Erratum

A Strong Szegő Theorem for Jacobi Matrices

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1. Introduction

In this erratum an error in [1] is pointed out and corrected. In that paper we were concerned with the spectral theory of Jacobi matrices, that is semi-infinite tridiagonal matrices

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 \\ a_1 & b_2 & a_2 & 0 \\ 0 & a_2 & b_3 & \ddots \\ 0 & 0 & \ddots & \ddots \end{pmatrix}$$

where $a_n > 0$ and $b_n \in \mathbb{R}$. We assume that $a_n^2 - 1$ and b_n are conditionally summable and define

$$\begin{aligned} \lambda_n &:= - \sum_{k=n+1}^{\infty} b_k, \\ \kappa_n &:= - \sum_{k=n+1}^{\infty} (a_k^2 - 1) \end{aligned} \tag{1.1}$$

for $n = 0, 1, \dots$

The goal of Sect. 5 of [1] was to prove the following result (which appears there as Proposition 5.1):

The online version of the original article can be found under doi:[10.1007/s00220-007-0195-5](https://doi.org/10.1007/s00220-007-0195-5).

Proposition 1.1. *There are solutions ψ_s and ψ_b to $J\psi = E\psi$ at energy $E = \pm 2$ such that*

$$\left| \frac{\psi_s(k)}{\psi_b(k)} \right| \rightarrow 0$$

and

$$\pm \frac{\psi_s(k+1)}{\psi_s(k)} = 1 + \ell_1^2.$$

Moreover, for either solution and for k sufficiently large,

$$(\pm 1)^k \psi(k)$$

is sign definite and non-zero.

In [1] we attempted to use Asymptotic Integration to prove this proposition. Due to two separate multiplication errors Eqs. (5.1) and (5.2) are incorrect, and the subsequent arguments are therefore not applicable. So far our attempts to resurrect the Asymptotic Integration argument have failed, so instead in the next section we present a complete proof of Proposition 1.1 based on a contraction mapping argument. With these corrections the remainder of the argument in [1] is valid as stated.

2. Proof of Proposition 1.1.

For this section all norms refer to ℓ_1^2 . We will use the notation $x \lesssim y$ if $x \leq cy$ for some constant $c > 0$ (which may change from one line to the next). We will also need the following lemma, which is a special case of Lemma 2.4 in [1]:

Lemma 2.1. *Let $\beta, \gamma \in \ell_1^2$, and define a sequence $\eta_n := \sum_{k=n}^{\infty} \beta_k \gamma_k$. Then $\eta \in \ell_1^2$ and $\|\eta\| \leq \|\beta\| \cdot \|\gamma\|$.*

The rest of this section is devoted to a proof of Proposition 1.1 for $E = 2$, the proof for $E = -2$ being analogous. We will begin by constructing the small solution ψ_s . By dividing the recurrence equation

$$a_{k+1}\psi(k+1) + (b_{k+1} - E)\psi(k) + a_k\psi(k-1) = 0$$

by $\psi(k)$ and writing

$$r(k) = \frac{\psi(k)}{\psi(k-1)}$$

we deduce the discrete Riccati equation

$$a_{k+1}r(k+1) + \frac{a_k}{r(k)} = 2 - b_{k+1}. \quad (2.1)$$

Lemma 2.2. *There is a $k_0 > 0$ and a solution $r(k)$, $k \geq k_0$, to the discrete Riccati equation (2.1) of the form $r(k) = 1 + \varepsilon(k)$ with $\|\varepsilon\| < 1$.*

Proof. Write $r(k) = 1 + \varepsilon(k)$ and

$$\frac{1}{r(k)} = 1 - \varepsilon(k) + \frac{\varepsilon(k)^2}{1 + \varepsilon(k)}$$

so (2.1) becomes

$$a_{k+1}\varepsilon(k+1) - a_k\varepsilon(k) + a_k \frac{\varepsilon(k)^2}{1 + \varepsilon(k)} = 2 - a_{k+1} - a_k - b_{k+1}.$$

Summing to infinity then yields

$$\varepsilon(k) = \frac{-1}{a_k} \left(\lambda_k + \kappa_k + \kappa_{k-1} - \sum_{l=k}^{\infty} a_l \frac{\varepsilon(l)^2}{1 + \varepsilon(l)} \right). \quad (2.2)$$

We will use a contraction mapping argument to show (2.2) has a solution $\varepsilon \in \ell_1^2$. To this end, let $[F(\varepsilon)](k)$ denote the right-hand side of (2.2). We must show that F maps some ball in ℓ_1^2 to itself and is a contraction there.

Notice that because $0 < a_k \rightarrow 1$ we have

$$\|F(\beta)\| \lesssim \|\lambda\| + \|\kappa\| + \left\| \sum_{l=k}^{\infty} \frac{\beta(l)^2}{1 + \beta(l)} \right\| \lesssim \|\lambda\| + \|\kappa\| + \|\beta\|^2,$$

where the second inequality follows from $\beta \in \ell_1^2$ and Lemma 2.1. So for any fixed $\delta < 1$, if $\|\lambda\|, \|\kappa\|$ are small enough (depending on δ) then

$$F : \{\beta \in \ell_1^2 : \|\beta\| \leq \delta\} \rightarrow \{\beta \in \ell_1^2 : \|\beta\| \leq \delta\} \quad (2.3)$$

as desired.

Next we show that F is a contraction:

$$\begin{aligned} |[F(\beta)](k) - [F(\gamma)](k)| &\lesssim \sum_{l=k}^{\infty} \left| \frac{\beta(l)^2}{1 + \beta(l)} - \frac{\gamma(l)^2}{1 + \gamma(l)} \right| \\ &= \sum_{l=k}^{\infty} \frac{|\beta(l)^2 - \gamma(l)^2 + \beta(l)^2\gamma(l) - \gamma(l)^2\beta(l)|}{|(1 + \beta(l))(1 + \gamma(l))|} \\ &\leq \sum_{l=k}^{\infty} \left(\frac{|\beta(l) + \gamma(l)| + |\beta(l)\gamma(l)|}{|(1 + \beta(l))(1 + \gamma(l))|} \right) |\beta(l) - \gamma(l)|. \end{aligned}$$

So from Lemma 2.1 we have

$$\|F(\beta) - F(\gamma)\| \lesssim \left\| \frac{|\beta + \gamma| + |\beta\gamma|}{|(1 + \beta)(1 + \gamma)|} \right\| \cdot \|\beta - \gamma\|.$$

In particular, if $\|\beta\|$ and $\|\gamma\|$ are small enough then

$$\|F(\beta) - F(\gamma)\| \leq \eta \|\beta - \gamma\| \quad (2.4)$$

with $\eta < 1$, and F is a contraction.

Now, choose δ small enough and k_0 large enough that (2.3) and (2.4) hold for the sequences $\{\lambda_k\}_{k=k_0}^\infty$ and $\{\kappa_k\}_{k=k_0}^\infty$ (which is possible since $\lambda, \kappa \in \ell_1^2$). Then by the Banach Fixed Point Theorem one finds a solution

$$\varepsilon \in \{\beta \in \ell_1^2(k_0, k_0 + 1, \dots) : \|\beta\| \leq \delta < 1\}$$

to the equation

$$\varepsilon(k) = [F(\varepsilon)](k).$$

That is, ε solves (2.2) for $k \geq k_0$, as claimed. $\square$

Corollary 2.3. *There is a solution $\psi_s(k)$ to $J\psi = 2\psi$ so that for $k \geq k_0$ we have*

$$\psi_s(k) = \prod_{l=k_0}^k (1 + \varepsilon(l)) \quad (2.5)$$

with $\|\varepsilon\| < 1$. In particular, for $k \geq k_0$, $\psi_s(k)$ is sign-definite and non-zero.

Proof. The first statement follows from

$$\frac{\psi_s(k)}{\psi_s(k-1)} = r(k) = 1 + \varepsilon(k)$$

and Lemma 2.2. The “in particular” part follows from $\|\varepsilon\| < 1$, which shows $|\varepsilon(k)| < 1$ for all $k \geq k_0$. $\square$

Now we construct the big solution ψ_b :

Lemma 2.4. *There is a solution ψ_b to $J\psi = 2\psi$ so that*

$$\left| \frac{\psi_s(k)}{\psi_b(k)} \right| \rightarrow 0$$

as $k \rightarrow \infty$.

Proof. Let ψ_s be the solution from Corollary 2.3 and let ψ_b be the solution with

$$1 = W_k(\psi_s, \psi_b) = a_k(\psi_s(k)\psi_b(k+1) - \psi_b(k)\psi_s(k+1))$$

and $\psi_b(k_0) = 1 + \varepsilon(k_0) = \psi_s(k_0)$. Then

$$\frac{\psi_b(k+1)}{\psi_s(k+1)} - \frac{\psi_b(k)}{\psi_s(k)} = \frac{W_k(\psi_s, \psi_b)}{a_k \psi_s(k)\psi_s(k+1)} = \frac{1}{a_k \psi_s(k)\psi_s(k+1)}$$

and so

$$\frac{\psi_b(k)}{\psi_s(k)} = 1 + \sum_{l=k_0}^{k-1} \frac{1}{a_l \psi_s(l)\psi_s(l+1)}.$$

Therefore to show

$$\left| \frac{\psi_s(k)}{\psi_b(k)} \right| \rightarrow 0$$

we must show

$$\sum_{l=k_0}^k \frac{1}{a_l \psi_s(l) \psi_s(l+1)} \rightarrow \infty$$

as $k \rightarrow \infty$.

From (2.5) and the expansion $\log(1+x) = x + O(|x|^2)$ we have

$$\begin{aligned} \psi_s(k) &= \exp\left(\sum_{l=k_0}^k \log(1 + \varepsilon(l))\right) \\ &= \exp\left(\sum_{l=k_0}^k \varepsilon(l) + O(|\varepsilon(l)|^2)\right) \\ &\lesssim \exp\left(\sum_{l=k_0}^k \varepsilon(l)\right) \exp(\|\varepsilon\|^2). \end{aligned}$$

Writing $\varepsilon(l) = (\varepsilon(l)\sqrt{l})(1/\sqrt{l})$ and using Cauchy-Schwarz and $\varepsilon \in \ell_1^2$ results in the bound

$$\psi_s(k) \lesssim \exp(\sqrt{\log(k)})$$

which we claim is bounded by k^γ for any $0 < \gamma < 1/2$. To see this simply notice that

$$\log\left(\frac{\exp(\sqrt{\log(k)})}{k^\gamma}\right) = \sqrt{\log(k)} - \gamma \log(k) \rightarrow -\infty$$

as $k \rightarrow \infty$.

So all together we have $\psi_s(k) \lesssim k^\gamma$ for any $0 < \gamma < 1/2$, and so

$$\left| \frac{\psi_b(k)}{\psi_s(k)} \right| = \sum_{l=k_0}^k \frac{1}{a_l \psi_s(l) \psi_s(l+1)} \gtrsim \sum_{l=k_0}^k l^{-2\gamma} \rightarrow \infty$$

as $k \rightarrow \infty$, as claimed. $\square$

Finally we have:

Proof of Proposition 1.1. The small solution ψ_s was constructed in Corollary 2.3 and the big solution ψ_b was constructed in Lemma 2.4. The ratio statements then follow from Lemmas 2.2 and 2.4. $\square$

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Reference

1. Ryckman, E.: A strong Szegő theorem for Jacobi matrices. *Comm. Math. Phys.* **271**(3), 791–820 (2007)

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