Erratum

2-Matrix versus Complex Matrix Model, Integrals over the Unitary Group as Triangular Integrals

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In the above article **Theorem 3.4** contains an error which is easily resolved through a slight reformulation of the statement, including in it part of **Theorem 4.1**. We keep the conventions and definitions given in the paper.

Original Version

This is **Theorem 3.4** as it appears in the published article.

Theorem 3.4. *For any polynomial invariant function $F(A, B)$, one has:*

$$\begin{aligned} & \frac{\tilde{J}_n^2}{n!^2 Z_H(n, \gamma, \alpha_1, \alpha_2)} \int_{D_n(\mathbb{R}) \times D_n(\mathbb{R})} dX dY e^{-\frac{\alpha_1}{2} \text{Tr } X^2} e^{-\frac{\alpha_2}{2} \text{Tr } Y^2} \tilde{W}_F(X, Y) \\ &= \frac{J_n}{n! Z_C(n, \gamma, \alpha_1, \alpha_2)} \int_{D_n(\mathbb{C})} dX e^{-\frac{\alpha_1}{2} \text{Tr } X^2} e^{-\frac{\alpha_2}{2} \text{Tr } \bar{X}^2} W_F(X, \bar{X}). \end{aligned} \quad (0-1)$$

Proof. Start from Theorem 3.3, diagonalize M_1 and M_2 on the hermitian side, and jordanize Z on the complex side,

$$\begin{aligned} \langle F \rangle_H &= \frac{\tilde{J}_n^2}{n!^2 Z_H(n, \gamma, \alpha_1, \alpha_2)} \int_{D_n(\mathbb{R}) \times D_n(\mathbb{R})} dX dY e^{-\frac{\alpha_1}{2} \text{Tr } X^2} e^{-\frac{\alpha_2}{2} \text{Tr } Y^2} \tilde{W}_F(X, Y) \\ &= \langle F \rangle_C = \frac{J_n}{n! Z_C(n, \gamma, \alpha_1, \alpha_2)} \int_{D_n(\mathbb{C})} dX e^{-\frac{\alpha_1}{2} \text{Tr } X^2} e^{-\frac{\alpha_2}{2} \text{Tr } \bar{X}^2} e^{-\gamma \text{Tr } X \bar{X}} \omega_F(X, \bar{X}) \end{aligned}$$

$$\begin{aligned}
&= \frac{J_n}{n! Z_C(n, \gamma, \alpha_1, \alpha_2)} \int_{D_n(\mathbb{C})} dX e^{-\frac{\alpha_1}{2} \text{Tr } X^2} e^{-\frac{\alpha_2}{2} \text{Tr } \bar{X}^2} e^{-\gamma \text{Tr } X_\sigma \bar{X}_\tau} \omega_F(X_\sigma, \bar{X}_\tau) \\
&= \frac{J_n}{n! Z_C(n, \gamma, \alpha_1, \alpha_2)} \int_{D_n(\mathbb{C})} dX e^{-\frac{\alpha_1}{2} \text{Tr } X^2} e^{-\frac{\alpha_2}{2} \text{Tr } \bar{X}^2} W_F(X_\sigma, \bar{X}_\tau). \quad (0-2)
\end{aligned}$$

The equality in the first line is obtained by diagonalizing M_1 and M_2 (with Jacobian given in Eq.2-17), the equality in the second line is obtained by Jordanizing Z (with Jacobian given in Eq.2-19), the equality between the second and third line holds for any pair of permutations σ and τ (it can be proven with Lemma A.1 given in the Appendix), and the equality of the last line comes from the definition of W_F . $\square$

The main mistake appears from second to third line, since at this point it is false that permuting the two matrices X and $\bar{X}$ to X_σ and $\bar{X}_\tau$ independently does not change the integral. However, a small rearrangement of **Theorems 3.4** and **4.1** yields the desired result.

Corrected Version

To fix the mistake, we reformulate **Theorem 3.4** as follows:

Theorem 3.4. *For any polynomial invariant function $F(A, B)$, one has:*

$$\begin{aligned}
&\frac{\tilde{J}_n^2}{n!^2 Z_H(n, \gamma, \alpha_1, \alpha_2)} \int_{D_n(\mathbb{R}) \times D_n(\mathbb{R})} dX dY e^{-\frac{\alpha_1}{2} \text{Tr } X^2} e^{-\frac{\alpha_2}{2} \text{Tr } Y^2} \tilde{W}_F(X, Y) \\
&= \frac{J_n}{n! Z_C(n, \gamma, \alpha_1, \alpha_2)} \int_{D_n(\mathbb{R}) \times D_n(\mathbb{R})} dX dY e^{-\frac{\alpha_1}{2} \text{Tr } X^2} e^{-\frac{\alpha_2}{2} \text{Tr } Y^2} W_F(X, Y). \quad (0-3)
\end{aligned}$$

Proof.

$$\begin{aligned}
\langle F \rangle_H &= \frac{\tilde{J}_n^2}{n!^2 Z_H(n, \gamma, \alpha_1, \alpha_2)} \int_{D_n(\mathbb{R}) \times D_n(\mathbb{R})} dX dY e^{-\frac{\alpha_1}{2} \text{Tr } X^2} e^{-\frac{\alpha_2}{2} \text{Tr } Y^2} \tilde{W}_F(X, Y) \\
= \langle F \rangle_C &= \frac{J_n}{n! Z_C(n, \gamma, \alpha_1, \alpha_2)} \int_{D_n(\mathbb{C})} dX e^{-\frac{\alpha_1}{2} \text{Tr } X^2} e^{-\frac{\alpha_2}{2} \text{Tr } \bar{X}^2} e^{-\gamma \text{Tr } X \bar{X}} \omega_F(X, \bar{X}) \\
&= \frac{J_n \left(\frac{\pi}{\sqrt{-\delta}} \right)^n}{n! Z_C(n, \gamma, \alpha_1, \alpha_2) \left(\frac{2\pi}{\sqrt{\delta}} \right)^n} \int_{D_n(\mathbb{R}) \times D_n(\mathbb{R})} dX dY e^{-\frac{\alpha_1}{2} \text{Tr } X^2} \\
&\quad \times e^{-\frac{\alpha_2}{2} \text{Tr } Y^2} e^{-\gamma \text{Tr } XY} \omega_F(X, Y) \\
&= \frac{J_n \left(\frac{\pi}{\sqrt{-\delta}} \right)^n}{n! Z_C(n, \gamma, \alpha_1, \alpha_2) \left(\frac{2\pi}{\sqrt{\delta}} \right)^n} \frac{1}{n!^2} \\
&\quad \times \sum_{\sigma, \tau} \int_{D_n(\mathbb{R}) \times D_n(\mathbb{R})} dX dY e^{-\frac{\alpha_1}{2} \text{Tr } X^2} e^{-\frac{\alpha_2}{2} \text{Tr } Y^2} e^{-\gamma \text{Tr } X_\sigma Y_\tau} \omega_F(X_\sigma, Y_\tau) \\
&= \frac{J_n \left(\frac{\pi}{\sqrt{-\delta}} \right)^n}{n! Z_C(n, \gamma, \alpha_1, \alpha_2) \left(\frac{2\pi}{\sqrt{\delta}} \right)^n} \frac{1}{n!^2} \\
&\quad \times \sum_{\sigma, \tau} \int_{D_n(\mathbb{R}) \times D_n(\mathbb{R})} dX dY e^{-\frac{\alpha_1}{2} \text{Tr } X^2} e^{-\frac{\alpha_2}{2} \text{Tr } Y^2} W_F(X, Y). \quad (0-4)
\end{aligned}$$

The equality in the first line is obtained by diagonalizing M_1 and M_2 (with Jacobian given in Eq.2-17), the equality in the second line is obtained by Jordanizing Z (with Jacobian given in Eq.2-19), the equality between the second and third line follows from Lemma A.1 given in the Appendix, the equality between the third and fourth line can be done thanks to the symmetry of the measure. It is convenient to do so in order to compare integrands later, since the integrand in the first line is symmetric by the action of the unitary integral inside $\tilde{W}_F(X, Y)$. Finally the equality between the fourth and fifth line comes from the definition of W_F . $\square$

Additional minor modification. Due to the inclusion of part of **Theorem 4.1** in the corrected **Theorem 3.4**, we must rewrite the proof of the former in the following terms

Theorem 4.1. *For any invariant function $F(A, B)$ one has:*

$$\begin{aligned} & \int_{U(n)} dU F(X, UYU^\dagger) e^{-\gamma \operatorname{Tr} XUYU^\dagger} \\ &= \frac{c_n}{n!} \frac{\sum_\sigma \sum_\tau (-1)^\sigma (-1)^\tau e^{-\gamma \operatorname{Tr} X_\sigma Y_\tau} \int_{T(n)} F(X_\sigma + T, Y_\tau + T^\dagger) e^{-\gamma \operatorname{Tr} TT^\dagger} dT}{\Delta(X)\Delta(Y)}, \end{aligned} \quad (0-5)$$

where

$$c_n = \frac{\prod_{k=0}^{n-1} k!}{(-2\pi)^{\frac{n(n-1)}{2}}}, \quad (0-6)$$

i.e.

$$\tilde{W}_F(X, Y) = n! c_n W_F(X, Y). \quad (0-7)$$

Proof. From Theorem 3.4 we have

$$0 = \int_{D_n(\mathbb{R}) \times D_n(\mathbb{R})} dX dY e^{-\frac{\alpha_1}{2} \operatorname{Tr} X^2} e^{-\frac{\alpha_2}{2} \operatorname{Tr} Y^2} (n! c_n W_F(X, Y) - \tilde{W}_F(X, Y)). \quad (0-8)$$

Notice that if $f(A)$ and $g(B)$ are invariant functions, i.e. $f(UAU^{-1}) = f(A)$ for all A and U (resp. $g(UBU^{-1}) = g(B)$ for all B and U), one has:

$$\begin{aligned} W_{f(X)g(Y)F(X,Y)}(X, Y) &= f(X)g(Y)W_F(X, Y), \\ \tilde{W}_{f(X)g(Y)F(X,Y)}(X, Y) &= f(X)g(Y)\tilde{W}_F(X, Y). \end{aligned} \quad (0-9)$$

Thus, for any symmetric polynomials $f(x_1, \dots, x_n)$ and $g(y_1, \dots, y_n)$, one has:

$$\begin{aligned} 0 &= \int_{D_n(\mathbb{R}) \times D_n(\mathbb{R})} dX dY e^{-\frac{\alpha_1}{2} \operatorname{Tr} X^2} e^{-\frac{\alpha_2}{2} \operatorname{Tr} Y^2} f(X)g(Y) \\ &\quad \times (n! c_n W_F(X, Y) - \tilde{W}_F(X, Y)). \end{aligned} \quad (0-10)$$

Notice that $e^{-\frac{\alpha_1}{2} \operatorname{Tr} X^2} e^{-\frac{\alpha_2}{2} \operatorname{Tr} Y^2} (n! c_n W_F(X, Y) - \tilde{W}_F(X, Y))$ is a continuous function quickly decreasing at ∞ in all variables, and it is symmetric in the x 's and in the y 's. Using the Stone-Weierstrass Theorem (polynomials are dense in the space of continuous functions), one gets that $(n! c_n W_F(X, Y) - \tilde{W}_F(X, Y)) = 0$. $\square$