Cellular resolutions of cointerval ideals

Anton Dochtermann · Alexander Engström

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Abstract Minimal cellular resolutions of the edge ideals of *cointerval* hypergraphs are constructed. This class of d -uniform hypergraphs coincides with the complements of interval graphs (for the case $d = 2$), and strictly contains the class of ‘strongly stable’ hypergraphs corresponding to pure shifted simplicial complexes. The polyhedral complexes supporting the resolutions are described as certain spaces of directed graph homomorphisms, and are realized as subcomplexes of mixed subdivisions of the Minkowski sums of simplices. Resolutions of more general hypergraphs are obtained by considering decompositions into cointerval hypergraphs.

Keywords Monomial ideals · Cellular resolutions · Stanley–Reisner rings · Hypergraph edge ideals · Interval graphs

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1 Introduction

An *edge ideal* I_H is an ideal in a polynomial ring generated by square-free monomials of a fixed degree d (the generators can be thought of as edges of a d -uniform hypergraph H , hence

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A. Dochtermann (✉)
Department of Mathematics, Stanford University, Stanford, CA, 94305, USA
e-mail: anton.dochtermann@gmail.com

A. Engström
Department of Mathematics, University of California, Berkeley, Berkeley, CA 94729, USA
e-mail: alex@math.berkeley.edu

the name). The study of edge ideals has recently enjoyed a surge of activity, and the most well-known results in this area relate algebraic properties of edge ideals to the combinatorial structure of the underlying (class of) graphs.

In this paper we study *resolutions* of edge ideals, and in particular give explicit descriptions of minimal cellular resolutions for edge ideals of a large class of hypergraphs. Given any ideal I in a polynomial ring $S = k[x_1, \dots, x_n]$, a *resolution* of I is an exact chain complex of free S -modules describing the generators, the relations, the relations among the relations (and so on) of the ideal I . A *cellular* resolution encodes these modules and maps as the chain complex computing the homology of a labeled polyhedral complex.

To any hypergraph H we associate a polytopal complex X_H which parameterizes the directed edges of H . Our main result is that this labeled polyhedral complex X_H supports a minimal cellular resolution of the edge ideal of H , whenever H is what we call a *cointerval* hypergraph (we refer to Theorem 4.1 for a precise formulation). The class of cointerval d -graphs can be seen to coincide with the complements of interval graphs (for the case $d = 2$), and in general strictly contains the class of ‘strongly stable’ hypergraphs corresponding to pure shifted simplicial complexes.

In this sense our constructions can be seen as an extension of the results of Corso and Nagel [7, 8] and Nagel and Reiner [19], where cellular resolutions of strongly stable edge ideals (and related ideals) are considered. In fact our complexes X_H can be seen as instances of the ‘complex-of-boxes’ introduced in [19]. However, viewing the complex X_H as a space of directed graph homomorphisms has the advantage that it connects us to the existing theory of Hom complexes of graph homomorphisms in the spirit of [3], which in turn allows us to use existing tools to study the topology of these spaces (see Sect. 3.1). In addition, we are led to a proof of our other main result, Theorem 5.1, which gives a particularly nice geometric embedding of X_H in terms of *mixed subdivisions* of dilated simplices (see Sect. 5). Finally, this perspective can lead to generalizations: recently Braun, Browder, and Klee have used these methods to construct cellular resolutions from complexes of maps between simplicial complexes (B. Braun, personal communication).

The facial structure of X_H is given by simple graph-theoretic data coming from the hypergraph H , and this allows us to provide transparent descriptions of certain algebraic invariants including Betti numbers, etc., of H . Furthermore, we can use the explicit description of the complexes to provide (not necessarily minimal) cellular resolutions of arbitrary hypergraphs by considering decompositions into cointerval graphs. Our results in this area are all independent of the characteristic of the coefficient field.

The rest of the paper is organized as follows. In Sect. 2 we review the basic definitions relating to edge ideals and cellular resolutions. In Sect. 3 we describe the complexes X_H that will support our resolutions, and establish some results regarding their topology. We provide the definition of cointerval graphs in Sect. 4. Here we also state and prove our main result, namely that the complex X_H supports a minimal cellular resolution of the ideal I_H , whenever H is a cointerval hypergraph. In Sect. 5 we describe how the complexes X_H can be realized as subcomplexes of certain well-studied mixed subdivisions of dilated simplices. We consider resolutions of more general hypergraph edge ideals in Sect. 6, and show how decompositions into cointerval subgraphs $H = H_1 \cup H_2$ lead to cellular resolutions obtained by gluing together the associated X_{H_i} . Here we also provide a thorough analysis of all three graphs on at most five vertices to illustrate our methods. We end in Sect. 7 with some comments regarding open questions and further study.

2 Definitions

We briefly discuss the main objects involved in our study. We begin with some graph-theoretic notions. For a finite subset $V \subseteq \mathbb{Z}$, a (uniform) d -hypergraph (or simply d -graph) H with vertex set $V(H) = V$ is a collection of subsets of V (called edges), each of which has cardinality d . We will often take $V = [n] := \{1, \dots, n\}$ and will suppress set notation in describing our edges, so that e.g. 245 will denote the edge $\{2, 4, 5\}$. The complete d -hypergraph K_n^d is the d -hypergraph on $[n]$ consisting of all possible d -subsets. Note that by definition our graphs come with integer labels on the vertices. If we want to consider the underlying (unlabeled) graph we will emphasize this distinction. In dealing with d -graphs we will often be interested in considering induced $(d - 1)$ -subgraphs in the following sense.

Definition 2.1 Let H be a d -graph and let $v \in V(H) \subseteq \mathbb{Z}$ be some vertex. Then the v -layer of H is a $(d - 1)$ -graph on $V \setminus v$ with edge set

$$\{v_1 v_2 \cdots v_{d-1} \mid vv_1 v_2 \cdots v_{d-1} \in E(H) \text{ and } v < v_1, v_2, \dots, v_{d-1}\}.$$

Note that if H is a 2-graph then one can think of the v -layer as simply the entries to the right of the v -column in the v -row of the adjacency matrix of H , the ‘edges’ of the resulting 1-graph are simply the entries that have a nonzero entry.

If $W \subseteq V(H)$ is a subset of the vertices of a d -graph H , the induced subgraph on W , denoted $H[W]$ (or sometimes simply W if the context is clear) is the d -graph with vertex set W and edges $\{E \subseteq W \mid E \in E(H)\}$.

We next turn to the algebraic notions, and refer to [18] for undefined terms and further discussion. Throughout the paper we let k denote a field, our results will be independent of the characteristic. Given a d -graph H on the vertices $V(H) = \{v_1, \dots, v_n\}$, the edge ideal I_H is by definition the monomial ideal in the polynomial ring $k[x_{v_1}, \dots, x_{v_n}]$ generated by the monomials corresponding to the edges of H ,

$$I_H = \left\langle \prod_{j=1}^d x_{v_{i_j}} \mid v_{i_1} v_{i_2} \cdots v_{i_d} \in E(H) \right\rangle.$$

We will usually take $V(H) = [n]$, so that $S = k[x_1, \dots, x_n]$, but it will be convenient to have the more general setup as well.

We will sometimes employ the Stanley–Reisner theory of face rings of simplicial complexes, and in this context we let Δ denote a simplicial complex on the vertices $[n]$. The Stanley–Reisner ideal of Δ , which we denote I_Δ , is the ideal in S generated by all monomials x_σ corresponding to nonfaces $\sigma \notin \Delta$. We let $R_\Delta = S/I_\Delta$, and recall that $\dim R_\Delta$, the (Krull) dimension of R_Δ , is equal to $\dim(\Delta) + 1$. We point out that the edge ideal of a d -hypergraph is the special cases of a Stanley–Reisner ideal generated in a fixed degree d . We recover the simplicial complex Δ as $\text{Ind}(H)$, the independence complex of the hypergraph H .

As monomial ideals, the edge ideals I_H are endowed with a fine $\mathbb{Z}^n$ -grading coming from the $\mathbb{Z}^n$ -grading on S . We will sometimes abuse notation and use $\alpha \in \mathbb{Z}^n$ to denote both a monomial degree (i.e. a vector in $\mathbb{Z}^n$), as well as a monomial with that degree. For example, if $n = 6$ and if 235 is an edge in H , the corresponding monomial $x_2 x_3 x_5$ will be regarded as the vector $(011010) \in \mathbb{Z}^6$. In this paper we will be interested in finely graded resolutions of the S -module I_H . If

$$\mathcal{F} : 0 \rightarrow \bigoplus_{\alpha} S[-\alpha]^{\beta_{\ell, \alpha}} \rightarrow \cdots \rightarrow \bigoplus_{\alpha} S[-\alpha]^{\beta_{0, \alpha}} \rightarrow I_H \rightarrow 0$$

is a *minimal* free resolution of I_H , then for $i \in \mathbb{N}$ and $\alpha \in \mathbb{Z}^n$ the numbers $\beta_{i,\alpha}$ are independent of the resolution and are called these *finely graded Betti numbers* of I_H . The *coarsely graded* Betti numbers of I_H are given by $\beta_{i,j} = \sum_{|\alpha|=j} \beta_{i,\alpha}$. The number ℓ (the length of a minimal resolution) is called the *projective dimension* of I_H , which we will denote $\text{pdim}(I_H)$. One can check that $\text{pdim}(S/I_H) = \text{pdim}(I_H) + 1$, and by the Auslander-Buchsbaum formula, we have $\dim S - \text{depth}(S/I_H) = \text{pdim}(S/I_H)$. The ideal I_H is said to have a *d-linear* resolution if $\beta_{i,j} = 0$ whenever $j - i \neq d - 1$. A ring $R = S/I$ is *Cohen-Macaulay* if $\dim R = \text{depth } R$.

Remark 2.2 As is typical in this area, when dealing with edge ideals of graphs we will often say that H has a certain algebraic property (e.g., ‘ H has a linear resolution’), by which we mean that the edge ideal I_H has this property.

We will be interested in resolutions of the edge ideals I_H which are supported on geometric complexes. Given an oriented polyhedral complex X with monomial labels on the faces, one constructs $\mathcal{F}_X$, a free graded chain complex of S -modules which computes the cellular homology of X . Under certain circumstances (see Proposition 2.3) this algebraic complex is a resolution of the ideal generated by the monomials corresponding to the labels of the vertices. This notion of a *cellular resolution* was introduced by Bayer and Sturmfels in [4] and generalizes several well-known resolutions of monomial ideals including the Taylor resolution and the Hull resolution. We will often use the following criteria (taken from [18]) as a way to check whether a labeled complex supports a cellular resolution of the associated ideal. Here for any $\alpha \in \mathbb{Z}^n$ we use the notation $X_{\leq \alpha}$ to denote the subcomplex of X induced by those faces with monomial labels which divide α .

Proposition 2.3 *Suppose X is a complex with vertices labeled by monomials, and label the higher dimensional faces F with $\text{lcm}\{\ell(v) : v \in F\}$, the least common multiple of the labels $\ell(v)$ on the vertices v of F . Then the cellular free complex $\mathcal{F}_X$ is a cellular resolution if and only if $X_{\leq \alpha}$ is acyclic over k for all $\alpha \in \mathbb{Z}^n$, in which case it is a free resolution of the ideal generated by all monomials corresponding to the vertex labels. Furthermore, the resolution is minimal if whenever $F \subsetneq G$ is a strict inclusion of faces, the monomial labels on those faces differ.*

Also from [18] we have the following. For any $\alpha \in \mathbb{Z}^n$ we here use $X_{< \alpha}$ to denote the subcomplex of X given by all faces with labels strictly less than α .

Proposition 2.4 *Let X be a cellular resolution of an ideal I . For $i \geq 1$ and $\alpha \in \mathbb{Z}^n$ the finely graded Betti numbers of I are given by*

$$\beta_{i,\alpha} = H_{i-1}(X_{< \alpha}; k).$$

If a labeled complex X supports a *minimal* resolution of an ideal I , then for any $i \in \mathbb{N}$ and $\alpha \in \mathbb{Z}^n$ the Betti numbers $\beta_{i,\alpha}$ can be read off from the labeled complex directly. This follows from the fact that each i -face of X with label α contributes a term $S[-\alpha]$ in homological degree i to the complex $\mathcal{F}_X$.

3 The labeled complex and some properties

In this Section we associate a polyhedral complex X_H to any d -graph. In what follows, a simplex with vertex set V is denoted Δ_V . Also, for subsets $\sigma, \tau \subseteq V \subseteq \mathbb{Z}$, we use $\sigma < \tau$ to denote $s < t$ for all $s \in \sigma$ and $t \in \tau$.

Fig. 1 A graph H along with the complex X_H

Definition 3.1 Let H be a d -graph on a finite vertex set $V \subseteq \mathbb{Z}$. The polyhedral complex X_H is the subcomplex of the product

$$\prod_{i=1}^d \Delta_V$$

satisfying

- (1) the vertices of X_H are $v_1 \times v_2 \times \cdots \times v_d$, where $v_1 v_2 \cdots v_d$ is an edge of H ;
- (2) for $\sigma_i \subseteq V$, the cells $\sigma_1 \times \sigma_2 \times \cdots \times \sigma_d$ satisfy $\sigma_1 < \sigma_2 < \cdots < \sigma_d$.

An example of the construction is given in Fig. 1. From Definition 3.1 one can see that the dimension of a cell $F = \sigma_1 \times \sigma_2 \times \cdots \times \sigma_d$ in X_H is given by $n - d$, where $n = |\sigma_1 \cup \cdots \cup \sigma_d|$. Here X_H is defined to be a subcomplex of a rather large ambient space; in Sect. 5 we will see a more convenient embedding.

Remark 3.2 For any d -graph H , the faces of the complex X_H are naturally labeled by monomials. In particular, the vertices are labeled by monomials corresponding to the edges of H (i.e. the generators of I_H), and the higher dimensional faces $F = \sigma_1 \times \sigma_2 \times \cdots \times \sigma_d$ are labeled by

$$\prod_{i=1}^d \prod_{v \in \sigma_i} x_v,$$

which can be seen to equal the least common multiple of the monomial labels on the vertices of F . As we remarked above, these monomials will sometimes be considered as vectors in $\mathbb{Z}^n$.

Remark 3.3 Viewing H as a directed d -graph (with orientation on the edges given by the integer labels on the vertices), one can regard X_H as a ‘space of directed edges’ of H . If we let E denote the d -graph with vertex set $[d]$ consisting of a single edge $\{1, 2, \dots, d\}$, then $X_H = \text{Hom}(E, H)$, a space of directed graph homomorphisms from E to H analogous to the undirected Hom complexes of [3]. We will have more to say regarding this perspective in Sect. 7.

Remark 3.4 Our construction of X_H was motivated by certain resolutions of bipartite edge ideals introduced by Corso and Nagel in [7]. While preparing this manuscript we were made aware of the results from [8], where a process called *specialization* allowed the authors to deal with resolutions of certain (in general non-bipartite) 2-graphs. This construction was further generalized to edge ideals of d -partite d -hypergraphs by Nagel and Reiner in [19], where the notion of a ‘complex-of-boxes’ was introduced. Each cell of our complex X_H is in this sense a ‘box’ (a product of simplices) but we point out that our construction makes sense for any hypergraph H , without any specialization needed.

In dealing with the topology of X_H it will often be convenient to work with its face poset, where tools from poset topology can be applied to determine homotopy type, etc. Since the order complex of the face poset of a polyhedral complex coincides with its barycentric subdivision, we are not losing any topological information. We record this as a proposition.

Proposition 3.5 *Let H be a d -graph on a finite vertex set $V \subseteq \mathbb{Z}$. Define P_H as the poset of all maps*

$$\phi : \{1, 2, \dots, d\} \rightarrow 2^V \setminus \emptyset$$

such that

- (1) *if $v_i \in \phi(i)$ for all $1 \leq i \leq d$, then $v_1 v_2 \cdots v_d$ is an edge of H ;*
- (2) *if $1 \leq i < j \leq d$, then $\phi(i) < \phi(j)$;*

and $\phi \geq v$ if $\phi(i) \supseteq v(i)$ for all i . Then P_H is the face poset of X_H and the order complex of P_H is the barycentric subdivision of X_H .

Proof Clear. □

3.1 Recursive topology and a folding lemma

We next turn to establishing certain properties of the complexes X_H . Our ultimate goal is to show that X_H supports a cellular resolution whenever H is cointerval, but we collect the necessary topological results in this section.

Proposition 3.6 *Let $v \in V(H) \subseteq \mathbb{Z}$ be the smallest vertex of H , and let G be the v -layer of H . If v is in every edge of H , then X_H and X_G are isomorphic.*

Proof Use the cellular isomorphism $v \times \sigma_2 \times \sigma_3 \times \cdots \times \sigma_d \rightarrow \sigma_2 \times \sigma_3 \times \cdots \times \sigma_d$. □

Our next result shows that a certain deformation of a graph H (related to a neighborhood containment of vertices) induces a homotopy equivalence of the associated complexes. Readers familiar with Hom complexes will see the similarity to the ‘folds’ of graphs which are important in that context (see for instance [2]). If G is the l -layer of H , then we denote the edges corresponding to H in G with $l * H$.

Theorem 3.1 *Let H be a d -graph with vertex set $V \subseteq \mathbb{Z}$. Suppose $i < j$ are vertices of H , and let G be the j -layer, and G' the i -layer, of H . If G is a subgraph of G' then the complexes X_H and $X_{H \setminus j * G}$ are homotopy equivalent.*

Proof We construct a poset P'_H along with two monotone surjective poset maps

$$P_H \rightarrow P'_H \rightarrow P_{H \setminus j * G},$$

which will show that the order complexes of P_H , P'_H , and $P_{H \setminus j * G}$ are all homotopy equivalent. Let

$$P'_H = \{\phi \in P_H \mid \text{if } j \in \phi(1) \text{ then } i \in \phi(1)\}$$

be a subposet of P_H . Define the map

$$\xi : P_H \rightarrow P'_H$$

by $\xi(\phi) = \phi$ if $j \notin \phi(1)$, and by

$$\xi(\phi)(l) = \begin{cases} \phi(1) \cup \{i\} & \text{if } l = 1 \\ \phi(l) & \text{if } l \neq 1 \end{cases}$$

if $j \in \phi(1)$.

We need to check that the map ξ is well defined. If $j \notin \phi(1)$ then $\xi(\phi) = \phi \in P'_H$. For $j \in \phi(1)$ the conditions in Proposition 3.5 need to be checked. The addition of i to $\phi(1)$ satisfies the second condition since $i < j$, and also satisfies the first condition since the j -layer is a subgraph of the i -layer. It's clear that ξ is monotone and surjective.

Let

$$P''_H = \{\phi \in P'_H \mid j \notin \phi(1)\}$$

be a subposet of P'_H and note that P''_H and $P_{H \setminus j * G}$ are isomorphic as posets. Define the map

$$\xi' : P'_H \rightarrow P''_H$$

by $\xi'(\phi) = \phi$ if $j \notin \phi(1)$, and by

$$\xi'(\phi)(l) = \begin{cases} \phi(1) \setminus \{j\} & \text{if } l = 1 \\ \phi(l) & \text{if } l \neq 1 \end{cases}$$

if $j \in \phi(1)$. The only obstruction to the map ξ' being well defined is if $\xi'(\phi)(1) = \emptyset$ for some ϕ . But $\phi \in P'_H$, so every $\phi(1)$ that contains j also contains i . The map ξ' is clearly both surjective and monotone. $\square$

We point out that since the surjective map $P_H \rightarrow P_{H \setminus j * G}$ is a composition of monotone maps, the induced map on the order complex of the underlying posets (which is the barycentric subdivision of the complexes X_H and $X_{H \setminus j * G}$) is a collapsing and in particular a simple homotopy equivalence.

4 Cointerval graphs and their cellular resolutions

In this Section we establish our main result, namely that the complexes X_H support minimal cellular resolutions whenever H is a cointerval graph. We discuss some consequences regarding combinatorial interpretations of the Betti numbers of cointerval graphs.

4.1 Cointerval graphs

We begin with the definition of cointerval graphs.

Definition 4.1 The class of *cointerval* d -graphs is defined recursively as follows.

Any 1-graph is cointerval. For $d > 1$, a finite d -graph H with vertex set $V(H) \subseteq \mathbb{Z}$ is *cointerval* if

- (1) for every $i \in V(H)$ the i -layer of H is cointerval;
- (2) for every pair $i < j$ of vertices, the j -layer of H is a subgraph of the i -layer of H .

When $d = 2$ the class of cointerval graphs defined here can be seen to coincide with the well-studied *complements of interval graphs* of structural graph theory (hence the name). By definition, an *interval graph* is a 2-graph with vertices given by closed intervals I in the real line, and with adjacency $I \sim I'$ if and only if $I \cap I' \neq \emptyset$. The *complement* of a 2-graph H is

Fig. 2 A cointerval graph and its interval representation

a 2-graph H^C with the same vertex set as H with adjacency $v \sim v'$ in H^C if and only if v and v' do not form an edge in H (note that $v \neq v'$, all graphs considered here do not have loops). Given a cointerval 2-graph as in Definition 4.1, one can obtain an interval representation as follows. Without loss of generality, assume that $V(H) = [n]$. To each vertex $i \in [n]$ assign the interval $[\ell_i + 1, i]$, where ℓ_i is the largest neighbor of i in H such that $\ell_i < i$; assign $[1, i]$ to the vertex if there is no such ℓ_i (in particular assign $[1, 1]$ to the vertex 1).

Conversely, suppose H is represented as the complement of an interval graph (so the vertices are given by intervals in the real line, with disjoint interval determining adjacency). Order the intervals according to the rightmost endpoint, so that $[a, b] < [a', b']$ if $b < b'$. One can check that this determines a cointerval graph as in Definition 4.1.

Cointerval d -graphs include the class of ‘strongly stable’ hypergraphs, considered for instance in an algebraic context in [19]. By definition a *strongly stable* d -hypergraph H on a vertex set $[n]$ has the property that whenever E is an edge in H with $i \in E$, then $E \setminus \{i\} \cup \{i-1\}$ is also an edge (whenever that set has the proper size). These are also called ‘shifted’ hypergraphs, or ‘square-free order ideals’ in the Gale order on d -subsets. When $d = 2$, strongly stable 2-graphs correspond to the well-known class of ‘threshold’ graphs. Note that if in Definition 4.1 we required that 1-graphs had the property that whenever $i \in E(H)$ then $j \in E(H)$ for all $j < i$ we would recover the class of strongly stable hypergraphs. The cointerval 2-graph depicted in Fig. 2 is also a threshold graph if one switches the labels 4 and 5.

Lemma 4.2 *Let H be a non-empty cointerval d -graph with $d > 0$. Let $j \in V(H)$ be the maximum vertex in H such that the j -layer (call it G) is non-empty. Then the d -graph $H \setminus j * G$ is cointerval.*

Proof We check the two conditions in Definition 4.1. The first requires that every layer of $H \setminus j * G$ is a cointerval graph, and this is satisfied since the layers of $H \setminus j * G$ are all layers of H . The second condition asks that for every pair $i < k$ of vertices, the k -layer of $H \setminus j * G$ should be a subgraph of the i -layer of $H \setminus j * G$. But again this is satisfied since the relevant (nonempty) layers are also layers in the original cointerval graph H . $\square$

In light of Proposition 2.3, the construction of our cellular resolutions will rely on the fact that our class of graphs is closed under taking induced subgraphs. Our next results shows that this is indeed the case for cointerval graphs.

Proposition 4.3 *Any induced subgraph of a cointerval d -graph is a cointerval d -graph.*

Proof We prove that if the d -graph H is cointerval, then $H[V]$ is cointerval.

The proof is by induction on d . Any 1-graph is cointerval, and hence assume $d > 1$. We need to check the conditions in Definition 4.1.

- (1) Let G be the i -layer of $H[V]$, and let G' be the i -layer of H . Then by definition, $G = G'[V]$. The $(d-1)$ -graph G' is cointerval since it is a layer of H . By induction every induced subgraph of G' is cointerval.

- (2) If $i < j$ are vertices of $H[V]$, then the j -layer of $H[V]$ is a subgraph of the i -layer of $H[V]$, since the j -layer of H is a subgraph of the i -layer of H . $\square$

4.2 Minimal cellular resolution of cointerval graphs

In this Section we establish our main result, Theorem 4.1. For the proof we will need the following observation.

Lemma 4.4 *If H is a non-empty cointerval d -graph with $d > 0$, then X_H is contractible.*

Proof We suppose H is a d -graph and t is the number of non-empty layers of H . The proof is by induction over d and t .

If $d = 1$ then X_H is a simplex and contractible. Now assume that $d > 1$.

If $t = 1$ and the i -layer G is the only non-empty one, then X_H and X_G are isomorphic, and X_H is contractible by induction on d . Now assume that $t > 1$.

Let j be the maximal number such that the j -layer G of H is non-empty, and let $i < j$ be such that the i -layer of H is non-empty. The d -graph H is cointerval, and hence by definition the j -layer is a subgraph of the i -layer. By Theorem 3.1 the space X_H is homotopy equivalent to $X_{H \setminus j * G}$. By Lemma 4.2 the d -graph $H \setminus j * G$ is also cointerval, but with $t - 1$ non-empty layers. By induction $X_{H \setminus j * G}$ is contractible, and hence so is X_H . $\square$

With these tools in place we can state and prove our main result. For this recall from Remark 3.2 that X_H is a labeled polyhedral complex with vertex labels corresponding to the monomial generators of the ideal I_H .

Theorem 4.1 *Let H be a cointerval d -graph. Then the polyhedral complex X_H supports a minimal cellular resolution of the edge ideal I_H .*

In particular, for $i \geq 0$ and $\alpha \subseteq V(H)$ the graded Betti numbers are given by $\beta_{i,\alpha}(I_H) = 0$ if $i \neq |\alpha| - d - 1$, and

$$\beta_{|\alpha|-d-1,\alpha}(I_H) = |\{\sigma \in X_{H[\alpha]} \mid \dim \sigma = |\alpha| - d - 1\}|.$$

In other words, the (i, α) -Betti numbers are given by the number of faces of dimension i in X_H with monomial label α .

Proof We will apply the conditions from Proposition 2.3. In particular let $n = |V(H)|$, and for any $\alpha \in \mathbb{Z}^n$ we consider the complex $(X_H)_{\leq \alpha}$. All labels are square-free, so it is enough to restrict to $\alpha \in \{0, 1\}^n$. For any such α , the complex $(X_H)_{\leq \alpha}$ is given by the complex $X_{H[V]}$ where $V = \{v \in V(H) \mid \alpha_v = 1\}$. We are assuming that H is cointerval, and hence by Proposition 4.3 so is $H[V]$. By Lemma 4.4 the complex $X_{H[V]}$ is contractible, and hence by Proposition 2.3 the complex X_H supports a cellular resolution of I_H .

We note that if $\sigma \subsetneq \tau$ is a strict containment of faces then the monomial labels on those faces differ since in particular the dimensions of the faces can be read off by the monomial label. Once again, from Proposition 2.3 we conclude that the resolution is minimal. $\square$

Example 4.5 In Fig. 2 we see a cointerval 2-graph along with its interval representation. The minimal cellular resolution X_H is depicted in Fig. 1. This graph is also the complement of a threshold graph and also appears in [8].

In independent work, Nagel and Reiner [19] construct cellular resolutions of the edge ideals of strongly stable hypergraphs (among other non-square free classes). As we mentioned above, the cointerval d -graphs form a strictly larger class than strongly stable graphs, and

Fig. 3 A cointerval graph which is not threshold (under any labeling), with its interval representation

Fig. 4 The cellular resolution of the graph in Fig. 3

our construction of X_H is an example of the ‘complex of boxes’ developed in [19]. For the case $d = 2$ it is known that strongly stable 2-graphs correspond to *threshold* graphs. The complement of a threshold graph is threshold, and threshold graphs are interval graphs, and hence our results are more general already in the case $d = 2$. In particular, there exist interval graphs which are not threshold, as the next example illustrates. For further examples of three graphs which are cointerval but not strongly stable we refer the reader to the Sect. 6.1.

Example 4.6 An example of a graph which is cointerval but not threshold is depicted in Fig. 3, along with its interval representation (threshold graphs have the property that all induced subgraphs have either a dominating or isolated vertex; here the subgraph induced on $\{2, 3, 4, 5\}$ does not have that property). This graph also appears in [8].

Its cellular resolution is depicted in Fig. 4, with seven 0-cells, eleven 1-cells, six 2-cells, and a single 3-cell. We have drawn the complex in such a way to emphasize the fact that it is a subcomplex of the subdivision depicted in Fig. 7 (details regarding this are in the next section).

Corollary 4.7 *If H is a cointerval d -graph, then the edge ideal I_H has a d -linear resolution.*

Corollary 4.8 *Let H be a cointerval d -graph, and let I_H^* denote the Alexander dual of I_H . Then the ring S/I_H^* is Cohen-Macaulay.*

Proof This follows from a result of Eagon and Reiner from [12]. $\square$

The Alexander dual of an edge ideal of a graph H is often called the *vertex cover algebra* of H . The fact that the Alexander duals of cointerval hypergraphs are Cohen-Macaulay has potential applications to face counting of simplicial complexes in the context of algebraic shifting. Algebraic shifting is a process which associates to an arbitrary simplicial complex Δ a *shifted* (strongly stable) complex Δ' , preserving much of the combinatorial data of Δ .

(see [16]). Shifted complexes are known to be vertex-decomposable (see [5]) and hence Cohen-Macaulay, and via Stanley–Reisner theory one can conclude certain things about its f -vector (e.g. non-negativity of the h -vector). The Alexander dual of the independence complex of a hypergraph is a complex whose facets are given by the complements of edges, and we have seen that if H is a cointerval graph this complex is already Cohen-Macaulay. Hence in general one will not need to ‘shift as far’ to obtain such a complex.

Furthermore, one can ask the question: When are the Alexander duals of cointerval hypergraphs shellable or vertex decomposable?

As was pointed out in [19], the cellular complex X_H leads to an easy combinatorial interpretation of the Betti numbers defined in Theorem 4.1.

Corollary 4.9 *Let H be a cointerval d -graph with vertex set $\{1, 2, \dots, n\}$. The hyper Ferrers diagram F_H of H is defined to be*

$$\{(v_1, v_2, \dots, v_d) \in \mathbb{N}^d \mid v_1 < v_2 < \dots < v_d \text{ and } v_1 v_2 \dots v_d \in E(H)\}.$$

A cube in $\mathbb{N}^d$ is a subset of the type $C = S_1 \times S_2 \times \dots \times S_d$ where all $S_i \subset \mathbb{N}$. The coordinates of C is the subset of $\mathbb{Z}$ given by $S_1 \cup S_2 \cup \dots \cup S_d$.

If V is a subset of $\{1, 2, \dots, n\}$ then $\beta_{|V|-d-1, V}(I_H)$ equals the number of cubes in F_H with V as coordinates.

Proof Use the definition of X_H and Theorem 4.1. □

5 Mixed subdivisions and a nice embedding

One particularly nice feature of the complexes X_H is that we can give explicit geometric embeddings, without resorting to the high-dimensional ambient space involved in Definition 3.1. It turns out that for the case of complete graphs K_n^d the complex $X_{K_n^d}$ can be realized as a particular *mixed subdivision* of a dilated simplex (definitions below). As any graph is a subgraph of some complete graph, the general complexes X_H are then subcomplexes of these subdivisions. This leads to useful geometric representations of our resolutions, and in fact it was these embeddings that led us to the construction of X_H described in the previous section.

5.1 Mixed subdivisions and the staircase triangulation

We begin with a brief review of some basic notions of polyhedral geometry (see for example [23]). In this section we let $e_1, \dots, e_{k+1}$ denote the standard basis vectors in $\mathbb{R}^{k+1}$ and let $\Delta_k = \text{conv}\{e_1, \dots, e_{k+1}\}$ denote the standard k -simplex. We fix $d \leq n$ and let $m = n - d$. We wish to realize $X_{K_n^d}$ as a certain mixed subdivision of $d\Delta_m$, the d -fold Minkowski sum of an m -simplex.

Recall that if $P_1, \dots, P_j$ are polytopes in $\mathbb{R}^{m+1}$, then the *Minkowski sum* is defined to be the polytope

$$P_1 + \dots + P_j := \{x_1 + \dots + x_j : x_i \in P_i\} \subseteq \mathbb{R}^{m+1}.$$

Here we will restrict ourselves to the case of $d\Delta_m$, the d -fold Minkowski sum of m -simplices.

To describe our desired subdivisions, we follow [1] for some definitions and notation. We define a *fine mixed cell* $X \subseteq d\Delta_m$ to be a Minkowski sum $B_1 + \dots + B_d$, where the B_i are

Fig. 5 Subdivision of $2\Delta_3$, for $n = 5$ and $d = 2$. The vertices are labeled by their realizations as Minkowski sums, so that for instance $0110 = e_2 + e_3$

Fig. 6 Subdivision of $3\Delta_2$, for $n = 5$ and $d = 3$

faces of Δ_m which lie in independent affine subspaces, and whose dimensions add up to m . A *fine mixed subdivision* of $d\Delta_m$ is then a subdivision of $d\Delta_m$ consisting of fine mixed cells.

Now we fix integers d and n , let $m = n - d$, and as above let K_n^d denote the complete d -graph on n vertices. We will construct a mixed subdivision $X_{d,n}$ of $d\Delta_m$ whose 0-dimensional cells naturally correspond to the vertices of our original complex $X_{K_n^d}$. For this, it will be convenient to use the following auxiliary construction. As above we use $\{e_1, \dots, e_{m+1}\}$ to denote the vertices of the simplex Δ_m and consider fine mixed cells of the following kind. Given a sequence $(b_1, b_2, \dots, b_{d+1})$ satisfying $1 = b_1 \leq b_2 < \dots < b_d \leq b_{d+1} = m + 1$ we let $B_i := \{e_{b_i}, e_{b_i+1}, \dots, e_{b_{i+1}}\}$ for $1 \leq i \leq d$, and use $(b_1, b_2, \dots, b_{d+1})$ to denote the corresponding (fine) mixed cell $B_1 + B_2 + \dots + B_d$ of $d\Delta_m$.

Example 5.1 For $n = 5, d = 2$, we have $m = 3$ so that our complex will be a certain mixed subdivision of $\Delta_3 + \Delta_3$. The maximal cells of the subdivision are encoded by the sequences $(1, 1, 4), (1, 2, 4), (1, 3, 4)$, and $(1, 4, 4)$, and each of these correspond to a fine mixed cell depicted in Fig. 5.

Example 5.2 For $n = 5, d = 3$, we have $m = 2$ so that our complex will be a mixed subdivision of $\Delta_2 + \Delta_2 + \Delta_2$. The maximal cells of the subdivision are encoded by the sequences $(1, 1, 1, 3), (1, 1, 2, 3), (1, 1, 3, 3), (1, 2, 2, 3), (1, 2, 3, 3)$, and $(1, 3, 3, 3)$, which correspond to the six cells in the subdivision pictured in Fig. 6.

We claim that the collection of fine mixed cells $\{(b_1, b_2, \dots, b_{d+1}) : 1 = b_1 \leq b_2 < \dots < b_d \leq b_{d+1} = m + 1\}$ forms a fine mixed subdivision of the complex $d\Delta_m$. One way to see this is to employ the *Cayley trick*, which (in this special case) gives a bijection between the set of mixed subdivisions of $d\Delta_m$ and the *triangulations* of the product of simplices $\Delta_{d-1} \times \Delta_m$. Under this bijection the mixed subdivision that we are describing here can

Fig. 7 The complex $X_{2,5}$, with vertex labels given by the edges of K_5^2

be seen to correspond to the ‘staircase’ triangulation of $\Delta_{d-1} \times \Delta_m$. We omit the details here, and refer to [1, 11] for further discussion regarding the Cayley trick and the staircase triangulation. We record this observation here.

Lemma 5.3 *The collection of fine mixed cells $\{(b_1, b_2, \dots, b_m) : 1 = b_1 \leq b_2 < \dots < b_{m-1} \leq b_m = m + 1\}$ forms a fine mixed subdivision of the complex $d\Delta_m$.*

For $m = n - d$, we use $X_{d,n}$ to denote this mixed subdivision.

5.2 The complexes X_H as mixed subdivisions

As Minkowski sums of the underlying simplex, the vertices of the mixed subdivision $X_{d,n}$ described above are labeled by all monomials of degree d among the variables $\{x_1, \dots, x_{m+1}\}$, where for instance the vertex $e_1 + e_3 + e_3$ is labeled $x_1 x_3^2$ (see Fig. 5). In fact these complexes support cellular resolutions of the d -th power of the maximal ideal $(x_1, \dots, x_{m+1})^d$ (see [11] for a proof of this as well as further discussion). Here we are interested in the associated square-free ideal and for this we relabel our complex according to the following well-known bijection between d -multisubsets of $[m + 1]$ with d -subsets of $[m + d]$.

Each monomial of degree d on the vertices $\{x_1, \dots, x_{m+1}\}$ can be thought of as a vector $\alpha \in \mathbb{N}^{m+1}$ with nonnegative coordinates α_i such $\sum \alpha_i = d$. The nonzero entries of this vector determine a multiset $\{i_1, i_2, \dots, i_d\}$ with $i_1 \leq i_2 \leq \dots \leq i_d$, where the exponent of x_j gives the number of occurrences of j . To each multiset of this kind we associate a set according to

$$\{i_1, \dots, i_d\} \mapsto \{i_1, i_2 + 1, \dots, i_d + d - 1\}.$$

The resulting set has d (distinct) nonzero elements of maximum size $m + 1 + d - 1 = n$, and hence this assignment labels the vertices of $X_{d,m}$ with square-free monomials corresponding to the edges of the complete hypergraph K_n^d . We note that this relabeling is equivalent to the ‘polarizations’ described by Nagel and Reiner in [19].

Example 5.4 In Fig. 7 we see the complex $X_{2,5}$ from Example 5.1, with vertex labels given by the edges of the graph K_5^2 .

Now, if H is a d -graph with vertex set $[n]$, we obtain a subcomplex $X_{d,n}[H]$ of the mixed subdivision $X_{d,n}$ by considering the subcomplex induced by those vertices corresponding to the edges of H . From this we obtain our other main result.

Fig. 8 A fine mixed subdivision Y , and a disconnected downset $Y_{\leq 1245}$

Theorem 5.1 For any d -graph H with vertex set $V(H) = [n]$, the complex $X_{d,n}[H]$ described above is isomorphic as a cell complex to X_H . In particular the complex X_H can be realized as a subcomplex of a mixed subdivision of the dilated simplex $d\Delta_{n-d}$.

Proof We have seen that the vertices of both complexes can be identified. One checks that this induces a polyhedral isomorphism $X_{d,n}[H] \rightarrow X_H$ which maps a mixed cell $(b_1, b_2, \dots, b_{d+1})$ to $\sigma_1 \times \sigma_2 \times \dots \times \sigma_d$, where $\sigma_i = \{b_i + i - 1, b_i + i, \dots, b_{i+1} + i - 1\}$, for $1 \leq i \leq d$. $\square$

Corollary 5.5 If H is a cointerval d -graph, then the edge ideal I_H has a minimal cellular resolution supported on a subcomplex of a mixed subdivision of a dilated simplex.

Remark 5.6 In [11] it is shown that any regular fine mixed subdivision of the dilated simplex $d\Delta_m$ supports a minimal cellular resolution of the ideal $\langle x_1, \dots, x_{m+1} \rangle^d$. Hence it is a natural question to ask whether any fine mixed subdivision can be used in the construction of resolutions of edge ideals of hypergraphs. In fact this is not the case, as the example in Fig. 8 illustrates. The particularly well-behaved properties of the staircase subdivision are really necessary here.

6 Constructing resolutions of more general graphs

Not all hypergraphs are cointerval (some examples are below) and in this section we discuss methods for building cellular resolutions for more general d -graphs. These resolutions will probably not be minimal, but they could be very good. The decomposition of ordinary 2-graphs into threshold, interval, or chordal graphs is an active area in theoretical computer science with deep connections to measure theory [6, 14].

The basic idea will be to decompose an arbitrary d -graph H as a union of cointerval d -graphs, and to glue together the associated complexes considered above. Even though basic large cellular resolutions as the Taylor resolution and its basic modifications can be constructed like this, the main point is that the graphs can, and should, be decomposed intelligently into cointerval graphs before their minimal resolutions are composed.

Theorem 6.1 Let $H_1, H_2, \dots, H_n$ be d -graphs on the same vertex set W . For all $i = 1, 2, \dots, n$ assume that there is a cellular resolution of I_{H_i} supported by the polyhedral complex X_i with vertices labeled by square-free ℓ_i . Assume that the higher dimensional cells are labeled by the least common multiple of their vertices.

If $H = H_1 \cup H_2 \cup \dots \cup H_n$, $X = X_1 * X_2 * \dots * X_n$, and $\ell(\sigma) = \text{lcm}(\ell_1(\sigma), \ell_2(\sigma), \dots, \ell_n(\sigma))$, then the complex X with labels ℓ supports a cellular resolution of I_H .

Proof Let the square-free monomial α support an edge of H . We will prove that $X_{\leq \alpha}$ is acyclic.

$$\begin{aligned} X_{\leq \alpha} &= \{\sigma \in X \mid \ell(\sigma) \leq \alpha\} \\ &= \{\sigma_1 * \cdots * \sigma_n \in X \mid \sigma_1 \in X_1, \dots, \sigma_n \in X_n, \ell(\sigma_1 * \cdots * \sigma_n) \leq \alpha\} \\ &= \{\sigma_1 * \cdots * \sigma_n \in X \mid \sigma_1 \in X_1, \ell_1(\sigma_1) \leq \alpha, \dots, \sigma_n \in X_n, \ell_n(\sigma_n) \leq \alpha\} \\ &= \{\sigma_1 * \cdots * \sigma_n \in X \mid \sigma_1 \in (X_1)_{\leq \alpha}, \dots, \sigma_n \in (X_n)_{\leq \alpha}\} \\ &= (X_1)_{\leq \alpha} * \cdots * (X_n)_{\leq \alpha} \end{aligned}$$

At least one of the $(X_i)_{\leq \alpha}$ is non-empty, and thus acyclic. Hence we conclude that $X_{\leq \alpha}$ is also acyclic. $\square$

The most basic example of Theorem 6.1 recovers what is known as the ‘Taylor resolution’ of I_H . For this note that a hypergraph H_i consisting of a single edge has a cellular resolution supported by a point. The join of $|E(H)|$ points is a $(|E(H)| - 1)$ -dimensional simplex supporting the resolution of I_H .

Definition 6.1 The *linear width* of a d -graph H , denoted $\omega_{\text{lin}}(H)$, is the smallest number k such that $H = H_1 \cup H_2 \cup \cdots \cup H_k$, with each H_i a cointerval d -graph.

The linear width is well defined and $\omega_{\text{lin}}(H) \leq |E(H)|$ since any hypergraph with one edge is a cointerval hypergraph. We have chosen the name *linear width* since for 2-graphs it is closely related to the path-width [20] and band-width [6, 14], and if the linear width of H is one, the ideal I_H has a linear resolution.

Bourgain [6] and Feige [14] have developed rather general theories regarding modifying combinatorial objects to obtain ‘perfect elimination orders’. If these ideas apply to decomposing hypergraphs into cointerval hypergraphs, then we expect the linear width to grow rather slowly. In fact we conjecture that for any a fixed d there is a constant C_d such that $\omega_{\text{lin}}(H) < C_d n$ for any d -graph on n vertices. A solution to this conjecture would give new general bounds on graded Betti numbers of hypergraph edge ideals.

In this paper all monomial ideals are generated in a fixed degree d , but there is a generalization of the previous Theorem to the corresponding non-uniform hypergraph case, since we never used that the edges are of the same order in the proof.

6.1 A case study: three graphs on at most five vertices

In this section we study (unlabeled) three graphs on at most five vertices. There are 34 of them. With an exhaustive computer search we find that 26 of these are cointerval under suitable labelings (the first 26 in the list below), 10 of which are *not* strongly stable (graphs 7, 10, 11, 17, 19, 21, 22, 23, 25, 26). The number of strongly stable graphs (16) is verified by the enumerative results presented in Theorem 3 of [17].

6.1.1 The cointerval three graphs on five vertices, with the vertices in the right order

In Fig. 9 we see minimal cellular resolutions of the cointerval three graphs on at most five vertices (Graph 1 is the empty graph). The graphs themselves can of course be recovered by recording the labels on the 0-cells.

6.1.2 The non-cointerval three graphs on five vertices

Next we turn to decompositions of non-cointerval graphs. Each of the graphs 27–34 are presented in Fig. 10 with a cellular resolution constructed as a join of minimal resolutions.

Fig. 9 Cellular minimal resolutions for cointerval graphs 2–26

We point out that using only strongly stable subgraphs in a decomposition, graph 28 needs to be decomposed into three graphs.

7 Further questions

7.1 A larger class of graphs

As we have seen, for any d -graph H the complex X_H that we construct has the property that the dimension of a face F is given by $i - d$, where i is the total degree of the monomial label on F . Hence whenever X_H supports a resolution of I_H , it is d -linear. It is a well-known result (see [15]) that a 2-graph H has a 2-linear resolution if and only if H is the complement of a *chordal* graph. Interval graphs (which correspond to cointerval 2-graphs) are a proper subset of chordal graphs, and in particular there exist graphs which are chordal but not interval. These include the graphs in Fig. 11.

A natural question to ask is whether our complexes X_H can be used to obtain resolutions of a more general class of graphs. It turns out that our construction will *not* work for the class of complements of chordal graphs. In fact, if G is taken to be the complement of the first graph in Fig. 11 (which happens to be isomorphic to the second graph in that list), one can check that no labeling of the vertices with $\{1, \dots, 6\}$ induces a complex X_G which supports

Fig. 10 Cellular resolutions for graphs 27–34 using Theorem 6.1

Fig. 11 Three graphs which are chordal but not interval

a resolution. However, it is still an open question to determine the largest class of graphs for which our construction (or the complex-of-boxes construction from [19]) do apply. We note that the classes recently defined by Emtander [13] and Woodroffe [22] could be good candidates.

7.2 Functoriality and more general complexes

Suppose that G and H are graphs on vertex sets $[m]$ and $[n]$, respectively. One can check that if $f : G \rightarrow H$ is a directed graph homomorphism then there is an induced polyhedral map $f_* : X_G \rightarrow X_H$. Furthermore, the map f gives rise to a map $f : S_m \rightarrow S_n$, where $S_j := k[x_1, \dots, x_j]$, and hence gives S_n (and in turn I_H) the structure of an S_m -module. The polyhedral map f_* then gives rise to a map of chain complexes of S_m -chain complexes. This functoriality then gives rise to the possibility of applications, where for instance algebraic invariants such as Betti numbers can be used to produce obstructions to the existence of graph

homomorphisms, in the spirit of equivariant topological obstructions in the context of Hom complexes.

As we have seen, when H is a cointerval hypergraph the complexes X_H are contractible. It would be interesting to determine which classes of spaces one can obtain as X_H for various choices of hypergraphs H . For instance, in his work on undirected Hom complexes Csorba [9] has shown that arbitrary free $\mathbb{Z}_2$ -spaces can be approximated up to $\mathbb{Z}_2$ -homotopy type by (undirected) complexes $\text{Hom}(K_2, G)$, for a suitably chosen graph G .

As we discussed in Sect. 3 the complexes X_H can be viewed as special case of a more general complex of homomorphisms between directed graphs. For this, suppose T and H are graphs with vertex sets $[m]$ and $[n]$, respectively. We can define a complex $X_{T,H} = \text{Hom}(T, H)$ parameterizing directed homomorphisms which, as above, gives rise to a chain complex. In this general case, the entries of the complex should no longer be considered as free modules over the polynomial ring, but instead as modules over the DG-algebra $\text{Hom}(E, T)$ (where, as above, E is the directed m -edge). We see further development in this area as a subject for future work.

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