dual. These incoming and outgoing complexes encode the gauge and gauge for gauge symmetries, and Bianchi as well as Bianchi for Bianchi identities of p -form electromagnetism.

A more sophisticated example is that of the Kähler detour complex; on these manifolds the exterior derivative and codifferential decompose into Dolbeault operators and their duals [50,51]

$$\mathbf{d} = \partial + \bar{\partial}, \quad \delta = \partial^* + \bar{\partial}^*,$$

subject to the superalgebra

$$\{\partial, \partial^*\} = \frac{1}{2} \Delta = \{\bar{\partial}, \bar{\partial}^*\}.$$

In addition, an $\mathfrak{sl}(2)$ Lefschetz algebra acts on the Dolbeault cohomology of a Kähler manifold M . This corresponds to the R symmetry algebra of the above $\mathcal{N} = 4$ superalgebra

$$\begin{aligned} \left[\Lambda, \begin{pmatrix} \partial \\ \bar{\partial} \end{pmatrix} \right] &= \begin{pmatrix} \bar{\partial}^* \\ -\partial^* \end{pmatrix}, & \left[\begin{pmatrix} \bar{\partial}^* \\ -\partial^* \end{pmatrix}, L \right] &= \begin{pmatrix} \partial \\ \bar{\partial} \end{pmatrix}, \\ [H, \Lambda] &= -2\Lambda, & [H, L] &= 2L, & [\Lambda, L] &= H. \end{aligned}$$

Differential forms on a Kähler manifold are bigraded by their holomorphic and anti-holomorphic degrees (p, q) in terms of which the eigenvalues of the operator H are

² The technique of split ghost polarizations is equivalent to the twisted adjoint representation of [37,38]. It has also been employed in [39–43].

$p + q - \frac{1}{2} \dim M$. The operator Λ maps (p, q) to $(p - 1, q - 1)$ -forms by contracting with the Kähler form and the operator L is its dual.

The Kähler analog of p -form electromagnetism [52] follows by a detour complex treatment of the spinning particle³ model obtained by gauging worldline translations, supersymmetries and the R -symmetry Λ . Nilpotency of $Q = \partial \frac{\partial}{\partial p} + \bar{\partial} \frac{\partial}{\partial \bar{p}}$ acting on polynomials in Grassmann even variables $p, \bar{p}$ with coefficients in $\wedge M$ yields the left-hand side of the complex.

Upon fixing a dimension for M and a bi-grading (p, q) this incoming complex becomes the Hodge diamond from complex manifold theory. It may be interpreted as gauge (and gauge for gauge) invariances of the “long” or detour operator G . Explicitly, gauge invariance reads

$$A \rightarrow A + \partial \alpha + \bar{\partial} \bar{\alpha}.$$

Clearly the equations $\partial \bar{\partial} A = 0$ are invariant, yet potentially over or underdetermined. Taking the Kähler trace yields the desired equations of motion $\Lambda \partial \bar{\partial} A = 0$. However, the operator $\Lambda \partial \bar{\partial}$ is not self-adjoint and so does not naturally connect the “incoming” Dolbeault complex with the “outgoing” dual complex depicted on the right hand side above.

The self adjoint operator

$$G = :I_0(2\sqrt{L\Lambda}) (\Lambda - 2\partial\bar{\partial}^* - 2\bar{\partial}\partial^*) + 2 \frac{I_1(2\sqrt{L\Lambda})}{\sqrt{L\Lambda}} (\partial\bar{\partial}\Lambda + L\partial^*\bar{\partial}^*):$$

found in [52] gives an equivalent equation of motion $GA = 0$. Here $:\bullet:$ denotes normal ordering of $\bullet$ by form degree and the functional dependence on $L\Lambda$ through the modified Bessel functions of the second kind is analytic at the origin.

In the special case that M is hyperKähler, replacing differential forms by sections of $\wedge E$ gives another representation of the above $\mathcal{N} = 4$ supersymmetry algebra: The

³ Supersymmetric mechanics on Kähler manifolds have been extensively studied in [53–56 and 57–60].

tangent bundle TM for $4n$ -dimensional manifolds M with quaternionic holonomy splits into a product of vector bundles

$$TM \cong H \otimes E,$$

of rank 2 and $2n$, respectively. The connection on a hyperKähler manifold acts on sections X^α and X^A of H and E , respectively, as

$$\nabla X^\alpha = dX^\alpha + \omega_\beta^\alpha X^\beta, \quad \nabla X^A = dX^A + \Omega_B^A X^B,$$

where the one-form Ω_B^A is $\mathfrak{sp}(2n)$ -valued. Writing the Levi-Civita connection as $\nabla^{\alpha A}$ in a basis for $H \otimes E$, there are $\mathfrak{sp}(2)$ doublets of exterior derivatives and codifferentials acting on $\wedge E$ via

$$\begin{aligned} \mathbf{d}^\alpha &: X^{A_1 \dots A_k} \mapsto \nabla^\alpha [A_1 X^{A_2 \dots A_{k+1}}], \\ \delta_\alpha &: X^{A_1 \dots A_k} \mapsto k \nabla_{\alpha A} X^{A A_1 \dots A_{k-1}}, \end{aligned}$$

in the index notation explained in Sect. 3. They obey the $\mathcal{N} = 4$ algebra

$$\begin{aligned} \{\mathbf{d}^\alpha, \mathbf{d}^\beta\} &= 0 = \{\delta_\alpha, \delta_\beta\}, \\ \{\delta_\alpha, \mathbf{d}^\beta\} &= -\frac{1}{2} \delta_\alpha^\beta \Delta, \end{aligned}$$

where Δ is the Bochner Laplacian $\nabla_\mu \nabla^\mu$. Only an $\mathfrak{sp}(2)$ subalgebra of the $\mathfrak{so}(2, 2)$ R -symmetry of this $\mathcal{N} = 4$ superalgebra acts non-trivially in this hyperKähler representation. The non-trivial R -symmetries are built from the $\mathfrak{sp}(2n)$ invariant tensor J ,

$$\begin{aligned} \mathbf{g} &: X^{A_1 \dots A_k} \mapsto J^{[A_1 A_2} X^{A_3 \dots A_{k+2}}], \\ \mathbf{N} &: X^{A_1 \dots A_k} \mapsto k X^{A_1 \dots A_k}, \\ \mathbf{tr} &: X^{A_1 \dots A_k} \mapsto k(k-1) J_{AB} X^{B A A_1 \dots A_{k-2}}, \end{aligned}$$

and obey the algebra

$$\begin{aligned} [\mathbf{tr}, \mathbf{N}] &= 2 \mathbf{tr}, & [\mathbf{tr}, \mathbf{g}] &= 4(\mathbf{N} - n), & [\mathbf{N}, \mathbf{g}] &= 2 \mathbf{g}, \\ [\delta_\alpha, \mathbf{N}] &= \delta_\alpha, & [\mathbf{N}, \mathbf{d}^\alpha] &= \mathbf{d}^\alpha, \\ [\mathbf{tr}, \mathbf{d}_\alpha] &= 2 \delta_\alpha, & [\delta^\alpha, \mathbf{g}] &= 2 \mathbf{d}^\alpha. \end{aligned}$$

The dictionary

$$\mathbf{d}^\alpha \leftrightarrow \left(\frac{\partial}{\partial} \right), \quad \delta_\alpha \leftrightarrow (-\partial^* - \bar{\partial}^*), \quad \mathbf{g} \leftrightarrow 2L, \quad \mathbf{tr} \leftrightarrow 2\Delta,$$

between the Kähler and hyperKähler representations of the $\mathcal{N} = 4$ superalgebra allows the Kähler detour complex to be translated directly to a hyperKähler one.

In particular, nilpotence of the operator $\mathcal{Q} = \mathbf{d}^\alpha \frac{\partial}{\partial p^\alpha}$ on polynomials in the Grassmann even variables p^α with coefficients in $\Gamma(\wedge E)$ gives gauge and gauge for gauge invariances of the over-determined, Maxwell like, and Einstein versions of the hyperKähler equations of motion

$$\begin{aligned} \mathbf{d}_\alpha \mathbf{d}^\alpha A = 0 &\Rightarrow \mathbf{tr} \mathbf{d}_\alpha \mathbf{d}^\alpha A = 0 \Leftrightarrow GA = 0, \\ G &= :I_0(\sqrt{\mathbf{g} \mathbf{tr}}) (\Delta + 2 \mathbf{d}_\alpha \delta^\alpha) - 2 \frac{I_1(\sqrt{\mathbf{g} \mathbf{tr}})}{\sqrt{\mathbf{g} \mathbf{tr}}} (\mathbf{d}_\alpha \mathbf{d}^\alpha \mathbf{tr} + \mathbf{g} \delta_\alpha \delta^\alpha) :, \end{aligned}$$

for gauge fields $A \in \Gamma(\wedge E)$. Explicitly, the gauge invariance reads

$$A \rightarrow A + \mathbf{d}^\alpha \alpha_\alpha.$$

The equation of motion $\mathbf{d}_\alpha \mathbf{d}^\alpha A = 0$ was first generalized to the more complicated quaternionic Kähler case by Baston [24], and later recovered in the context of BPS, $\mathcal{N} = 2$ supersymmetric black hole systems in [25]. The main result of this paper is to further extend this generalization to the full “Einstein” equations of motion $GA = 0$ in the quaternionic Kähler setting. It relies on a trio of geometric operators (one of which is Baston’s original second order operator) transforming as a triplet under $\mathfrak{sp}(2)R$ -symmetries. We now present the basic geometric data on quaternionic Kähler manifolds needed for this paper.

3. Special Geometry

HyperKähler and quaternionic Kähler manifolds in dimension $4n$ and signature $(2n, 2n)$ enjoy $\mathfrak{sp}(2n)$ and $\mathfrak{sp}(2) \otimes \mathfrak{sp}(2n)$ holonomy, respectively.⁴ In either case, this implies that the tangent bundle splits into a product of vector bundles [6]

$$TM \cong H \otimes E$$

of rank 2 and $2n$, respectively. Therefore, we denote curved and flat indices by $\mu, \nu, \dots$ and $m, n, \dots$ respectively, and decompose tangent space indices as

$$m = \alpha A,$$

where $A = 1, \dots, 2n$ and $\alpha = 1, 2$ label the fundamental representations of $\mathfrak{sp}(2n)$ and $\mathfrak{sp}(2)$, respectively.

The invariant $\mathfrak{so}(2n, 2n)$ metric decomposes this way as

$$\eta_{mn} = \varepsilon_{\alpha\beta} J_{AB},$$

where $\varepsilon_{\alpha\beta}$ and J_{AB} are the $\mathfrak{sp}(2)$ and $\mathfrak{sp}(2n)$ invariant, antisymmetric tensors. This allows for all indices to be raised and lowered independently. For example, $v_A \equiv J_{AB} v^B$, $v^\alpha \equiv v_\beta \varepsilon^{\beta\alpha}$ and $\varepsilon^\alpha_\beta = \delta^\alpha_\beta = -\varepsilon_\beta^\alpha$. Note that we use an uphill convention.

The action of the connection on sections of H and E , respectively, is given by

$$\nabla X^\alpha = dX^\alpha + \omega^\alpha_\beta X^\beta, \quad \nabla X^A = dX^A + \Omega^A_B X^B,$$

where both $\omega_{\alpha\beta}$ and Ω_{AB} are symmetric. On hyperKähler manifolds, only the latter is non-zero. This may be extended to arbitrary tensor products of sections of H and E in the obvious way. For the purposes of calculations involving such products, we specify this action by introducing representations of the $\mathfrak{sp}(2n)$ and $\mathfrak{sp}(2)$ subalgebras of the full local Lorentz algebra $\mathfrak{so}(2n, 2n)$. The generators of these algebras are represented as operators T^{AB} and $t^{\alpha\beta}$, indexed by symmetric pairs of indices, that act on $\mathfrak{sp}(2n)$ and $\mathfrak{sp}(2)$ indices by

$$\begin{aligned} T^{AB} X^C &= J^{CA} X^B + J^{CB} X^A, \\ t^{\alpha\beta} X^\gamma &= \epsilon^{\gamma\alpha} X^\beta + \epsilon^{\gamma\beta} X^\alpha. \end{aligned} \tag{2}$$

⁴ The maximally split signature corresponds to paraquaternionic holonomy – all our results apply to general signatures, this choice being a matter of notational convenience.

These operators satisfy

$$\begin{aligned} [T^{AB}, T^{CD}] &= J^{CA} T^{BD} + J^{CB} T^{AD} + J^{DA} T^{BC} + J^{DB} T^{AC}, \\ [t^{\alpha\beta}, t^{\gamma\delta}] &= \varepsilon^{\gamma\alpha} t^{\beta\delta} + \varepsilon^{\gamma\beta} t^{\alpha\delta} + \varepsilon^{\delta\alpha} t^{\beta\gamma} + \varepsilon^{\delta\beta} t^{\alpha\gamma}, \end{aligned}$$

their extension to higher tensors is by the usual Leibnitz rule, and thus

$$\nabla = d + \frac{1}{2} \omega_\beta^\alpha t_\alpha^\beta + \frac{1}{2} \Omega_B^A T_A^B.$$

Throughout this paper, the symbol ∇ will refer to this definition.

The final geometric ingredient needed here is the Riemann tensor. As a result of special holonomy it has the decomposition [6]

$$R_{\alpha A \beta B \gamma C \delta D} = \Lambda \varepsilon_{(\alpha|\gamma|} \varepsilon_{\beta)\delta} J_{AB} J_{CD} + \varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta} [\Lambda J_{(A|C|} J_{B)D} + \Omega_{ABCD}]. \quad (3)$$

Hence, the commutator of covariant derivatives on sections of H and E follows from:

$$[\nabla_{A\alpha}, \nabla_{B\beta}] \phi_{C\gamma} = \Lambda J_{BA} \varepsilon_{\gamma(\alpha} \phi_{C\beta)} + \Lambda \varepsilon_{\beta\alpha} J_{C(A} \phi_{B)\gamma} + \varepsilon_{\beta\alpha} \Omega_{ABC}^D \phi_{D\gamma}.$$

This specifies an action on higher rank tensors which can be succinctly expressed in terms of the operators

$$[\nabla_{A\alpha}, \nabla_{B\beta}] = \frac{1}{2} J_{BA} t_{\alpha\beta} + \frac{1}{2} \varepsilon_{\beta\alpha} \left(T_{AB} + \Omega_{ABC}^D T_D^C \right).$$

The tensor Ω_{ABCD} is totally symmetric and will appear only seldomly in this paper since it cannot couple to the antisymmetric sections of $\wedge E$ which appear in our models.

The terms proportional to the constant Λ are present only on quaternionic Kähler manifolds and vanish for the hyperKähler case.⁵ Finally, note that the Ricci and scalar curvatures are $R_{mn} = -\Lambda(n+2)\eta_{mn}$ and $R = -4\Lambda n(n+2)$.

4. $\mathcal{N}=2$ Supersymmetric Black Holes and Quaternionic Geometry

Breitenlohner, Maison and Gibbons [60] showed that Kaluza–Klein reduction along a single isometry of a four dimensional, curved space non-linear sigma models coupled to Maxwell fields

$$S = -\frac{1}{2} \int \left[d^4 x \sqrt{-g} R + g_{\mathcal{AB}}^{(4)}(\varphi) d\varphi^{\mathcal{A}} \wedge *d\varphi^{\mathcal{B}} + \frac{1}{2} F^{\mathcal{I}} \wedge \left(M_{\mathcal{IJ}} *F^{\mathcal{J}} + N_{\mathcal{IJ}} F^{\mathcal{J}} \right) \right],$$

(where $\mathcal{A}, \mathcal{B} = 1, \dots, n_S$ the number of scalar fields and $\mathcal{I}, \mathcal{J} = 1, \dots, n_V$ the number of vector fields) yields a three dimensional curved space non-linear sigma model

$$S = -\frac{1}{2} \int \left[d^3 x \sqrt{-g} R + g_{\mu\nu}(\phi) d\phi^\mu \wedge *d\phi^\nu \right].$$

The metric $g_{\mu\nu}$ on the moduli space of the three dimensional non-linear sigma model depends on that of the four dimensional sigma model $g_{\mathcal{AB}}^{(4)}$ as well as the couplings $M_{\mathcal{IJ}}$ and $N_{\mathcal{IJ}}$ of the Maxwell field strengths $F^{\mathcal{I}}$ to the four dimensional scalars $\varphi^{\mathcal{A}}$. We refer

⁵ Note that these are not proportional to $\eta_{r[m}\eta_{n]s}$ – the constant curvature Riemann tensor – since general quaternionic Kähler manifolds are not constant curvature.

to the original paper [60] for the precise formulæ. Suffice it to say, that the n_S scalars in four dimensions are enlarged to a set of $n_S + 2n_V + 2$ scalars coming from the dilaton, dualized graviphoton, Maxwell Kaluza–Klein scalar modes and dualized three Maxwell fields. They span the moduli space $\mathcal{M}$ of the three dimensional sigma model, and in this paper we will be primarily interested in the case that $\dim \mathcal{M} = 4n$. In particular when the original four dimensional theory is the bosonic sector of $\mathcal{N} = 2$ SUGRA, the four dimensional scalar moduli space is a Kähler manifold and its image under dimensional reduction is a (para)quaternionic Kähler manifold. This correspondence is known as the *c*-map [61–64].

When the reduction isometry is generated by a timelike Killing vector, solutions of the three dimensional sigma model correspond to stationary solutions of the four dimensional theory. If we make the additional assumption of spherical symmetry of the three dimensional stationary slices

$$ds^2 = N^2(\rho)d\rho^2 + r^2(\rho)(d\theta^2 + \sin^2 \theta d\varphi^2),$$

solutions then derive from a one dimensional action

$$S = -\frac{1}{2} \int d\rho \left[N + N^{-1}(r'^2 - r^2 \phi'^\mu g_{\mu\nu} \phi'^\nu) \right],$$

where primes denote ρ -derivatives. This model can be interpreted as a relativistic particle moving in a cone metric

$$dr^2 - r^2 d\phi^\mu g_{\mu\nu} d\phi^\nu,$$

over the quaternionic Kähler moduli space $\mathcal{M}$. Classical solutions separate into radial motion and geodesics on the moduli space $\mathcal{M}$. Of these, the extremal black hole solutions of the original four-dimensional theory are necessarily in correspondence with lightlike geodesics [60]; the radial quantization of static, spherically symmetric black holes in Einstein and Einstein-Maxwell gravity has been studied in [65–70]. The consequences of the four dimensional local supersymmetry of the underlying $\mathcal{N} = 2$ SUGRA can be incorporated in this minisuperspace approximation by computing the dimensional reduction of the supersymmetry transformations (see [8,9]). BPS states follow by requiring that the transformations of the fermions vanish. This requirement splits into a radial condition

$$dr = N d\rho,$$

as well as the BPS conditions of a (worldline) locally supersymmetric extension of a relativistic, massless particle moving in the moduli space $\mathcal{M}$. Indeed, imposing $r' = N$ on the constraint $N^2 = r'^2 - r^2 \phi'^\mu g_{\mu\nu} \phi'^\nu$ implied by the N -variation of the above action yields

$$r^2 \phi'^\mu g_{\mu\nu} \phi'^\nu = 0.$$

Therefore we can reinterpret $r^2 = 1/e$ as the inverse einbein of a massless relativistic particle moving in $\mathcal{M}$. The coupling of this particle to worldline fermions $\theta_A^i = (\theta_A^*, \theta_A)$ is determined by requiring that their supersymmetry variations coincide with those obtained by dimensional reduction of the four dimensional SUGRA variations. This leads to a one dimensional SUGRA with action principle

$$S = \int dt \left\{ \frac{1}{2e} \dot{x}^\mu g_{\mu\nu} \dot{x}^\nu + \frac{1}{2} \theta_A^i \frac{\nabla \theta_i^A}{dt} + \frac{\Lambda}{4} e \theta_A^i \theta_{iB} \theta^{jA} \theta_j^B \right\}.$$

In this formula

$$\overset{\circ}{x}{}^\mu \equiv \dot{x}^\mu - V^\mu{}_\alpha \theta_A^i \psi_i^\alpha,$$

is the supercovariantized tangent vector and ψ_i^α are worldline gravitini; the gauge fields for the four local worldline supersymmetries. The BRST quantization of this supersymmetric spinning particle model is a central focus of this paper.

5. HyperKähler Sigma Model

We now construct a supersymmetric, non-linear sigma model in a $4n$ -dimensional, hyperKähler target space $(M, g_{\mu\nu})$. The field content of the model consists of bosonic worldline embedding coordinates $x^\mu(t)$, and fermionic spinning degrees of freedom $\theta_A^i(t)$. Their dynamics are governed by the simple action

$$S = \frac{1}{2} \int dt \left\{ \dot{x}^\mu g_{\mu\nu} \dot{x}^\nu + \theta_A^i \frac{\nabla \theta_A^i}{dt} \right\}. \quad (4)$$

The (rigid) symmetries of the model are

1. *Worldline translations*:

$$\delta x^\mu = \xi \dot{x}^\mu, \quad \delta \theta_A^i = \xi \dot{\theta}_A^i. \quad (5)$$

2. *$Sp(2)R$ -symmetry*:

$$\delta \theta_A^i = \lambda^{ij} \theta_{Aj}, \quad \lambda^{ij} = \lambda^{ji}. \quad (6)$$

3. $\mathcal{N} = 4$ *supersymmetry*:

$$\delta x^\mu = V^\mu{}_\alpha \theta_A^i \varepsilon_i^\alpha, \quad \mathcal{D} \theta_A^i = -\dot{x}^\mu V_{\mu A}{}^\alpha \varepsilon_i^\alpha. \quad (7)$$

Here $V_m^\mu = V_{A\alpha}^\mu$ are the inverse vielbeine⁶ written with split flat indices and $\mathcal{D}$ is the covariant variation: $\mathcal{D} \theta_A^i \equiv \delta \theta_A^i - \delta x^\mu \Omega_{\mu A}{}^B \theta_B^i$. On functions of x^μ it equals $\delta x^\rho \nabla_\rho$; it obviates the requirement to vary covariantly constant quantities. In this regard it helps to observe that $\delta = \mathcal{D}$ when varying scalars (such as the action).

To see explicitly that the action (4) is supersymmetric, we note the identities

$$\begin{aligned} \mathcal{D} \dot{x}^\mu &= \frac{\nabla \delta x^\mu}{dt}, \\ \left[\mathcal{D}, \frac{\nabla}{dt} \right] \theta_i^A &= \delta x^\mu \dot{x}^\nu R_{\mu\nu B}{}^A \theta_i^B = \delta x^{C\alpha} \dot{x}^D \Omega_{BCD}^A \theta_i^B. \end{aligned} \quad (8)$$

Variations linear in fermions cancel by virtue of the first identity, but there are potentially cubic fermion terms proportional to $\frac{1}{2} \theta_A^i [\mathcal{D}, \frac{\nabla}{dt}] \theta_i^A$. Using the second identity we see that these vanish since $\Omega_{ABCD} \theta_i^A \theta_j^B \theta_k^C \equiv 0$.

⁶ The vielbeine/orthonormal frames, denoted $V_\mu{}^m$ obey

$$V_{\mu\alpha}{}^A V_{\nu}{}^\alpha{}_A = -g_{\mu\nu}, \quad V_{\mu\alpha}{}^A V^\mu{}_\beta{}^B = -\delta_B^A \delta_\alpha^\beta.$$

Special holonomy dictates that in addition to these identities for $V_{\mu\alpha}{}^A$ (jocularly, the “zweimalhalbsovielbein”) it is also true that:

$$V_{(\mu\alpha}{}^A V_{\nu)}{}^\beta{}_A = -\frac{1}{2} g_{\mu\nu} \delta_\alpha^\beta, \quad V_{(\mu\alpha}{}^A V_{\nu)}{}^\alpha{}_B = -\frac{1}{2n} g_{\mu\nu} \delta_B^A.$$

5.1. *Quantization.* To quantize the model we write it in first order form

$$S^{(1)} = \int dt \left\{ p_\mu \dot{x}^\mu + \frac{1}{2} \theta_A^i \dot{\theta}_i^A - \frac{1}{2} \pi_\mu g^{\mu\nu} \pi_\nu \right\},$$

where $\pi_\mu = p_\mu + \theta_A^i \Omega_{\mu B}^A \theta_i^B$, and directly impose the canonical commutation relations dictated by the Darboux form of the first order kinetic terms:

$$[p_\mu, x^\nu] = -i \delta_\mu^\nu, \quad \{\theta_A^i, \theta_B^j\} = -i \epsilon^{ij} J_{AB}. \quad (9)$$

We introduce a Fock representation on a vacuum state $|0\rangle$ as⁷

$$\theta_A^i \mapsto \left(\frac{\eta_A}{-i \frac{\partial}{\partial \eta^A}} \right), \quad p_\mu |0\rangle = 0 = \frac{\partial}{\partial \eta^A} |0\rangle.$$

The fermionic anticommutator (9) implies

$$\left\{ \frac{\partial}{\partial \eta^A}, \eta^B \right\} = \delta_A^B,$$

so the creation operators η^A produce Fock states which may be identified with sections of the bundle $\wedge E$:

$$\Gamma(\wedge E) \ni \Phi \equiv \phi_{A_1 \dots A_k}(x) \eta^{A_1} \dots \eta^{A_k} |0\rangle \equiv |\phi_{A_1 \dots A_k}\rangle. \quad (10)$$

The form of π_μ in the action above may be understood in terms of this representation; in general the covariant momentum is

$$\pi_\mu = p_\mu - \frac{i}{2} P_{\mu mn} M^{mn},$$

where M^{mn} generate the local Lorentz algebra

$$[M^{mn}, M^{rs}] = M^{ms} \eta^{nr} - M^{ns} \eta^{mr} + M^{nr} \eta^{ms} - M^{mr} \eta^{ns}.$$

For hyperKähler manifolds the spin connection acts as $P_{mn} M^{mn} = \Omega_{AB} T^{AB}$, where T^{AB} , defined in (2), generate $\mathfrak{sp}(2n)$. On $\wedge E$ one may alternatively represent $\mathfrak{sp}(2n)$ by bilinears in the spinning degrees of freedom;

$$T_{AB} \equiv -2 \eta_{(A} \frac{\partial}{\partial \eta^{B)}} \quad (11)$$

acts identically on Φ to the operator introduced in (2). This explains the form of π_μ ; acting on $\wedge E$ -valued states it produces the covariant derivative⁸

$$\pi_\mu = p_\mu - \frac{i}{2} \Omega_{\mu AB} T^{AB} = -i \nabla_\mu.$$

⁷ The positive definite quantum mechanical inner product for the spinning degrees of freedom is defined by taking $\eta^{A\dagger} = \frac{\partial}{\partial \eta^A}$.

⁸ As usual for first quantized models, $\pi_\mu \pi_\nu \neq \nabla_\mu \nabla_\nu$ because π_μ does not see the open index of π_ν .

5.2. Charges. Our next task is to write down charges generating the symmetries (5)–(7). At the quantum level these are subject to ordering ambiguities which we resolve by relating symmetry charges and geometric operations. Firstly, we expect the Hamiltonian – the generator of worldline translations – to correspond to the Laplacian $\Delta \equiv \nabla_\mu \nabla^\mu$:

$$-2H\Phi = \Delta\Phi.$$

This is true so long as we adopt the quantum ordering

$$H = \frac{1}{2}\pi_{A\alpha}\pi^{A\alpha} - \frac{i}{2}\Omega_{A\alpha}{}^B\pi^{B\alpha}, \quad \pi_\alpha^A \equiv V^\mu{}_\alpha{}^A\pi_\mu.$$

The four supercharges transform as a doublets under the $\mathfrak{sp}(2)$ holonomy subalgebras as well as under a Lefschetz-Verbitsky $\mathfrak{sp}(2)$ algebra which we introduce below. They are built from the $\mathfrak{sp}(2n)$ contraction of the spinning degrees of freedom θ_A^i with the covariant momenta. On states they act as

$$Q_\alpha^i \equiv \begin{pmatrix} \mathbf{d}_\alpha \\ \delta_\alpha \end{pmatrix} \equiv \begin{pmatrix} \eta^A \nabla_{\alpha A} \\ -\nabla_\alpha^A \frac{\partial}{\partial \eta^A} \end{pmatrix},$$

where, again, the operator ordering is chosen based on the natural geometric action:

$$Q_\alpha^i \Phi = \begin{pmatrix} \mathbf{d}_\alpha \\ \delta_\alpha \end{pmatrix} \Phi = \begin{pmatrix} |\nabla_{\alpha[A_1}\phi_{A_2\dots A_{k+1}]}\rangle \\ |k\nabla_{\alpha A}\phi^A{}_{A_2\dots A_k}\rangle \end{pmatrix}.$$

The operator

$$\mathbf{d}^\alpha : \Lambda^k E \rightarrow H \otimes \Lambda^{k+1} E,$$

belongs to a sequence of Dirac operators introduced by Baston in a study of quaternionic complexes [24]. Indeed the operators $\mathbf{d}_\alpha$ and δ_α are analogous to the Dolbeault operators on forms, but they act on $\Gamma(\wedge E)$ instead of $\Gamma(TM)$.

Next, we present the R -symmetry charges generating (6). They can be derived from geometric grounds alone as follows: Firstly observe that since we deal with wavefunctions (10), there is no prohibition on adding anti-symmetric E -tensors with differing number of indices. The state Φ in (10) is in fact an eigenstate of the number or “index” operator

$$\mathbf{N} = \eta^A \frac{\partial}{\partial \eta^A}. \quad (12)$$

The invariant tensor J_{AB} allows us to construct two further bilinears,

$$\mathbf{tr} = \frac{\partial}{\partial \eta_A} \frac{\partial}{\partial \eta^A}, \quad \mathbf{g} = \eta^A \eta_A. \quad (13)$$

These act on states as suggested by their names; the operator $\mathbf{tr}$ removes a pair of indices by tracing with the invariant tensor J_{AB} :

$$\mathbf{tr} |\phi_{A_1\dots A_k}\rangle = k(k-1)|\phi_{A_1\dots A_{k-2}}^A{}_{A_{k-1}A_k}\rangle.$$

Conversely, its adjoint, $\mathbf{g}$ adds a pair of indices by multiplying by J_{AB} and antisymmetrizing:

$$\mathbf{g} |\phi_{A_1\dots A_k}\rangle = |J_{[A_1A_2}\phi_{A_3\dots A_{k+2}]}\rangle.$$

We arrange these generators in a symmetric 2×2 matrix

$$f^{ij} = \begin{pmatrix} \mathbf{g} & \mathbf{N} - n \\ \mathbf{N} - n & \mathbf{tr} \end{pmatrix}. \quad (14)$$

These are precisely the charges corresponding to the R -symmetries (6) and obey the $\mathfrak{sp}(2)$ algebra

$$[f^{ij}, f^{kl}] = \epsilon^{ki} f^{jl} + \epsilon^{kj} f^{il} + \epsilon^{li} f^{jk} + \epsilon^{lj} f^{ik}.$$

We note that one may view this representation of $\mathfrak{sp}(2)$ as the Howe dual of the representation of $\mathfrak{sp}(2n)$ generated by T_{AB} (i.e., $\mathfrak{sp}(2)$ and $\mathfrak{sp}(2n)$ are the commutants of one another in $\mathfrak{so}(2n, 2n)$). In an equation

$$[f^{ij}, T_{AB}] = 0.$$

Moreover, the quadratic Casimirs of these two algebras are related by

$$\mathbf{c} = \mathbf{g} \mathbf{tr} - \mathbf{N}(\mathbf{N} - 2n - 2) = \frac{1}{2} f^{ij} f_{ij} + n(n + 2) = -\frac{1}{2} T^{AB} T_{AB}. \quad (15)$$

The above geometric operators are closely related to the $\mathfrak{so}(4, 1)$ Verbitsky algebra acting on differential forms on hyperKähler manifolds. (An elegant description of this algebra from a supersymmetric quantum mechanical viewpoint is given in [51].) In fact $\{\mathbf{g}, \mathbf{N}, \mathbf{tr}\}$ generate an $\mathfrak{sp}(2)$ subalgebra of $\mathfrak{so}(4, 1)$ corresponding to writing dx^μ as dx_α^A and studying Verbitsky transformations which do not act on the H -index α . Alternatively, we may view this algebra as a generalization of the Lefschetz subalgebra that acts on forms on a Kähler manifold. Henceforth we adopt the hybrid designation “Lefschetz–Verbitsky algebra”.

After some calculation we find⁹

$$\begin{aligned} \{Q_\alpha^i, Q_\beta^j\} &= \frac{1}{2} \epsilon^{ij} \varepsilon_{\alpha\beta} \Delta, \quad [f^{ij}, Q_\alpha^k] = 2\epsilon^{k(i} Q_\alpha^{j)}, \\ [f^{ij}, f^{kl}] &= \epsilon^{ki} f^{jl} + \epsilon^{kj} f^{il} + \epsilon^{li} f^{jk} + \epsilon^{lj} f^{ik}, \quad [\Delta, f^{ij}] = 0 = [\Delta, Q_\alpha^i]. \end{aligned} \quad (16)$$

5.3. Summary. The hyperKähler sigma model presented in this section (and summarized in Fig. 1) provides a geometric representation of the algebra

$$\{Q_I, Q_J\} = J_{IJ} \mathfrak{D},$$

with J the invariant rank two tensor of $\mathfrak{so}(2, 2)$. This algebra belongs to the family of orthosymplectic algebras for which the BRST detour quantization procedure [36] was developed.

The most general R -symmetry of this algebra is $\mathfrak{so}(2, 2)$, with generators R_{IJ} acting as

$$[R_{IJ}, Q_K] = 2J_{K[I} Q_{J]}.$$

⁹ It is interesting to note that this algebra is an Inönü–Wigner contraction of the $\mathfrak{osp}(2|2)$ superalgebra where the bosonic $\mathfrak{sp}(2)$ and $\mathfrak{so}(1, 1)$ blocks are generated by f_{ij} and H respectively while Q_α^i belong to off diagonal fermionic blocks. The rescaling of $\mathfrak{osp}(2|2)$ generators $H \rightarrow \lambda^2 H$ and $Q_\alpha^i \rightarrow \lambda Q_\alpha^i$, and the limit $\lambda \rightarrow \infty$ recovers the algebra above.

<i>Action</i>		
$S = \frac{1}{2} \int \{ \dot{x}^\mu g_{\mu\nu} \dot{x}^\nu + \theta_A^i \frac{\nabla \theta_A^i}{dt} \}$		
<i>States</i>		
$\Gamma(\wedge E) \ni \phi_{[A_1 \dots A_k]}$		
<i>Charges and Geometry</i>		
SUSY	Hamiltonian	<i>R</i> -symmetry
$Q_\alpha^i = \begin{pmatrix} \mathbf{d}_\alpha \\ \delta_\alpha \end{pmatrix}$	$-2H = \Delta$	$f^{ij} = \begin{pmatrix} \mathbf{g} & \mathbf{N} - n \\ \mathbf{N} - n & \mathbf{tr} \end{pmatrix}$
Quaternionic Dirac	Laplacian	Lefschetz–Verbitsky
<i>Algebra</i>		
$[\mathbf{tr}, \mathbf{N}] = 2\mathbf{tr}$	$[\mathbf{tr}, \mathbf{g}] = 4\mathbf{N} - 4n$	$[\mathbf{N}, \mathbf{g}] = 2\mathbf{g}$
$[\delta_\alpha, \mathbf{N}] = \delta_\alpha$	$\{\delta_\alpha, \mathbf{d}_\beta\} = \frac{1}{2} \varepsilon_{\alpha\beta} \Delta$	$[\mathbf{N}, \mathbf{d}_\alpha] = \mathbf{d}_\alpha$
$[\mathbf{tr}, \mathbf{d}_\alpha] = 2\delta_\alpha$	Δ central	$[\delta_\alpha, \mathbf{g}] = 2\mathbf{d}_\alpha$

Fig. 1. Geometric data for the quantized hyperKähler sigma model

Upon breaking the index $I = i_\alpha$, so that $J_{IJ} = \epsilon_{\alpha\beta} \epsilon^{ij}$, a Howe dual pair of $\mathfrak{sp}(2)$ subalgebras generated by $R^i_{(\alpha\beta)i}$ and $R^{\alpha(ij)}_\alpha$ are readily identified. In our hyperKähler sigma model, only the Lefschetz–Verbitsky $\mathfrak{sp}(2)$ part of the *R*-symmetry algebra acts non-trivially and is identified by $R^{\alpha(ij)}_\alpha \mapsto f^{ij}$.

The model we have written down makes sense also on a quaternionic Kähler manifold. The geometric interpretations of the charges and wavefunctions is unaltered. What does change however is the algebra of charges which is no longer a super Lie algebra, but receives deformations from the non-vanishing $\mathfrak{sp}(2)$ holonomy of a quaternionic Kähler manifold. Fortunately however, these deformations produce a first class constraint algebra. Therefore local, or *spinning* particle models can be constructed by gauging supersymmetries. These are the subject of the next section.

6. Quaternionic Kähler, $\mathcal{N} = 4$, $d = 1$ SUGRA

Upon replacing the hyperKähler target space with a quaternionic Kähler one, it is no longer possible to maintain the rigid $\mathcal{N} = 4$ supersymmetry algebra (16). However, by requiring the algebra to hold only weakly we may instead study local symmetries. There are various choices for first class algebras built from the generators H , Q^i_α and f_{ij} . Gauging the Hamiltonian H yields a model which is worldline reparameterization independent—generally a desirable feature. Local, $\mathcal{N} = 4$, worldline supersymmetry is achieved by gauging the supercharges Q^i_α . Thereafter, one can also consider gauging some combination of *R* symmetry generators. From a spinning particle perspective gauging $\{H, Q^i_\alpha\}$ and $\{H, Q^i_\alpha, f_{ij}\}$ might seem most natural. In general the choice depends on the particular physical or geometric application one has in mind. Also, in general, when quantizing a first class constraint algebra, one needs to keep in mind what quantization

procedure will be employed. Possibly the simplest choice is a naïve Dirac quantization where one attempts to impose the constraints directly as operator relations on the physical Hilbert space. Often however, this is not the most interesting choice, and far more can be learned from a BRST approach.

In this section we construct the classical spinning particle models corresponding to the $\{H, Q_\alpha^i\}$ and $\{H, Q_\alpha^i, f_{ij}\}$ gaugings. In the remainder of the paper, we will be primarily concerned with the BRST quantization of the former of these. In particular we show, motivated by ideas from higher spin theories, that gauging only a single R symmetry generator $\mathbf{tr}$ within a BRST detour setting produces a gauge invariant quantum field theoretical model on quaternionic Kähler spaces.

The first step is to introduce Lagrange multipliers (gauge fields) for each constraint

Constraints	Gauge Fields
$H \approx 0$ $Q_\alpha^i \approx 0$	Lapse N Gravitini ψ_i^α
$H \approx 0$ $Q_\alpha^i \approx 0$ $f^{ij} \approx 0$	Lapse N Gravitini ψ_i^α Yang–Mills A_{ij}

In this one-dimensional setting, these gauge fields have no dynamics. The charges Q_α^i and f^{ij} are the same as those of the hyperKähler sigma model in Sect. 5, while we add curvature corrections to the Hamiltonian H reflecting that the background is now quaternionic Kähler. These are determined by ensuring that the algebra of charges is first class. Let us give details for each model separately.

6.1. Rigid Lefschetz–Verbitsky model. Gauging only the Q_α^i and H yields a model with rigid Lefschetz–Verbitsky symmetries. Since we work in a quaternionic Kähler target space as described in Sect. 3 the connection ∇ now is both $\mathfrak{sp}(2)$ and $\mathfrak{sp}(2n)$ -valued. There are two easy methods to compute the (second order) action and its symmetries. The first is to start with the sigma model action (4) and to proceed using the Noether method, whose first step couples the gravitini to the supersymmetry current/charges Q_α^i . This computation is analogous to the one employed by Bagger and Witten [6] to compute matter couplings to $\mathcal{N} = 2, d = 4$ SUGRA. Alternatively, we can begin with a first order action given by the sum of the standard symplectic current $\int dt \{p_\mu \dot{x}^\mu + \frac{1}{2} \theta_A^i \dot{\theta}_i^A\}$ and the product of Lagrange multipliers (N, ψ_i^α) with their corresponding constraint. Thereafter, a Legendre transformation yields the second order action. The results are equivalent and we find

$$S = \int dt \left\{ \frac{1}{2N} \dot{x}^\mu g_{\mu\nu} \dot{x}^\nu + \frac{1}{2} \theta_A^i \frac{\nabla \theta_i^A}{dt} + \frac{\Lambda N}{4} \theta_A^i \theta_{iB} \theta^{jA} \theta_j^B \right\}, \quad (17)$$

which enjoys symmetries:

1. *Local worldline reparameterizations:*

$$\delta x^\mu = \xi \dot{x}^\mu, \quad \delta \theta_A^i = \xi \dot{\theta}_A^i, \quad \delta N = \frac{d(\xi N)}{dt}, \quad \delta \psi_i^\alpha = \frac{d(\xi \psi_i^\alpha)}{dt}.$$

2. *$Sp(2)R$ -Rigid symmetry:*

$$\delta \theta_A^i = \lambda^{ij} \theta_{Aj}, \quad \delta \psi_\alpha^i = \lambda^{ij} \psi_{\alpha j}.$$

3. *Local $\mathcal{N} = 4$ supersymmetry:*

$$\begin{aligned} \delta x^\mu &= V_\alpha^{\mu A} \theta_A^i \varepsilon_i^\alpha, \\ \mathcal{D} \theta_A^i &= -\frac{1}{N} \overset{\circ}{x}^\mu V_{\mu A}^\alpha \varepsilon_\alpha^i, \\ \delta N &= \psi_\alpha^i \varepsilon_i^\alpha, \\ \mathcal{D} \psi_\alpha^i &= \frac{\nabla \varepsilon_\alpha^i}{dt} + \frac{\Lambda N}{2} \theta_A^i \theta_j^A \varepsilon_\beta^j. \end{aligned}$$

In these formulæ, $\mathcal{D}$ is again the covariant variation, but just like the connection ∇ , it too is now $\mathfrak{sp}(2)$ covariant so that, for example, $\mathcal{D} \psi_\alpha^i = \delta \psi_\alpha^i - \delta x^\mu \omega_{\mu\alpha}^\beta \psi_\beta^i$. Also, we have introduced the supercovariant tangent vector

$$\overset{\circ}{x}^\mu \equiv \dot{x}^\mu - V_\alpha^{\mu A} \theta_A^i \psi_i^\alpha.$$

To verify invariance of this action, notice that the supercovariant tangent vector transforms as

$$\mathcal{D} \overset{\circ}{x}^\mu = \frac{\delta N}{2N} \overset{\circ}{x}^\mu + V_\alpha^{\mu A} \left\{ \frac{\nabla \theta_A^i}{dt} \varepsilon_i^\alpha - \theta_A^i \Delta \psi_i^\alpha \right\} + \frac{1}{N} \overset{\circ}{x}_\nu V^{[\mu A} V^{\nu] \beta} \varepsilon_\beta^i \psi_i^\alpha.$$

Here $\Delta \psi_i^\alpha \equiv \mathcal{D} \psi_i^\alpha - \frac{\nabla \varepsilon_\alpha^i}{dt}$ is shorthand for the two fermion gravitini variations. The last terms are of the form $\overset{\circ}{x}_\nu A^{[\mu \nu]}$ so do not contribute to the variation of the bosonic matter kinetic term $\frac{1}{2N} \overset{\circ}{x}^2$, while the leading term perfectly ensures the kinetic terms vary into

$$\delta \int \left\{ \frac{1}{2N} \overset{\circ}{x}_\mu \overset{\circ}{x}^\mu + \frac{1}{2} \theta_A^i \frac{\nabla \theta_A^i}{dt} \right\} = \int \left[-\frac{1}{N} \overset{\circ}{x}_A^\alpha \Delta \psi_\alpha^i + \frac{1}{2} \theta_A^i \left[\mathcal{D}, \frac{\nabla}{dt} \right] \right] \theta_i^A. \quad (18)$$

These cancel the variation of the four point fermi coupling to the Riemann tensor. This relies on the quaternionic Kähler analog of the identity (8) which yields $\delta x \dot{x}$ times the Riemann tensor for the commutator of covariant worldline derivatives and variations. Trading $\dot{x}$ for $\overset{\circ}{x}$ yields exactly the terms required to cancel the variation of the lapse N multiplying the four point coupling.

A final point worth stressing is that the parameter Λ is not fixed by the requirement of local supersymmetry in one dimension. In dimension four, coupling $\mathcal{N} = 2$ SUGRA to matter fixes the scalar curvature in terms of Newton's constant κ [6]. (This follows by requiring variations of the Einstein–Hilbert and Rarita–Schwinger terms to cancel at order κ^0 in the Noether procedure.) Both these terms are absent in our one dimensional model.

6.2. Gauged Lefschetz–Verbitsky model. To gauge the Lefschetz–Verbitsky $\mathfrak{sp}(2)$ symmetry we need only replace the covariant derivative ∇ in (17) by its $\mathfrak{sp}(2)$ covariantization $\overset{\text{A}}{\nabla}$ defined by

$$\frac{\overset{\text{A}}{\nabla} v^i}{dt} \equiv \frac{\nabla v^i}{dt} + A^{ij} v_j.$$

Therefore the gauged action reads

$$S = \int dt \left\{ \frac{1}{2N} \overset{\circ}{x}^\mu g_{\mu\nu} \overset{\circ}{x}^\nu + \frac{1}{2} \theta_A^i \frac{\overset{\text{A}}{\nabla} \theta_i^A}{dt} + \frac{\Lambda N}{4} \theta_A^i \theta_{iB} \theta^{jA} \theta_j^B \right\}, \quad (19)$$

which differs from (17) by a Lagrange multiplier term $\int \frac{1}{2} \theta_A^i A_{ij} \theta^{jA}$ (so the gauge field A_{ij} is a unit weight, worldline tensor density or volume form). In addition to the new local Lefschetz–Verbitsky symmetry,

$$\delta \theta_A^i = \lambda_j^i \theta_A^j, \quad \delta \psi_\alpha^i = \lambda_j^i \psi_\alpha^j, \quad \delta A^{ij} = \dot{\lambda}^{ij} + 2A^{k(i} \lambda_{k}^{j)},$$

the supersymmetry transformations are modified to read

$$\begin{aligned} \delta x^\mu &= V_\alpha^{\mu A} \theta_A^i \varepsilon_i^\alpha, \\ \mathcal{D} \theta_A^i &= -\frac{1}{N} \overset{\circ}{x}^\mu V_{\mu A}^\alpha \varepsilon_\alpha^i, \\ \delta N &= \psi_\alpha^i \varepsilon_i^\alpha, \\ \mathcal{D} \psi_\alpha^i &= \frac{\overset{\text{A}}{\nabla} \varepsilon_\alpha^i}{dt} + \frac{\Lambda N}{2} \theta_A^i \theta_j^A \varepsilon_\alpha^j, \\ \delta A^{ij} &= 0. \end{aligned}$$

These results and other gaugings follow easily from the canonical analysis of the next section.

6.3. Dirac quantization. To perform a canonical analysis and Dirac quantization of the rigid Lefschetz–Verbitsky model we first note that the symplectic structure $\int dt \{ p_\mu \dot{x}^\mu + \frac{1}{2} \theta_A^i \dot{\theta}_i^A \}$ implies the same Fock space structure as in the hyperKähler case (see in particular formulæ (9)–(10)). The Dirac Hilbert space is therefore again sections of the antisymmetric $\mathfrak{sp}(2n)$ tensor bundle $\wedge E$.

The (quantized) supercharges Q_α^i and Lefschetz–Verbitsky generators take the same form as in the analysis of the hyperKähler sigma model in Sect. 5.2. The Hamiltonian H receives a curvature correction term (implied by the four-fermi term in the action (17) proportional to the lapse N). Again these charges may all be quantized with orderings obtained by ensuring that the quantum algebra of constraints is first class. The Dirac quantization of the model then amounts simply to imposing the conditions $H\Psi = Q_\alpha^i \Psi = 0$ on wavefunctions Ψ valued in $\Gamma(\wedge E)$. (The gauged Lefschetz–Verbitsky model incurs the additional constraint $f_{ij}\Psi = 0$.) We pay little attention to an analysis of this quantum system because it suffers a certain deficiency which we now explain, and will remedy in the next section by means of a BRST analysis:

On a quaternionic Kähler manifold we must remember that the spin connection has both $\mathfrak{sp}(2n)$ and $\mathfrak{sp}(2)$ valued parts which couple naturally to the respective generators T_{AB} and $t_{\alpha\beta}$. However, from the spinning degrees of freedom θ_A^i of this model, we can *only* build a representation of the $\mathfrak{sp}(2n)$ generators T_{AB} . On the one hand, this seems sufficient because acting on $\wedge E$ -sections, we still have $i\pi_\mu = \nabla_\mu$.

But, acting with a supersymmetry generator Q_α^i introduces an $\mathfrak{sp}(2)$ index α , and we seem to have no way, in the spinning particle model context, to obtain further covariant derivatives acting correctly on α . A geometer might consider constructing supersymmetry-like operators built from the covariant derivative by *fiat* (and in fact, the geometric calculus Sect. 8 of this paper can be taken on its own and read this way). However, there is a very natural physical mechanism to introduce additional spinning degrees of freedom that can represent the $\mathfrak{sp}(2)$ generators $t_{\alpha\beta}$. In fact, this is precisely what BRST quantization of the model does.

7. BRST and the Geometry of Ghosts

The one dimensional quaternionic Kähler spinning particle model enjoys local worldline supersymmetry and reparameterization invariances. This implies that they form a first class algebra (even though the supercharges do not commute with the Hamiltonian unlike those in the hyperKähler sigma model where they generate genuine symmetries). In this section we present the nilpotent, quantum, BRST charge for this algebra. Again, unlike the hyperKähler model, this constraint algebra is higher rank; it does not form a Lie algebra. This means that, in principle, we need to resort to homological perturbation methods to construct the BRST charge. (The reader may consult [15] for a detailed account of the analysis of gauge theories using BRST techniques and in particular the construction of a nilpotent BRST charge for higher rank algebras.) Although standard, such a computation is rather involved, so instead we present a solution relying on the underlying quaternionic geometry.

The general structure of the BRST charge we search for is given by expanding it in powers of the worldline reparameterization ghost c and its antighost b represented as $\frac{\partial}{\partial c}$,

$$Q_{\text{BRST}} = c \mathcal{D} + \mathcal{Q} - M \frac{\partial}{\partial c}. \quad (20)$$

If our constraint algebra were a Lie algebra (as it is in the hyperKähler case), the operator $\mathcal{D}$ would be the worldline Hamiltonian and $\mathcal{Q}$ the contraction of the supercharges with commuting supersymmetry ghosts c_i^α . However, since we have a higher rank constraint algebra, we must add terms with higher powers of ghosts and antighosts. We determine these by making a simple geometric *ansatz* for $\mathcal{Q}$ and then requiring nilpotency of Q_{BRST} .

The key geometric idea is that ghosts and antighosts can be used to represent the $\mathfrak{sp}(2)$ special holonomy generators. The quantized commuting superghosts c_i^α and superantighosts b_α^i with algebra

$$[b_\alpha^i, c_j^\beta] = \delta_j^i \delta_\alpha^\beta \quad (21)$$

allow formation of bilinears $c_\alpha^i b_\beta^j - c_\beta^j b_\alpha^i$ that generate a faithful representation of $\mathfrak{so}(2, 2)$, the R -symmetry algebra of our first class constraint superalgebra, on the ghosts

(and/or antighosts). Specializing to the Howe dual subalgebras generated by

$$\begin{aligned} f_{ij}^{\text{gh}} &= -2c_{(i}^{\alpha} b_{j)\alpha}, \\ t_{\text{gh}}^{\alpha\beta} &= -2c^{i(\alpha} b_i^{\beta)}, \end{aligned} \quad (22)$$

we obtain representations of the Lefschetz–Verbitsky and H -bundle special holonomy $\mathfrak{sp}(2)$ algebras, respectively. (We will discuss the precise definition of the superghost Hilbert space at the end of this section, but for now concentrate on building a nilpotent BRST charge.)

This means that we can solve the problem of the covariant momentum operator π_{μ} discussed in the previous section—namely that it was not covariantized with respect to the $\mathfrak{sp}(2)$ holonomy—by using the above ghost representation for $t_{\alpha\beta}$. So we now construct a covariant momentum operator

$$\Pi_{\mu} \equiv p_{\mu} - \frac{i}{2} \Omega_{\mu}{}^A{}_B T_A^B - \frac{i}{2} \omega_{\mu}{}^{\alpha}{}_{\beta} t_{\alpha}^{\beta}, \quad (23)$$

which acts on both E and H bundles. (In some sense, the ghosts play the *rôle* of frames for the bundle H .) In turn we introduce BRST-extended supersymmetry charges $\theta_A^i V^{\mu A}{}_{\alpha} \Pi_{\mu}$ and consider the ansatz

$$\mathcal{Q} \equiv i c_i^{\alpha} \left(\eta^A \frac{\partial}{\partial \eta_A} \right)^i V^{\mu}{}_{\alpha A} \Pi_{\mu}$$

for the form of Eq. (20).

Before proceeding, it is worth noting that we have actually found a new Dirac operator: Reunifying $\mathfrak{sp}(2)$ and $\mathfrak{sp}(2n)$ indices as a single $\mathfrak{so}(2n, 2n)$ index $m = A\alpha$ and forming the combination

$$\gamma^m = c^{\alpha i} \left(\eta^A \frac{\partial}{\partial \eta_A} \right)_i,$$

we find a Clifford algebra

$$\begin{aligned} \{\gamma^m, \gamma^n\} &= M \eta^{mn}, \\ M &\equiv \frac{1}{2} c^{\alpha i} c_{\alpha i}. \end{aligned}$$

Since the covariant momentum (23) acts as the covariant derivative, a Dirac-type operator follows

$$\mathcal{Q} = \gamma^m \nabla_m. \quad (24)$$

Returning to our BRST charge computation, a simple Weitzenböck-like calculation¹⁰ shows

$$\mathcal{Q}^2 = M \mathfrak{D}, \quad (25)$$

where the BRST-extended Hamiltonian is

$$2 \mathfrak{D} = \square - \frac{1}{4} (f^{ij} + f_{\text{gh}}^{ij}) (f_{ij} + f_{ij}^{\text{gh}}) - \frac{n}{2} (n+2).$$

¹⁰ Note that the computation of the term coupling the curvature to two Dirac matrices relies heavily on γ^m being a composite built from ghosts and spinning degrees of freedom.

In this expression, $\square = \Delta + \frac{1}{4}(T^2 + t^2)$ is a quaternionic Kähler Lichnerowicz wave operator, which will be introduced in Sect. 8. It satisfies $[\square, Q] = 0$. Further, since f^{ij} and f_{gh}^{ij} obey $[f_{ij} + f_{ij}^{\text{gh}}, c_k^\alpha Q_\alpha^k] = 0$ and the latter commutes with¹¹ M , we have the following identities

$$[\mathcal{D}, M] = [Q, \mathcal{D}] = [Q, M] = Q^2 - M\mathcal{D} = 0. \quad (26)$$

These immediately imply that the BRST charge (20) is nilpotent. The form of this BRST charge is exactly suited to the detour quantization methods of [36]. To that end we next specify our choice of ghost vacuum.

We represent the ghost algebra (21) in a Fock representation by splitting the ghosts and antighosts into derivatives and power series coordinate coefficients. The choice of vacuum is determined by splitting the Verbitsky–Lefschetz doublets as

$$c_i^\alpha = \left(z^\alpha \frac{\partial}{\partial p_\alpha} \right), \quad b_i^\alpha = \left(-p^\alpha \frac{\partial}{\partial z_\alpha} \right). \quad (27)$$

Therefore we may view (z^α, p^α) as creation operators for symmetric H -bundle indices. So states Φ in the superghost extended Hilbert space are sections of

$$\begin{aligned} \Gamma(\wedge E \otimes (\odot H)^{\otimes 2}) \ni \Phi &\equiv \phi_{A_1 \dots A_k \alpha_1 \dots \alpha_s}^{\beta_1 \dots \beta_t}(x) \eta^{A_1} \dots \eta^{A_k} z^{\alpha_1} \dots z^{\alpha_s} p_{\beta_1} \dots p_{\beta_t} |0\rangle \\ &= |\phi_{[A_1 \dots A_k](\alpha_1 \dots \alpha_s)}^{(\beta_1 \dots \beta_t)}\rangle = \Phi \end{aligned}$$

In the Young diagram notation the column denotes antisymmetrized E -indices while the rows are symmetrized H -indices.

We now have a well-defined BRST cohomology. Before analyzing it via BRST detour methods, we take a short geometric excursion to develop a quaternionic calculus of the various operators that will appear in those results.

8. A Quaternionic Geometric Calculus

On a d -dimensional Einstein manifold the Riemann tensor decomposes as

$$R_{\mu\nu\rho\sigma} = \underbrace{\frac{2\Lambda}{(d-1)(d-2)}(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma})}_{\text{Constant Curvature}} + \underbrace{W_{\mu\nu\rho\sigma}}_{\text{Weyl}}.$$

The special constant curvature case—when the Weyl tensor vanishes—enjoys many distinguishing properties, including a Lichnerowicz wave operator which commutes with generalized gradient and divergence operators acting on tensors of very general types. Comparing this formula with the one for the quaternionic Kähler Riemann tensor in (3) we see that the totally symmetric tensor Ω_{ABCD} plays a *rôle* similar to the Weyl tensor;¹²

¹¹ In fact, linear combinations of the ghost bilinears mentioned below Eq. (21) are precisely those which commute with M .

¹² In fact, in four dimensions it plays the *rôle* of the anti-self dual Weyl tensor [1, 24].

if we could somehow find a “regime” in which it did not contribute we might be able to analyze quaternionic Kähler geometry along lines similar to the constant curvature case.

In fact, exactly such a regime does exist, namely sections of the product of $\wedge E$ with the tensor bundle $\mathcal{TH}$ (with sections being arbitrary H -tensors)

$$\Gamma(\wedge E \otimes \mathcal{TH}) \ni \phi_{[A_1 \dots A_k]}^{\alpha_1 \dots \alpha_s},$$

the idea being that antisymmetry in $\mathfrak{sp}(2n)$ indices prevents the totally symmetric tensor Ω_{ABCD} from contributing.

In particular, the central operations will be the quaternionic generalizations of the Dolbeault operators

$$\begin{aligned} \left(\begin{array}{c} \mathbf{d}^\alpha \\ \delta^\alpha \end{array} \right) : \Gamma(\wedge E \otimes \mathcal{TH}) &\longrightarrow \Gamma(\wedge E \otimes \mathcal{TH})^{\otimes 2} \\ \Psi &\qquad \qquad \qquad \Psi \\ \phi_{[A_1 \dots A_k]}^{\alpha_1 \dots \alpha_s} &\mapsto \left(\begin{array}{c} \nabla_{[A_1}^\alpha \phi_{A_2 \dots A_{k+1}]}^{\alpha_1 \dots \alpha_s} \\ k \nabla_A^\alpha \phi_{[A_2 \dots A_k]}^{\alpha_1 \dots \alpha_s} \end{array} \right). \end{aligned}$$

These operators are motivated by the quantized supersymmetry charges of the previous sections, but are more general since they can act on arbitrary H -tensors. For computations, it is often useful to adopt a hybrid E -index free notation where

$$\begin{aligned} \phi_{[A_1 \dots A_k]}^{\alpha_1 \dots \alpha_s} &\rightarrow \Phi^{\alpha_1 \dots \alpha_s} = \phi_{A_1 \dots A_k}^{\alpha_1 \dots \alpha_s} \eta^{A_1} \dots \eta^{A_k}, \\ \mathbf{d}^\alpha &= \eta^A \nabla_A^\alpha, \\ \delta^\alpha &= -\nabla^{\alpha A} \frac{\partial}{\partial \eta^A}, \end{aligned}$$

and the Grassmann variables η^A play the *rôle* of the anticommuting differentials dx^μ employed in the theory of differential forms.

The non-dynamical Lefschetz–Verbitsky charges

$$f^{ij} = \left(\begin{array}{cc} \mathbf{g} & \mathbf{N} - n \\ \mathbf{N} - n & \mathbf{tr} \end{array} \right)$$

act exactly as described in 5.2 on the antisymmetric E -indices (with the same expressions in terms of η ’s), namely adding or removing pairs of antisymmetrized indices using the invariant tensor J_{AB} or counting indices. In terms of these $\mathbf{d}^\alpha$, δ^α obey a very elegant algebra

$$\begin{aligned} \{\mathbf{d}^\alpha, \mathbf{d}^\beta\} &= -\frac{1}{2} \mathbf{g} \, t^{\alpha\beta}, \\ \{\mathbf{d}^\alpha, \delta^\beta\} &= \frac{1}{2} \varepsilon^{\alpha\beta} (\Delta - \mathbf{c}) - \frac{1}{2} t^{\alpha\beta} (\mathbf{N} - n), \\ \{\delta^\alpha, \delta^\beta\} &= -\frac{1}{2} \mathbf{tr} \, t^{\alpha\beta}, \end{aligned} \tag{28}$$

where $\mathbf{c}$ is again the Lefschetz–Verbitsky $\mathfrak{sp}(2)$ Casimir operator of (15).

These formulæ can be repackaged even more simply by noticing that the operator

$$\square = \Delta + \frac{1}{4} T^2 + \frac{1}{4} t^2, \quad \text{with} \quad \begin{cases} T^2 = T_{AB} T^{AB} \\ t^2 = t_{\alpha\beta} t^{\alpha\beta} \end{cases}$$

commutes with $\mathbf{d}^\alpha$ and δ^α . This is an extremely important result, so we shall call $\square$ a *quaternionic Kähler Lichnerowicz wave operator*. Its existence validates our claim that by studying the bundle $\wedge E \otimes TH$, quaternionic Kähler geometry could be made to mimic its constant curvature counterpart.

Specialized to totally symmetric H -tensors, the operators $(\mathbf{d}^\alpha, \delta^\alpha)$ coincide with the action of the BRST-extended supersymmetry charges in Sect. 7, therefore we adopt the suggestive notation

$$\mathcal{Q}_\alpha^i = \begin{pmatrix} \mathbf{d}_\alpha \\ \delta_\alpha \end{pmatrix},$$

and call these operators *generalized supercharges*. We may now unify the algebra (28) as

$$\{\mathcal{Q}_\alpha^i, \mathcal{Q}_\beta^j\} = \frac{1}{2} \varepsilon_{\alpha\beta} \epsilon^{ij} \widetilde{\square} - \frac{1}{2} f^{ij} t_{\alpha\beta},$$

with

$$\widetilde{\square} \equiv \square - \frac{1}{4} f_{ij} f^{ij} - \frac{1}{4} t_{\alpha\beta} t^{\alpha\beta} - \frac{n}{2}(n+2).$$

It is interesting to note that these formulæ enjoy a complete symmetry when all H -indices $\alpha, \beta, \dots$ are exchanged with their Lefschetz–Verbitsky counterparts $i, j, \dots$. This symmetry appears more starkly when we compute the products of generalized supercharges

$$\mathcal{Q}_\alpha^i \mathcal{Q}_\beta^j = \frac{1}{4} \varepsilon_{\alpha\beta} \epsilon^{ij} \widetilde{\square} - \frac{1}{4} f^{ij} t_{\alpha\beta} - \frac{1}{2} \varepsilon_{\alpha\beta} \mathbf{b}^{ij} - \frac{1}{2} \epsilon^{ij} \mathbf{b}_{\alpha\beta},$$

where we have defined the bilinears

$$\mathbf{b}^{ij} \equiv \mathcal{Q}_\alpha^{(i} \mathcal{Q}^{j)\alpha}, \quad \mathbf{b}_{\alpha\beta} \equiv \mathcal{Q}_{i(\alpha} \mathcal{Q}_{\beta)}^i.$$

Observe that, since the generalized supercharges form $\mathfrak{sp}(2)$ doublets under Lefschetz–Verbitsky and H -symmetries

$$[f^{ij}, \mathcal{Q}_\alpha^k] = \epsilon^{ki} \mathcal{Q}_\alpha^j + \epsilon^{kj} \mathcal{Q}_\alpha^i, \quad [t_{\alpha\beta}, \mathcal{Q}_\gamma^i] = \varepsilon_{\gamma\alpha} \mathcal{Q}_\beta^i + \varepsilon_{\gamma\beta} \mathcal{Q}_\alpha^i,$$

the six charge bilinears $\mathbf{b}_{\alpha\beta}$ and $\mathbf{b}_{ij}$ form two adjoint $\mathfrak{sp}(2)$ triplets. This leads one to wonder whether these operators form a pair of $\mathfrak{sp}(2)$ algebras when commuted among themselves. This question is particularly pressing when we observe that the operator

$$\mathbf{d}_\alpha \mathbf{d}^\alpha + \mathbf{g},$$

coincides with that introduced by Baston in his construction of quaternionic analogues of Dolbeault cohomology on quaternionic Kähler manifolds. In fact, this operator is one of a triplet of operators

$$\mathbf{B}_{ij} = \mathbf{b}_{ij} + f_{ij}$$

which we shall call *Baston operators*. This structure of R -symmetry groups represented in terms of bilinears in supercharges has appeared before [24]. For example, for differential forms on a Kähler manifold, bilinears in the Dolbeault operators $\{\bar{\partial}\bar{\partial}, \Delta - 2\partial\bar{\partial} - 2\bar{\partial}\partial, \partial\bar{\partial}\}$ obey an $\mathfrak{sp}(2)$ Lie algebra (up to an overall factor of the central form Laplacian on the right-hand side of commutators). Also a similar phenomenon holds for more

Quaternionic Dolbeault Operators

$$\mathcal{Q}_\alpha^i = \begin{pmatrix} \mathbf{d}_\alpha \\ \delta_\alpha \end{pmatrix}$$

Quaternionic Dolbeault Algebra

$$\{\mathcal{Q}_i^\alpha, \mathcal{Q}_j^\beta\} = \frac{1}{2} \varepsilon^{\alpha\beta} \epsilon_{ij} \tilde{\square} - \frac{1}{2} f_{ij} t^{\alpha\beta}$$

Quaternionic Kähler Lichnerowicz wave operator

$$\square = \Delta + \frac{1}{4}(T^2 + t^2) = \tilde{\square} + \frac{1}{4}f^2 + \frac{1}{4}t^2 + \frac{n}{2}(n+2)$$

Baston operators

$$\mathbf{B}^{ij} = Q_\alpha^{(i} Q^{j)\alpha} + f^{ij}$$

$$= \begin{pmatrix} \mathbf{d}_\alpha \mathbf{d}^\alpha + \mathbf{g} & \mathbf{d}_\alpha \delta^\alpha + \delta_\alpha \mathbf{d}^\alpha + 2(\mathbf{N} - n) \\ \mathbf{d}_\alpha \delta^\alpha + \delta_\alpha \mathbf{d}^\alpha + 2(\mathbf{N} - n) & \delta_\alpha \delta^\alpha + \mathbf{tr} \end{pmatrix}$$

Baston Algebra

$$[\mathbf{B}_{ij}, \mathbf{B}_{kl}] = \epsilon_{(i(k} [\mathbf{B}_{j)l}] \tilde{\square} + \tilde{\square} \mathbf{B}_{j)l})$$

$$[f^{ij}, \mathbf{B}^{km}] = 2 \epsilon^{k(j} \mathbf{B}^{i)m} + 2 \epsilon^{m(j} \mathbf{B}^{i)k}$$

Fig. 2. The quaternionic Kähler calculus

general orthosymplectic algebras [7]. Moreover, the Kähler result immediately implies the same algebra for the $\mathbf{b}_{ij}$ on hyperKähler manifolds. In the more general quaternionic Kähler case one no longer finds a Lie algebra built from $\mathbf{b}_{ij}$ but instead the following rather interesting deformation thereof:¹³

$$[\mathbf{B}^{ij}, \mathbf{B}^{kl}] = \epsilon^{(i(k} \left[(\tilde{\square} - 2) \mathbf{B}^{j)l} + \mathbf{B}^{j)l} (\tilde{\square} - 2) - f^{j)l} \right] (\mathbf{b}_{\alpha\beta} t^{\alpha\beta} + \frac{1}{2} t^2) \Big].$$

The Weyl ordering on the right-hand side is necessary because (as opposed to the quaternionic Kähler Lichnerowicz wave operator $\square$) the operator $\tilde{\square}$ is not central. Note that the operators $\mathbf{b}_{\alpha\beta} + t_{\alpha\beta}$ obey an analogous algebra, thanks to the aforementioned symmetry between H -indices and Lefschetz–Verbitsky ones. The main formulæ of this section are summarized in Fig. 2. We now orchestrate these geometric results with our BRST detour techniques to construct our main result, a gauge invariant quaternionic Kähler quantum field theory.

9. The Quaternionic Kähler Detour Complex

The BRST detour quantization formalism presented in [36], takes as its input a BRST charge of the form (20), together with a representation of the underlying constraint algebra acting on sections of a bundle over some manifold M , and outputs a classical field

¹³ It would be interesting to investigate whether the last terms in this formula can be absorbed by replacing the operator $\tilde{\square}$ with the BRST Hamiltonian. Of course, this could only be the case specializing to the BRST superghost Hilbert space of the previous section.

theory on M . The equation of motion, gauge invariances, and Bianchi identities are concisely summarized in a detour complex

$$\begin{array}{ccccccc} \cdots & \xrightarrow{Q} & \left(\begin{array}{c} \text{Gauge} \\ \text{parameters} \end{array} \right) & \xrightarrow{Q} & \left(\begin{array}{c} \text{Gauge} \\ \text{fields} \end{array} \right) & \left(\begin{array}{c} \text{Equations of} \\ \text{motion/currents} \end{array} \right) & \xrightarrow{Q} \left(\begin{array}{c} \text{Bianchi/Noether} \\ \text{identities} \end{array} \right) \xrightarrow{Q} \cdots \\ & & & & \uparrow \mathcal{D} - QM^{-1}Q & & \end{array}$$

The $\cdots$ on the ends of the complex describe any gauge for gauge symmetries and their accompanying Bianchi for Bianchi identities.

The models described by the above complex depend on towers of gauge fields (possibly infinitely many for the case when the constraint algebra contains Grassmann odd generators). There are cases when these towers of gauge fields have a simple geometric interpretation (including the quaternionic Kähler models described here—see our conclusions for a discussion of this point). These towers of gauge fields arise because the physical cohomology retains a dependence on certain bilinears in ghosts. Generically it is desirable to remove this ghost dependence; this can be achieved by gauging further combinations of R symmetries (the “ghostbusting” procedure of [36]). This leads to more standard physical models with equations of motion and local invariances of the form

$$(\Delta + \cdots)A = 0, \quad \delta A = D\alpha,$$

where Δ is typically the Laplace operator, A denotes some type of gauge field, and the operator D generates its gauge invariance. The $\cdots$ ’s stand for terms required for the equation of motion to be gauge invariant. The operator $\Delta + \cdots$ can be expressed in a simple “Labastida” form (a name which refers to its origin in the theory of higher spin theories) or equivalently as a self-adjoint “Einstein operator” (this name was chosen since the linearized Einstein tensor is one of the simplest examples). The latter form immediately implies a gauge invariant action principle. Let us now apply these results to the model at hand, we focus on the main formulæ, referring the reader to the articles [36] for detailed derivations of the underlying methodology.

Firstly the “long operator” $\mathcal{D} - QM^{-1}Q$ can be defined as acting on wavefunctions

$$\Psi(y) \in \wedge E[y]$$

built from polynomials in a commuting bilinear in superghosts $y = 2z^\alpha p_\alpha$ with coefficients in $\Gamma(\wedge E)$ (because this space forms the ghost number zero kernel of the operator M). Explicitly it yields a gauge invariant equation of motion

$$\mathbf{B}^{ij} f_{ij}^{\text{gh}} \Psi = 0, \tag{29}$$

where, acting on functions of only y , the operators f_{ij}^{gh} have the simple expression

$$f_{ij}^{\text{gh}} = \begin{pmatrix} y & -2(y\partial_y + 1) \\ -2(y\partial_y + 1) & 4(y\partial_y^2 + 2\partial_y) \end{pmatrix}.$$

This model is but a stepping stone to our theory of interest, obtained by also gauging the Lefschetz–Verbitsky generator $\mathbf{tr}$. This choice may seem *ad hoc*, but is well known in the higher spin literature (for example, it is necessary to obtain the linearized Einstein tensor in the case of a spin 2 theory). In particular it removes all dependence of the

physical cohomology on the ghost bilinear y . The physical gauge fields now take values in $\wedge E$ only.

In fact, gauging the R -symmetry $\mathbf{tr}$ amounts to restricting the y dependence of $\Psi(y)$ in the detour complex to

$$\Psi = \frac{I_1(\sqrt{y\mathbf{tr}})}{\sqrt{y\mathbf{tr}}} \varphi, \quad \varphi \in \wedge E,$$

and pushing the long operator in (29) past the operator-valued Bessel function yields the very simple “Labastida” equation of motion

$$\mathbf{tr} (\mathbf{d}_\alpha \mathbf{d}^\alpha + \mathbf{g}) \varphi = 0. \quad (30)$$

In particular, notice that this equation factorizes as the product of $\mathbf{tr}$ with the operator discovered long ago by Baston [24]. In fact this gauge theory, on a quaternionic Kähler manifold mimics the higher form (p, q) -form Kähler Electromagnetism theory presented in [52] (observe the correspondence between the Dolbeault bilinear $\partial\bar{\partial}$ and the Baston operator $\mathbf{d}_\alpha \mathbf{d}^\alpha + \mathbf{g}$).

The Labastida equation of motion enjoys the Maxwell like gauge invariance

$$\delta\varphi = \mathbf{d}^\alpha \xi_\alpha,$$

thanks to the identity

$$(\mathbf{d}_\alpha \mathbf{d}^\alpha + \mathbf{g}) \mathbf{d}^\beta \xi_\beta = 0,$$

first uncovered by Baston [24]. In fact the Labastida equation of motion has further gauge for gauge symmetries and accompanying Bianchi for Bianchi identities. These are most easily displayed by writing the Labastida equation of motion in a form following from the variation of an action. This is achieved by constructing the self-adjoint Einstein operator¹⁴

$$\mathbf{G} = : \frac{I_1(\sqrt{\mathbf{g}\mathbf{tr}})}{2\sqrt{\mathbf{g}\mathbf{tr}}} : \mathbf{tr} (\mathbf{d}_\alpha \mathbf{d}^\alpha + \mathbf{g}) = (\delta_\alpha \delta^\alpha + \mathbf{tr}) \mathbf{g} : \frac{I_1(\sqrt{\mathbf{g}\mathbf{tr}})}{2\sqrt{\mathbf{g}\mathbf{tr}}} : = \mathbf{G}^*,$$

in terms of which the Labastida equation of motion is equivalent to the “Einstein” equation of motion $\mathbf{G}\varphi = 0$.

The Einstein operator has the compact, and manifestly self-adjoint expression

$$\begin{aligned} \mathbf{G} = & : I_0(\sqrt{\mathbf{g}\mathbf{tr}}) [\mathbf{d}_\alpha \delta^\alpha + \delta_\alpha \mathbf{d}^\alpha + 2\Lambda (N - n)] \\ & - 2 \frac{I_1(\sqrt{\mathbf{g}\mathbf{tr}})}{\sqrt{\mathbf{g}\mathbf{tr}}} [(\mathbf{d}_\alpha \mathbf{d}^\alpha + \Lambda \mathbf{g}) \mathbf{tr} + \mathbf{g} (\delta_\alpha \delta^\alpha + \Lambda \mathbf{tr})] : \end{aligned}$$

In all the above formulæ, normal ordering denoted by $: \bullet :$ puts all factors of $\mathbf{g}$ and $\mathbf{tr}$ to the far left and right, respectively and we have restored the dependence on the scalar

¹⁴ The derivation of this result is described in [36, 42, 43] and amounts to composing the long operator with the Bessel series to balance its appearance on the right in [42, 43] and fixing y -independent representatives of coker $(y + \mathbf{g})$.

curvature through Λ so that the $\Lambda \rightarrow 0$ hyperKähler limit is manifest. It is important to note that this operator acts on sections of $\wedge E$ of arbitrary degree. Therefore, the equation of motion we write down is really the generating function for the equations valid at any degree and in arbitrary dimensions, this is what necessitates the operator-valued Bessel functions.

Given the Einstein operator, we can now express the equations of motion, gauge and gauge for gauge invariances, Bianchi and Bianchi for Bianchi identities neatly in a single complex

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\mathbf{D}} & \wedge E \otimes \odot H & \xrightarrow{\mathbf{D}} & \cdots & \xrightarrow{\mathbf{F}} & \wedge E \otimes \odot H \xrightarrow{\mathbf{F}} \cdots \\ & & & \searrow \mathcal{G} & & & \nearrow \end{array} \quad (31)$$

Here the operators $\mathbf{D}$ and $\mathbf{F}$ are closely related to the Dirac and Dirac–Fueter operators introduced by Baston [24]. Explicitly, they act on sections of $\wedge E \otimes \odot H$ as

$$\begin{aligned} \mathbf{D} : \phi_{A_1 \dots A_k}^{\alpha_1 \dots \alpha_s} &\mapsto s \nabla_{[A_1}^{\alpha} \phi_{A_2 \dots A_{k+1}] \alpha}^{\alpha_1 \dots \alpha_{s-1}}, \\ \mathbf{F} : \phi_{A_1 \dots A_k}^{\alpha_1 \dots \alpha_s} &\mapsto k \nabla_A^{(\alpha_1} \phi_{A_1 \dots A_{k-1}}^{\alpha_2 \dots \alpha_{s+1})}. \end{aligned} \quad (32)$$

In an index free notation where $\Phi = \sum_{k,s} \phi_{A_1 \dots A_k}^{\alpha_1 \dots \alpha_s} \eta^{A_1} \dots \eta^{A_k} z_{\alpha_1} \dots z_{\alpha_s} \in \wedge E \otimes \odot H$, we may simply write

$$\mathbf{D} = \eta^A \nabla_{\alpha A} \frac{\partial}{\partial z_{\alpha}} = \mathbf{d}_{\alpha} \frac{\partial}{\partial z_{\alpha}}, \quad \mathbf{F} = z_{\alpha} \nabla_A^{\alpha} \frac{\partial}{\partial \eta_A} = z_{\alpha} \delta^{\alpha}.$$

Both these operators are nilpotent by virtue of the algebra (28) and the identity $t^{\alpha\beta} \psi_{\alpha\beta\gamma_1 \dots \gamma_s} = 0$. Moreover,

$$(\mathbf{d}_{\alpha} \mathbf{d}^{\alpha} + \mathbf{g}) \mathbf{D} = 0 = \mathbf{F} (\delta_{\alpha} \delta^{\alpha} + \mathbf{tr}),$$

verify the veracity of the complex (31).

The incoming complex with differential $\mathbf{D}$ can be viewed as the quaternionic generalization of the Dolbeault complex [24], while the outgoing complex with differential $\mathbf{F}$ is its dual (*i.e.* the Dirac–Fueter type operator $\mathbf{F}$ is a codifferential). Physically they encode gauge invariances and Bianchi identities. The Einstein operator $\mathcal{G}$ gives the detour connecting the two complexes and, physically, the equations of motion. Notice also, that it can connect the equations of motion at any degree in $\wedge E$ or $\odot H$, so gauge potentials are generic sections of $\wedge E \otimes \odot H$. The mathematical elegance of this model is perhaps surprising, but even more remarkable is its *rôle* as the arena for a minisuperspace quantization of $\mathcal{N} = 2$ supersymmetric black holes. We further discuss this and other possible applications of our theory in the conclusions.

10. Conclusions

The results presented in this paper rely on an analogy between (i) differential forms on a Kähler manifold, (ii) tensors on a constant curvature manifold and (iii) the bundle

$$\wedge E \otimes \mathcal{T}H$$

over a quaternionic manifold obtained by splitting its tangent bundle using the $\mathfrak{sp}(2n) \otimes \mathfrak{sp}(2)$ special holonomy and then taking antisymmetric sections of the $\mathfrak{sp}(2n)$ part E

Fig. 3. A map of the physical models encountered in this paper

along with arbitrary H -tensors. The analogy with Kähler differential forms holds because the natural geometric operators on this bundle are in correspondence with the Dolbeault operators and the generators of the Lefschetz symmetry of Dolbeault cohomology. There is a relation to constant curvature manifolds because, acting on sections of $\wedge E$, only the covariantly constant part of the quaternionic Kähler Riemann tensor contributes. This means that the properties of the geometric operators we have studied are algebraically similar to the Lichnerowicz wave operator and the set of geometric operators that commute with it on a constant curvature manifold. In fact a main result of this paper is the geometric calculus of operators, including a central wave operator, acting on $\Gamma(\wedge E \otimes \mathcal{T}H)$. Remarkably, this seemingly purely mathematical structure was motivated by a study of supersymmetric black holes in four dimensional spacetime.

The route from four dimensional black holes to local quantum field theories on quaternionic Kähler manifolds is sketched in Fig. 3. It began with $\mathcal{N} = 2$ SUGRA in four dimensions. Reducing along an isometry and specializing to spherical symmetry led to a spinning model with four local worldline supersymmetries. Thanks to the c -map this spinning particle moves in a quaternionic Kähler manifold. Moreover, fermionic degrees of freedom were retained in order that the BPS conditions of the spinning particle model corresponded to the reduced ones of the four dimensional SUGRA, and therefore in turn to the linear evolution equations of the attractor mechanism. We then studied the quantization of this model through BRST detour methods. This led to the gauge invariant equation of motion (29). Let us make a few remarks on this model.

Given a $4n$ -dimensional quaternionic Kähler manifold, it is always possible to find a $4n + 4$ dimensional hyperKähler manifold whose metric is a quaternionic cone over the

original $4n$ -dimensional model [71–75]. In the work [71], the dimensionally reduced supersymmetry parameters of the four dimensional SUGRA were shown to correspond to the extra four coordinates required to build a $4n + 4$ dimensional hyperKähler cone over the quaternionic Kähler, stationary, spherically symmetric, black hole moduli space. However, in BRST quantization the ghosts correspond to the local gauge parameters, in particular the superghosts play the *rôle* of the supersymmetry parameters. Hence, the model (29), where we made no additional gaugings to eliminate ghosts, really should be viewed as a model on the hyperKähler cone. This explains the third signpost on the roadmap 3.

The next stop on the roadmap was motivated by ideas from higher spin models. In particular, our aim was to write down a model where all ghosts had been eliminated from the physical cohomology. Based on ideas coming from our earlier work on orthosymplectic constraint algebras, we suspected that gauging the Lefschetz–Verbitsky trace operator would lead to a gauge invariant quantum field theory generalizing both p -form electromagnetism and (p, q) -form Kähler electromagnetism to quaternionic Kähler manifolds. This hunch was correct and led to the model (31). Interestingly enough, it could have been the case that this choice of route would lead to a model that did *not* describe supersymmetric black holes. However, it is clear that in fact the quaternionic Kähler model does so, and in a fascinating way. Examining the Labastida form of the equation of motion (30) we see that it is a product of the Baston operator and the Lefschetz–Verbitsky trace operator. As shown in [25], by explicitly constructing the quaternionic Penrose transform underlying Baston’s quaternionic generalization of the Dolbeault complex, at least in the scalar sector of $\wedge E$, zero modes of the Baston operator correspond to supersymmetric black hole states. We suspect that within BRST quantization, this picture can be extended to a general correspondence with the Baston complex. In this case, solutions to our quaternionic Kähler electromagnetism theory would fall into two classes:

1. BPS solutions in the kernel of $\mathbf{d}_\alpha \mathbf{d}^\alpha + \mathbf{g}$.
2. Solutions whose non-vanishing image under $\mathbf{d}_\alpha \mathbf{d}^\alpha + \mathbf{g}$ lies in the kernel of $\mathbf{tr}$.

This explains the last signpost of the roadmap (3). Clearly our work opens many avenues for further study:

Firstly, since our BRST quantization methods produce a gauge theory on the hyperKähler cone and furthermore rely on a polarization where one Fourier transforms over half the ghost variables (alias quaternionic cone coordinates), there should exist a rather direct relationship between BRST quantization and the quaternionic twistor methods of [25].

Secondly, our quaternionic Kähler higher form electromagnetism may provide an interesting arena for further studies of minisuperspace black hole quantization. One might hope that constructing interactions for this abelian gauge theory could lead to a far more detailed understanding of these theories (perhaps along the lines of the multi-centered configuration and attractor flow trees—“third quantization” [76]). This might sound extremely ambitious, since higher spin interactions are fraught with inconsistencies.

However, it is possible that some of the methods of Vasiliev, who has constructed three point higher spin interaction using a combination of unfolding techniques (which are closely related to our BRST framework) and Chern–Simons like equations of motions based on a star product, could solve this problem. Also, we cannot help but remark that whenever two seemingly disparate fields (such as higher spin interactions and four dimensional black hole physics) turn out to be related, oftentimes the flow of new ideas is

bidirectional. In fact, we suspect that higher quantum corrections to $\mathcal{N} = 2$ supergravities in four dimensions could even have implications for possible higher spin interactions.

Finally, another topic that is worth further investigation is the novel Dirac operator in (24). This operator acts on the BRST superghost Hilbert space; in the context of this paper it was merely a tool for constructing a nilpotent BRST charge. However, we suspect that it might have a distinguished *rôle* to play. In particular, it would be fascinating to compute the Witten index of this operator. Given that it was built from a supersymmetric quantum mechanical model, standard quantum methods may suffice for this.

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