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About de Smit's question on flatness

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Abstract Let $\varphi : A \to B$ be a flat morphism of Artin local rings with the same embedding dimension. Denote by $\mathfrak{m}_A$ the maximal ideal of A. Bart de Smit asked whether any finite B-module that is A-flat is B-flat. We prove the conjecture in embedding dimension one or two. In embedding dimension n, we prove the conjecture under an additional assumption on $B/\mathfrak{m}_A B$.

Keywords Flatness · Artin rings · Complete intersections · Gorenstein rings

1 Introduction

Notations and conventions All rings are commutative rings with unit. If A is a local ring, its maximal ideal and residue field are always denoted by $\mathfrak{m}_A$ and $\kappa(A)$. If M is an A-module, the minimal number of generators of M is denoted by $\mu_A(M)$. We will say that an element of M is a minimal generator of M if it is part of a minimal system of generators for M. The *embedding dimension* of A is the dimension of the $\kappa(A)$-vector space $\mathfrak{m}_A/\mathfrak{m}_A^2$. It is denoted by edim (A). Note that edim (A) is equal to $\mu_A(\mathfrak{m}_A)$. A morphism $\varphi : A \to B$ of local rings is a *complete intersection morphism* (resp. a *Gorenstein morphism*) if $B/\mathfrak{m}_A B$ is a complete intersection ring (resp. a Gorenstein ring). (Unlike [4, IV 19.3.6], a complete intersection morphism is not necessarily flat. Note also that a Gorenstein or complete intersection morphism between local rings is local by definition, since Gorenstein or complete intersection local rings are not null.)

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Statement of the conjecture and results Let us first consider a very easy example. Take $A = k[x]/(x^2)$ and $B = A[u]/(u^2 - x) \simeq k[u]/(u^4)$ (where k is a field). Let M be a finite B-module, that is free over A. Let $(e_1, \ldots, e_n)$ be an A-basis of M and let U be the matrix of multiplication by u. We can write $U = U_0 + U_1 x$ where U_0 and U_1 have coefficients in k. Then the relation $U^2 = x.\mathrm{Id}$ yields

$$\begin{cases} U_0^2 = 0 \\ U_0 U_1 + U_1 U_0 = \mathrm{Id}. \end{cases}$$

Using these relations, it is an easy exercise in linear algebra over a field to verify that n is necessarily even, and that M is actually free over B.

In the 1990s, Bart de Smit raised a question: could the following generalization be true?

Conjecture 1.1 (de Smit) Let $\varphi : A \to B$ be a flat morphism of Artin local rings with the same embedding dimension. Then every B-module M that is flat and of finite type over A, is flat over B.

Remark 1.2 Note that, since the maximal ideal of an Artin local ring is nilpotent, a morphism between to such rings is necessarily local. Recall also the following from [2, chap. II, Sect. 3, n°2, corollary 2 of proposition 5]. Let M be a module over a Noetherian local ring A, and assume that M is of finite type or that A is Artin. Then it is equivalent to say that M is flat, projective, or free. The case where M is of finite type is well-known, the case where A is Artin follows from 2.1 (iv).

To explain the results contained in our paper, we propose the following definition.

Definition 1.3 Let $\varphi : A \to B$ be a local morphism of Noetherian local rings. We say that φ is a *de Smit* morphism if every B-module M that is flat and of finite type over A is flat over B.

Our first observation is that a morphism satisfying the hypothesis of de Smit's conjecture is necessarily a *complete intersection* morphism (see 2.5), hence a Gorenstein morphism. We introduce below another class of morphisms, which we call *nice* morphisms. The precise definition is given in 4.5. We then prove the following results:

Theorem 1.4 (see 4.7) *A nice and Gorenstein morphism of Artin local rings is de Smit.*

Theorem 1.5 (see 4.10) *Let $\varphi : A \to B$ be a complete intersection morphism of Noetherian local rings. Assume that* $\mathrm{edim}\,(B) < \mathrm{edim}\,(A)$. *Then the only B-module* (*of finite type over A if A is not Artin*) *that is flat over A is zero. In particular φ is nice and de Smit.*

Theorem 1.6 (see 5.1, 5.11 for (i), see 5.12 for (ii) and 5.13 for (iii)) *Let $\varphi : A \to B$ be a complete intersection morphism of Artin local rings. Assume that A and B have same embedding dimension n and that one of the following holds.*

- (i) *The embedding dimension n is* 0, 1 *or* 2.
- (ii) $n = 3$ *and there is a minimal generator u of $\mathfrak{m}_B$ that divides* (*in B*) *an element of* $\mathfrak{m}_A B \backslash \mathfrak{m}_B \mathfrak{m}_A B$ (*i.e. a minimal generator of $\mathfrak{m}_A B$, for instance the image of a minimal generator of $\mathfrak{m}_A$ if φ is flat, see* 2.5).
- (iii) *Let $B_0 = B/\mathfrak{m}_A B$. For some minimal system of generators $u = (u_1, \ldots, u_l)$ of $\mathfrak{m}_{B_0}$, there is an upper triangular u-Wiebe matrix for B_0* (*the definition of Wiebe matrices is recalled in* 5.2).

Then φ is nice (*hence de Smit*).

Remark 1.7 The additional assumption given in embedding dimension 3 (in (ii)) can be replaced by any of the equivalent conditions given in Lemma 5.6. The authors initially hoped that this condition would always hold. This is unfortunately not the case as we can see in Example 6.1.3. In practice, this condition is used to reduce the statement to embedding dimension $n-1$ (see the Lemma 5.7). Actually, the condition given in (iii) more or less means that the condition discussed above "recursively holds". There is a Maple file available on the web page of the first author for testing whether this condition does, or does not, hold (in the case $n=3$, φ being given with explicit equations, and with the assumption that A and B have residue field equal to $\mathbb{C}$).

Example 1.8 If in de Smit's Conjecture 1.1 we drop the condition $\mathrm{edim}\,(A) = \mathrm{edim}\,(B)$, then there are trivial counter-examples. For instance let A be a field and let $B = A[\varepsilon]/(\varepsilon^2)$. Then the ideal (ε) of B is obviously flat over A, and not flat over B.

The text is organised as follows: the first paragraph is devoted to basic consequences of Nakayama's lemma and to basic facts on embedding dimension and complete intersection rings of dimension zero (Sect. 2). There is nothing original in this section, but we remark that the hypotheses of de Smit's conjecture imply that B is a relative complete intersection of dimension zero over A. The discussion of de Smit's conjecture is carried out in the following. First (Sect. 3) we introduce the notion of weakly torsion-free modules and give an equivalent condition for flatness over B in these terms (see 3.7 and 3.9). Then (Sect. 4) we define and study the flatness in first order over a Noetherian local ring A. It is weaker than flatness, and equivalent to flatness if $\mathfrak{m}_A^2 = 0$ (see 4.3). We prove the conjecture for embedding dimension less than two and the other results of 1.6 in Sect. 5. At last, we discuss some examples, some generalizations of de Smit's conjecture and possible applications (Sect. 6).

2 Flatness and complete intersections

We prove in this section that a morphism that satisfies the hypothesis of the Conjecture 1.1 is always a complete intersection morphism. Let us first recall briefly the following useful lemma and give some consequences (note that M is not necessarily of finite type).

Lemma 2.1 [2, chap. II, Sect. 3, n°2, prop. 4 and 5] *Let A be a ring, $\mathfrak{a}$ an ideal that is contained in the radical of A, and M an A-module. Assume that $\mathfrak{a}$ is nilpotent or that M is of finite type.*

(i) *If $M = \mathfrak{a}M$ then $M = 0$.*
(ii) *Let N be a submodule of M. Then $M = N + \mathfrak{a}M$ implies $M = N$.*
(iii) *Let $(x_i)_{i\in I}$ be a family of elements of M, the images of which generate the $A/\mathfrak{a}$-module $M/\mathfrak{a}M$. Then the x_i's generate M.*
(iv) *Let $(x_i)_{i\in I}$ be a family of elements of M, the images of which form a basis of the $A/\mathfrak{a}$-module $M/\mathfrak{a}M$. Assume that M is flat over A. Then the x_i's form a basis of M.*

Remark 2.2 In the whole paper, the finiteness assumption on the module M is only used in Nakayama's lemma. Thus it is useless if A is Artin, and in this case we can remove the sentence "of finite type over A" in the Definition 1.3 of de Smit morphisms.

Corollary 2.3 *Let $\varphi : A \to B$ be a local morphism of Noetherian local rings. Let M be a B-module, flat over A. Assume that M is of finite type over B or that A is Artin. Then the following conditions are equivalent.*

(i) *The module M is flat over B.*
(ii) *The module $M/\mathfrak{m}_A M$ is flat over $B/\mathfrak{m}_A B$.*

Proof (i) $\Rightarrow$ (ii) is obvious, since $M/\mathfrak{m}_A M$ is isomorphic to $M \otimes_B (B/\mathfrak{m}_A B)$. Conversely, let $(f_i)_{i\in I}$ be a family of elements of M, the images of which in $M/\mathfrak{m}_A M$ form a basis of $M/\mathfrak{m}_A M$ over $B/\mathfrak{m}_A B$. We will prove that the f_i's form a basis of M over B. We already know that they generate M (use 2.1 with the ideal $\mathfrak{a} = \mathfrak{m}_A B$). Let us consider the short exact sequence

$$0 \longrightarrow K \longrightarrow B^{(I)} \xrightarrow{\psi} M \longrightarrow 0$$

where ψ is given by $(\lambda_i) \mapsto \sum \lambda_i f_i$ and K is its kernel. Since M is flat over A, this sequence remains exact after applying the functor $. \otimes_A (A/\mathfrak{m}_A)$. Thus $K \otimes_A (A/\mathfrak{m}_A) = 0$. Applying 2.1 to the B-module K and the ideal $\mathfrak{m}_A B$ (note that if M is of finite type over B, then I is finite and K is of finite type over B as well) we deduce that $K = 0$, proving that ψ is an isomorphism. □

Corollary 2.4 *To prove the Conjecture 1.1, we can assume that the maximal ideal $\mathfrak{m}_A$ of A is a square-zero ideal.*

Proof Assume that the conjecture is true with the additional hypothesis $\mathfrak{m}_A^2 = 0$. Let M be an A-flat B-module. Using our assumption and tensoring by $A/\mathfrak{m}_A^2$, we see that $M/\mathfrak{m}_A^2 M$ is a flat $(B/\mathfrak{m}_A^2 B)$-module. Now tensoring by $A/\mathfrak{m}_A$, we deduce that $M/\mathfrak{m}_A M$ is flat over $B/\mathfrak{m}_A B$. Hence M is flat over B owing to 2.3. □

Proposition 2.5 *Let $\varphi : A \to B$ be a flat local morphism of Noetherian local rings.*

(i) *The $B/\mathfrak{m}_A B$-module $\mathfrak{m}_A B/\mathfrak{m}_A^2 B$ is free, and*

$$\mathrm{rk}_{\,B/\mathfrak{m}_A B} \frac{\mathfrak{m}_A B}{\mathfrak{m}_A^2 B} = \dim_{\kappa(B)} \frac{\mathfrak{m}_A B}{\mathfrak{m}_B \mathfrak{m}_A B} = \mathrm{edim}\,(A).$$

(ii) *We have* $\mathrm{edim}\,(A) \le \mathrm{edim}\,(B)$.
(iii) *Assume moreover that* $\mathrm{edim}\,(B) \le \mathrm{edim}\,(A) + 1$. *Then φ is a complete intersection morphism.*

Proof (i) First we note that, as B is flat over A, we have canonical isomorphisms:

$$\begin{aligned}\mathfrak{m}_A B/\mathfrak{m}_A^2 B &\simeq (\mathfrak{m}_A/\mathfrak{m}_A^2) \otimes_A B \\ &\simeq (\mathfrak{m}_A/\mathfrak{m}_A^2) \otimes_{\kappa(A)} (B/\mathfrak{m}_A B)\end{aligned}$$

so that $\mathfrak{m}_A B/\mathfrak{m}_A^2 B$ is free of rank $\mathrm{edim}\,(A)$ over $B/\mathfrak{m}_A B$. We also have

$$\begin{aligned}\left(\frac{\mathfrak{m}_A B}{\mathfrak{m}_B.(\mathfrak{m}_A B)}\right) &\simeq (\mathfrak{m}_A B) \otimes_B \kappa(B) \\ &\simeq (\mathfrak{m}_A B) \otimes_B (B/\mathfrak{m}_A B) \otimes_{(B/\mathfrak{m}_A B)} \kappa(B) \\ &\simeq (\mathfrak{m}_A B/\mathfrak{m}_A^2 B) \otimes_{(B/\mathfrak{m}_A B)} \kappa(B)\end{aligned}$$

so that the dimension of $\frac{\mathfrak{m}_A B}{\mathfrak{m}_B.(\mathfrak{m}_A B)}$ over $\kappa(B)$ is also equal to $\mathrm{edim}\,(A)$.

(ii) and (iii). If A and B are Artin, the ideal $\mathfrak{m}_A B$ is $\mathfrak{m}_B$-primary and we can apply [12, 2.1 and 2.3]. In the general case, we will see that the same proofs (more or less) still work. First we replace φ by its composition with the natural map from B into its completion for its

$\mathfrak{m}_B$-adic topology. This affects neither the hypothesis, nor the conclusions. We are thus reduced to the case where B is a complete Noetherian local ring. With the Cohen structure theorem for complete Noetherian local rings we now have a surjective map $\pi : R \to B$ where R is a regular local ring with $\operatorname{edim}(B) = \operatorname{edim}(R)$. Write $I = \pi^{-1}(\mathfrak{m}_A B)$ and $J = \pi^{-1}(\mathfrak{m}_A^2 B)$. We have $I \supset J \supset I^2$ and $R/I \simeq B/\mathfrak{m}_A B$. Owing to (i), we also have that $I/J \simeq \mathfrak{m}_A B/\mathfrak{m}_A^2 B$ is a free R/I-module of rank $r = \operatorname{edim}(A)$. Since every ideal in a regular local ring has finite projective dimension, we conclude with [12, Theorem 1.1] that $I = (x_1, \ldots, x_r) + J$, where $(x_1, \ldots, x_r)$ is a regular sequence in R. As obviously $r \leq \dim(R) = \operatorname{edim}(R)$, we obtain (ii).

With the remark in [12] before its Corollary 1, we also know that the module $I/(x_1, \ldots, x_r)$ has finite projective dimension over $R/(x_1, \ldots, x_r)$.

In the case $\operatorname{edim}(A) = \operatorname{edim}(B)$, that is, in the case $r = \operatorname{edim}(R)$, the sequence $(x_1, \ldots, x_r)$ is a maximal regular sequence in R. This implies that $\dim(R/(x_1, \ldots, x_r)) = 0$ and, as in [12], Proof of Theorem 2.3, we observe that the ideal of finite projective dimension $I/(x_1, \ldots, x_r)$ of $R/(x_1, \ldots, x_r)$ must be null (remember that a nonnull ideal of finite projective dimension always contains a regular element). Thus $I = (x_1, \ldots, x_r)$, so that $R/I \simeq B/\mathfrak{m}_A B$ is a complete intersection of dimension zero. In the case $\operatorname{edim}(B) = \operatorname{edim}(A) + 1$, we refer to [12] and its Proof of Theorem 2.3 again to reach the conclusion. □

3 Weakly torsion-free modules

Let $\varphi : A \to B$ be a Gorenstein morphism of Artin local rings. Let us consider a B-module M that is flat over A. We will give a necessary and sufficient condition for the module M to be flat over B (3.9). For that purpose, we introduce the notion of weakly torsion-free modules.

Definition 3.1 Let R be a local ring and M be an R-module. We say that M is weakly torsion-free if, for every λ in R and every m in M, the relation $\lambda m = 0$ implies $\lambda = 0$ or $m \in \mathfrak{m}_R M$.

The main result of this section is the flatness criterion 3.7 for modules over a Gorenstein, Artin, local k-algebra. First, we need the following lemma.

Lemma 3.2 *Let k be a field and R an Artin local k-algebra. Assume that R is Gorenstein and that its residue field is k. Then there exists a basis $(e_1, \ldots, e_n)$ of R as a k-vector space, together with elements $a_1, \ldots, a_{n-1}$ of R such that the following holds.*

1. *for all i we have $a_i e_i = e_n$*
2. *for all $j > i$, $a_i e_j = 0$.*

Example 3.3 For the k-algebra $k[u, v]/(u^3, v^3)$, the basis

$$(1, u, u^2, v, uv, u^2v, v^2, uv^2, u^2v^2)$$

works. We can take as a_i's the elements $u^2v^2, uv^2, v^2, u^2v, uv, v, u^2, u$.

Remark 3.4 Recall that, in a Gorenstein local ring R of dimension 0, there is a unique minimal ideal, called the *socle* of R. If $\mathfrak{m}_R$ is the maximal ideal of R, then the socle of R is equal to the annihilator $\operatorname{Ann}(\mathfrak{m}_R)$ of $\mathfrak{m}_R$ (see for example [11] for a formal computation of this annihilator). Now, if $(e_1, \ldots, e_n)$ is as in the lemma, then e_n is necessarily a generator of the socle of R. Indeed, e_n is different from zero since it is part of a basis of R, and if v is a nonzero element of R, we can write $v = \alpha_i e_i + \cdots + \alpha_n e_n$ with α_i an invertible element.

Multiplying by a_i, we see that e_n belongs to (v), so that the principal ideal generated by e_n is the minimal ideal of R.

Moreover, the element e_1 is necessarily invertible in R, and this is the only one. Indeed, we know that $a_1 e_1$ is different from zero, so that $a_1 \neq 0$. But we also have $a_1 e_j = 0$ for all $j > 1$ so that the e_j's cannot be invertible for $j \geq 2$. It follows that e_1 is invertible, since the basis necessarily contains at least one invertible element.

Proof Let e_n be a generator of the socle of R, and let us take elements $e_1, \dots, e_{n-1}$ such that $(e_1, \dots, e_n)$ is a basis of R. We proceed by induction. Assume that we already have constructed elements $a_1, \dots, a_{i-1}$ with the properties of the lemma $(1 \leq i \leq n-1)$. As e_n is in the socle of the ring, there exists an element a_i such that $a_i e_i = e_n$. Note that a_i is not invertible, so $a_i e_n = 0$. If there is an e_j, $j > i$ such that $a_i e_j$ does not belong to (e_n), then we exchange e_i and e_j and we replace a_i with some multiple of a_i in such a way that $a_i e_i$ is still equal to e_n. Now, for every $j > i$, $a_i e_j$ belongs to (e_n), so we can write $a_i e_j = \lambda_j e_n$. We replace e_j with $e_j - \lambda_j e_i$. This gives the result. □

Remark 3.5 The Gorenstein assumption cannot be removed. Let R be equal to $k[x, y]/(x, y)^2 = k[x, y]/(x^2, xy, y^2)$. Assume that there exists a basis (e_1, e_2, e_3) and elements a_1, a_2 as in the lemma. We have seen in the preceding remark that e_2 is in the maximal ideal $\mathfrak{m}_R$ of R. Moreover, the relation $a_2 e_3 = 0$ proves that a_2 is also in the maximal ideal. Then $a_2 e_2 = 0$, which yields a contradiction.

Remark 3.6 The hypothesis on the residue field cannot be removed. For instance if R is a finite (nontrivial) extension field of k, the lemma is obviously false for R.

Proposition 3.7 *Let R be an Artin local ring. Assume that R is Gorenstein and contains a field. Then an R-module M is flat over R if and only if it is weakly torsion-free.*

Proof Assume that M is flat over R. Let λ be a nonzero element in R and $m \in M$ such that $\lambda m = 0$. As M is flat over R, we can write $m = \sum_i \alpha_i e_i$ where the e_i's belong to an R-basis of M. Thus, for every i, $\lambda \alpha_i = 0$, so that α_i cannot be invertible and belongs to $\mathfrak{m}_R$.

Conversely, assume that M is weakly torsion-free. Recall that R necessarily contains its residue field (see for instance [4, 0, 19.6.3]). Let $(f_i)_{i \in \Lambda}$ be a family of elements of M the images of which form a basis of the $\kappa(R)$-vector space $M/\mathfrak{m}_R M$. We will prove that this is a basis of M over R. We have to prove that the morphism

$$\Phi : \begin{cases} R^{(\Lambda)} & \longrightarrow M \\ (b_i)_{i \in \Lambda} & \longrightarrow \sum b_i f_i \end{cases}$$

is an isomorphism. Owing to Lemma 2.1, we already know that it is surjective. Let us consider a relation $\sum_i b_i f_i = 0$. We have to prove that all the b_i are zero. Let $(e_1, \dots, e_n)$ be a $\kappa(R)$-basis of R given by the Lemma 3.2. We can write $b_i = \sum_j \alpha_{ij} e_j$ with α_{ij} in $\kappa(R)$. Thus we get the relation

$$\sum_i \sum_{j=1}^{n} \alpha_{ij} e_j f_i = 0.$$

Let a_1, ..., a_{n-1} be the elements of R given together with the basis $(e_1, \dots, e_n)$ in the Lemma 3.2. Multiplying the relation by a_1, we get:

$$e_n \sum_i \alpha_{i1} f_i = 0.$$

Since M is weakly torsion-free, we deduce that $\sum_i \alpha_{i1} f_i$ belongs to $\mathfrak{m}_R M$. But the f_i's form a basis of $M/\mathfrak{m}_R M$, so for every i, $\alpha_{i1} = 0$. Proceeding by induction (using $a_2, \ldots, a_{n-1}$) we prove that all the α_{ij} are equal to zero. □

Example 3.8 If we drop the Gorenstein assumption, 3.7 is not true any more. Let k be a field, let $R = k[x, y]/(x, y)^2$, and let M be the quotient R^2/I, where I is the submodule of R^2 generated by the element $\binom{x}{y}$. Then M is not flat over R since it is of dimension 5 as a k-vector space. But M is weakly torsion-free. Indeed, let $\lambda \in R$ and $m = \binom{a}{b} \in R^2$ such that $\lambda m \in I$. If λ is invertible it is obvious that $m \in \mathfrak{m}_R R^2$. If $\lambda m = 0$ it is obvious that either $\lambda = 0$ or $m \in \mathfrak{m}_R R^2$. Now if $\lambda \in \mathfrak{m}_R$ and $\lambda m \neq 0$, we have $\lambda a = \alpha x$, $\lambda b = \alpha y$ with $\alpha \in R^\times$ so a and b are invertible and $\lambda \in (x) \cap (y) = (0)$, a contradiction.

Corollary 3.9 *Let $\varphi : A \to B$ be a Gorenstein morphism of Artin local rings. Let M be a B-module, flat over A. Then the following conditions are equivalent.*

(i) *The module M is flat over B.*
(ii) *The module $M/\mathfrak{m}_A M$ is flat over $B/\mathfrak{m}_A B$.*
(iii) *The module $M/\mathfrak{m}_A M$ is weakly torsion-free over $B/\mathfrak{m}_A B$.*

Proof The equivalence of (i) and (ii) is 2.3. The equivalence of (ii) and (iii) is a consequence of the previous proposition. (Note that the Gorenstein ring $B/\mathfrak{m}_A B$ contains the field $\kappa(A)$.) □

4 Flatness in first order

We begin with a technical lemma that will be used twice: the first time to define the notion of flatness in order one below (4.2 (i)), and the second time in the Proof of Lemma 4.11 (that will be useful in Sect. 5 to modify the morphism φ when necessary).

Lemma 4.1 *Let A be a Noetherian local ring, I an ideal of A, and M an A-module. Let $x = (x_1, \ldots, x_n)$ be a sequence generating I minimally. Let us denote by N_x the submodule of M generated by the elements m_i for all the relations*

$$x_1 m_1 + \cdots + x_n m_n = 0.$$

Then the module N_x does not depend on the choice of x.

Proof Let $y = (y_1, \ldots, y_n)$ be another sequence generating I minimally. It is enough to prove that $N_x \subset N_y$. Let

$$x_1 m_1 + \cdots + x_n m_n = 0$$

be a relation as in the statement of the lemma.

We still write x (resp. y) for the row matrix $(x_1, \ldots, x_n)$ (resp. $(y_1, \ldots, y_n)$) and m for the column matrix $(m_1, \ldots, m_n)^t$. The above relation becomes $x.m = 0$. Since x and y are minimal systems of generators of I, there is a matrix C of size $n \times n$ with coefficients in A such that $x = yC$. Since the images of x and y in $I/\mathfrak{m}_A I$ are both $\kappa(A)$-basis of $I/\mathfrak{m}_A I$, this matrix C is invertible modulo $\mathfrak{m}_A$. This implies that C is also invertible in A since A is local. The relation $xm = yCm = 0$ implies that Cm has its entries in N_y. Hence $m = C^{-1}Cm$ also has its entries in N_y, this proves the assertion. □

Definition 4.2 (i) Let A be a Noetherian local ring and M an A-module. Then M is said to be flat in first order (or flat in order one, or simply 1-flat) if for a (resp. for any, see 4.1) minimal system of generators $(x_1, \dots, x_n)$ of $\mathfrak{m}_A$, and for elements $m_1, \dots, m_n$ in M the relation

$$x_1 m_1 + \cdots + x_n m_n = 0$$

implies $m_i \in \mathfrak{m}_A M$ for all i. (In other words, if $N_x \subset \mathfrak{m}_A M$ with the notations of the above lemma.)

(ii) Let $\varphi : A \to B$ be a morphism of Noetherian local rings and M a B-module. Then M is said to be φ-1-flat if it is 1-flat over A *via* φ.

(iii) Let $\varphi : A \to B$ be a morphism of Noetherian local rings and M be a B-module. Then M is said to be φ-weakly torsion-free if $M/\mathfrak{m}_A M$ is a weakly torsion-free $B/\mathfrak{m}_A B$-module.

Proposition 4.3 *Let A be a Noetherian local ring and M an A-module. Then M is flat in first order if and only if $M/\mathfrak{m}_A^2 M$ is flat over $A/\mathfrak{m}_A^2$. In particular, any flat A-module is 1-flat, and the converse is true if $\mathfrak{m}_A^2 = 0$.*

Proof Let $(e_i)_{i \in I}$ be an $A/\mathfrak{m}_A^2$-basis of $M/\mathfrak{m}_A^2 M$. Let $x_1 m_1 + \cdots + x_n m_n = 0$ be a relation as in the Definition 4.2 and consider its projection in $M/\mathfrak{m}_A^2 M$. Let us denote by λ_{ij} the coefficient of $\overline{m_j}$ along e_i. Then we have the relation in $A/\mathfrak{m}_A^2$:

$$x_1 \lambda_{i1} + \cdots + x_n \lambda_{in} = 0$$

Since the x_i's form a basis of $\mathfrak{m}_A/\mathfrak{m}_A^2$ this implies that all the λ_{ij} belong to $\mathfrak{m}_A$, hence all the m_j belong to $\mathfrak{m}_A M$.

Conversely, assume that M is 1-flat. Let $(e_i)_{i \in I}$ be a family of elements of $M/\mathfrak{m}_A^2 M$, the images of which form a basis of $M/\mathfrak{m}_A M$ over $\kappa(A)$. The e_i's generate $M/\mathfrak{m}_A^2 M$ owing to 2.1 (note that $A/\mathfrak{m}_A^2$ is Artin). To prove they are linearly independent over $A/\mathfrak{m}_A^2$, let us consider a relation $\sum \lambda_i e_i = 0$ with $\lambda_i \in A$. Viewing the relation in $M/\mathfrak{m}_A M$, we immediately see that λ_i belongs to $\mathfrak{m}_A$. Thus we can write $\lambda_i = x_1 \alpha_1^i + \cdots + x_n \alpha_n^i$ and we have

$$x_1 \left(\sum \alpha_1^i e_i \right) + \cdots + x_n \left(\sum \alpha_n^i e_i \right) = 0.$$

Since M is 1-flat, this implies that for all j, $\sum \alpha_j^i e_i = 0$ in $M/\mathfrak{m}_A M$, thus the α_j^i belong to $\mathfrak{m}_A$ and the λ_i belong to $\mathfrak{m}_A^2$. □

Remark 4.4 If $\mathfrak{m}_A^2 \neq 0$, a 1-flat module does not need to be flat. For instance in the ring $k[x]/(x^3)$, the ideal (x) is 1-flat but not flat. Actually, since $A/\mathfrak{m}_A^2$ is 1-flat, we have $\mathfrak{m}_A^2 = 0$ if and only if every 1-flat module is flat.

Definition 4.5 Let $\varphi : A \to B$ be a morphism of Noetherian local rings. Then φ is called *nice* if every φ-1-flat B-module is φ-weakly torsion-free.

Remark 4.6 A morphism $\varphi : A \to B$ of Noetherian local rings is nice if and only if the induced morphism $\varphi \otimes_A (A/\mathfrak{m}_A^2)$ is nice. (Immediate from the Definitions 4.2, 4.5 and the Proposition 4.3.) Note also that if φ is not local then it is obviously nice since $B/\mathfrak{m}_A B = 0$.

Remark 4.7 If φ is a nice Gorenstein morphism of Artin local rings then using 4.3 and 3.9, we see that every B-module M that is flat over A, is flat over B. In particular this proves 1.4.

Proposition 4.8 *Let $\varphi : A \to B$ be a local morphism of Noetherian local rings and let M be a B-module. Assume that M is of finite type over A or that A is Artin. Assume also that $\mu_B(\mathfrak{m}_A B) < \operatorname{edim}(A)$, and that M is φ-1-flat. Then $M = 0$.*

Proof By 2.1, we only have to prove that $M = \mathfrak{m}_A M$. Let $(x_1, \ldots, x_n)$ be a minimal system of generators of $\mathfrak{m}_A$. By assumption, one of the $\varphi(x_i)$'s, say $\varphi(x_1)$, is a linear combination of the others:

$$\varphi(x_1) = \sum_{i=2}^{n} b_i \varphi(x_i)$$

with $b_i \in B$. Now if m is an element of M, multiplying the above relation by m and using the first order flatness of M over A, we see that $m \in \mathfrak{m}_A M$. □

Corollary 4.9 *Let $\varphi : A \to B$ be a local morphism of Noetherian local rings. Assume that $\mu_B(\mathfrak{m}_A B) < \operatorname{edim}(A)$. Then φ is nice and de Smit.*

Corollary 4.10 *Let $\varphi : A \to B$ be a local complete intersection morphism of Noetherian local rings. Assume that $\operatorname{edim}(B) < \operatorname{edim}(A)$. Then the only φ-1-flat B-module (of finite type over A if A is not Artin) is zero. In particular φ is nice and de Smit.*

Proof Indeed, using the Lemma 5.4 below, we see that $\mu_B(\mathfrak{m}_A B) \leq \operatorname{edim}(B)$. Note that to prove that φ is nice, we can reduce to $\mathfrak{m}_A^2 = 0$ using 4.6, so A is Artin. □

Lemma 4.11 *Let $\varphi, \psi : A \to B$ be two local morphisms of Noetherian local rings. Assume that $\varphi(\mathfrak{m}_A)B = \psi(\mathfrak{m}_A)B$. Let M be a B-module. Then*:

(i) $\varphi(\mathfrak{m}_A)M = \psi(\mathfrak{m}_A)M$.
(ii) *M is φ-weakly torsion-free if and only if it is ψ-weakly torsion-free.*
(iii) *M is φ-1-flat if and only if it is ψ-1-flat.*

Proof (i) This is obvious.
(ii) This is an immediate consequence of the definition and (i).
(iii) In view of 4.3, we can assume that $\mathfrak{m}_A^2 = 0$, so A is Artin. Now if $\mu_B(\mathfrak{m}_A B) < \operatorname{edim}(A)$ the statement is obvious since the only φ-(resp. ψ-) 1-flat B-module is zero, see 4.8. Otherwise, let $(x_1, \ldots, x_n)$ be a minimal system of generators of $\mathfrak{m}_A$. Then $(\varphi(x_1), \ldots, \varphi(x_n))$ and $(\psi(x_1), \ldots, \psi(x_n))$ are both minimal systems of generators of $\mathfrak{m}_A B$, and the result is a consequence of Lemma 4.1. □

5 Sufficient conditions for the conjecture to be verified

We will give in this section some particular cases in which we can prove the Conjecture 1.1. The results 5.1, 5.11, 5.12 and 5.13 give the proof of 1.6. We first get rid of the trivial cases of embedding dimension zero or one.

Theorem 5.1 *Let $\varphi : A \to B$ be a morphism of Artin local rings with same embedding dimension n equal to zero or one. Then φ is nice.*

Proof If n equals 0, the rings A and B are fields and the statement is obvious. Now assume n is equal to 1. Let M be a φ-1-flat B-module and let x (resp. u) be a minimal generator of $\mathfrak{m}_A$ (resp. $\mathfrak{m}_B$). Let $\lambda \in B$ and $m \in M$ be elements such that $\lambda \notin \varphi(x)B$ and $\lambda m \in \varphi(x)M$.

We have to prove that $m \in uM$. We can assume $\varphi(x) \neq 0$ (see 4.9). Let p (resp. q) be the greatest integer such that $\varphi(x) \in (u^p)$ (resp. $\lambda \in (u^q)$). Then we have $\varphi(x) = au^p$ and $\lambda = bu^q$ with $a, b \in B^\times$, and $p > q$ since λ does not belong to $\varphi(x)B$. There is an m' in M such that $\lambda m = \varphi(x)m'$. Then:

$$u^q(bm - au^{p-q}m') = 0$$
$$\varphi(x)(bm - au^{p-q}m') = 0.$$

Since M is φ-1-flat, this implies that $bm - au^{p-q}m'$ belongs to $\varphi(x)M$, thus $m \in uM$. □

Before going further, we recall some facts about Wiebe matrices and consequences of Wiebe's criterion (which recognizes complete intersections of dimension zero among the class of Noetherian local rings).

Definition 5.2 [11, 2.6] Let A be a Noetherian local ring with a sequence $x = (x_1, \ldots, x_n)$ generating its maximal ideal (not necessarily minimally). An x-Wiebe matrix for the ring A is a square matrix ψ of size n such that $x.\psi = 0$ and $\det(\psi) \neq 0$. (Here the sequence x is also viewed as a row matrix.)

Proposition 5.3 (Wiebe, see [11, 2.7]) *A Noetherian local ring A is a complete intersection of dimension zero if and only if it has an x-Wiebe matrix for some (every) sequence x generating its maximal ideal. When this is the case, the determinant of an x-Wiebe matrix generates the socle of the ring.*

Lemma 5.4 *Let A be a Noetherian local ring, and I a proper ideal of A. Assume that the ring A/I is a complete intersection.*

(i) *Then*

$$\mu_A(I) \leq \text{edim}\,(A) - \dim(A/I).$$

(ii) *If moreover $I \subset \mathfrak{m}_A^2$, then*

$$\mu_A(I) \leq \text{edim}\,(A/I) - \dim(A/I).$$

Remark 5.5 The inequalities are not equalities in general (consider for instance $A = \mathbb{C}[x]/(x^2)$ and $I = 0$). But if A is regular, then it is well known that (ii) actually is an equality.

Proof We first prove (ii). As the minimal number of generators of any A-module is impervious to the change of rings $A \to \hat{A}$, where $\hat{A}$ is the completion of A in its $\mathfrak{m}_A$-adic topology, we are reduced to the case where A is complete. Using Cohen's theory, we know that there exists a regular Noetherian local ring R and a surjective local morphism $\pi : R \to A$ with $\text{Ker}\,\pi \subset \mathfrak{m}_R^2$. Write $J = \pi^{-1}(I)$. Then, since $I \subset \mathfrak{m}_A^2$ and $\text{Ker}\,\pi \subset \mathfrak{m}_R^2$, we have $J \subset \mathfrak{m}_R^2$. Moreover, by assumption the ring $R/J \simeq A/I$ is a complete intersection. Thus by [10, (21.2)] the ideal J of R is generated by a regular sequence, so that $\mu_R(J) = \text{edim}\,(R/J) - \dim(R/J)$. But $\mu_A(I) \leq \mu_R(J)$ (because π is surjective). This gives the result.

Now let us prove (i), using (ii). The inclusion of the ideal I in $\mathfrak{m}_A$ induces a map of $\kappa(A)$-vector spaces $\psi : \frac{I}{\mathfrak{m}_A I} \to \frac{\mathfrak{m}_A}{\mathfrak{m}_A^2}$. Let $x_1, \ldots, x_i$ be a sequence of elements of I, the images of which form a basis of the image of ψ. We complete it to form a minimal system of generators $(x_1, \ldots, x_n)$ of $\mathfrak{m}_A$. Let $J \subset I$ be the ideal generated by the elements $x_1, \ldots, x_i$. Let us denote by A' the quotient ring A/J and I' the ideal I/J in A'. Then we see that $I' \subset \mathfrak{m}_{A'}^2$, and the ring $A'/I' \simeq A/I$ is a complete intersection. Thus using (ii), we get

$$\mu_{A'}(I') \leq \text{edim}\,(A'/I') - \dim(A'/I')$$
$$\leq \text{edim}\,(A') - \dim(A/I)$$

Moreover, we obviously have

$$\mu_A(I) \leq \mu_A(J) + \mu_A(I').$$

Now we have $\mu_A(J) \leq i$ by definition of J, and $\mu_A(I') = \mu_{A'}(I')$. We conclude

$$\begin{aligned}\mu_A(I) &\leq i + \operatorname{edim}(A') - \dim(A/I)\\ &\leq \operatorname{edim}(A) - \dim(A/I)\end{aligned}$$

□

The existence of an element $u \in B$ verifying the equivalent conditions (i)–(vi) given in the following lemma is precisely the condition under which we (the authors) can go from embedding dimension n to embedding dimension $n-1$ when trying to prove de Smit's Conjecture 1.6 (see 5.7 and the proof of 5.13). Unfortunately such an element u does not always exist (see 6.1.3). Note that, as mentioned in the introduction, there is a Maple program on the webpage of the first author for testing (in some cases) if such an element exists.

Lemma 5.6 *Let $\varphi : A \to B$ be a complete intersection morphism of Artin local rings. Assume that*

$$\operatorname{edim}(A) = \operatorname{edim}(B) = \dim_{\kappa(B)} \frac{\mathfrak{m}_A B}{\mathfrak{m}_B \mathfrak{m}_A B}.$$

Let n denote this dimension and B_0 denote the quotient ring $B/\mathfrak{m}_A B$. Let u be a minimal generator of $\mathfrak{m}_B$ such that its image u_0 in B_0 is a minimal generator of $\mathfrak{m}_{B_0}$ (i.e. u is not in $\mathfrak{m}_A B + \mathfrak{m}_B^2$). Let $I = \operatorname{Ann}(u_0)$ denote the annihilator of u_0. The following are equivalent:

- (i) *There is an element y in $\mathfrak{m}_A B \backslash \mathfrak{m}_B \mathfrak{m}_A B$ such that u divides y in B.*
- (ii) *The ring $B_0/(u_0)$ is a complete intersection.*
- (iii) *The ring B_0/I is a complete intersection.*
- (iv) *The ideal I of B_0 is principal.*
- (v) *For any (for some) sequence $v = (v_1, \ldots, v_r)$ generating $\mathfrak{m}_{B_0}$ (not necessarily minimally), there is a v-Wiebe matrix ψ of B_0 such that its entries in the first column are all multiples of u.*
- (vi) *For any (for some) sequence $v = (u_0, v_2, \ldots, v_n)$ generating $\mathfrak{m}_{B_0}$ minimally, there is a v-Wiebe matrix for B_0 of the form*

$$\psi = \left(\begin{array}{c|c} z_1 & \cdots \\ \hline 0 & \psi^* \end{array}\right).$$

Moreover if these conditions hold, and if ψ is a matrix as in (vi), then ψ^* is a $(v_2, \ldots, v_n)$-Wiebe matrix for the ring $B_0/(u_0)$.

Proof The equivalence of conditions (ii) to (vi) and the last statement are proved in [11, 5.5 and 5.7]. So we only have to prove the equivalence of (i) and (ii).

Assume (i). Using Cohen's theory, there is a Noetherian regular local ring R and a surjective local morphism $\pi : R \to B$, with $\operatorname{Ker} \pi \subset \mathfrak{m}_R^2$. In R, there are liftings y' and u' of y and u such that y' is a multiple of u'. Let $J = \pi^{-1}(\mathfrak{m}_A B)$. Since $\pi(\mathfrak{m}_R) \subset \mathfrak{m}_B$, we easily see that y' and u' are minimal generators of J and $\mathfrak{m}_R$ respectively. Let $(y', y_2, \ldots, y_s)$ be a minimal system of generators of J. This is a regular sequence since the ring R/J, isomorphic to B_0, is a complete intersection. This means that the sequence $(y_2, \ldots, y_s)$ is regular and that y' is not a zero divisor in $R/(y_2, \ldots, y_s)$. Neither is u' since y' is a multiple of u'. Thus

$(u', y_2, \dots, y_s)$ is a regular sequence and the ring $R/(u', y_2, \dots, y_s) = R/(J + (u'))$ is a complete intersection. This last ring is isomorphic to $B_0/(u_0)$, proving (ii).

Conversely, assume (ii). Let π be the projection map $\pi : B \to \overline{B} = B/(u)$. The quotient ring $\overline{B}/\mathfrak{m}_A\overline{B}$ is isomorphic to $B_0/(u_0)$, and so is a complete intersection. Applying the previous Lemma 5.4 to $\overline{B}$ and its ideal $\mathfrak{m}_A\overline{B}$, we get

$$\mu_{\overline{B}}(\mathfrak{m}_A\overline{B}) \leq \operatorname{edim}(\overline{B}) = n - 1.$$

Let $(x_1, \dots, x_n)$ be a sequence in A generating $\mathfrak{m}_A$ minimally. With the above, the sequence $\pi(\varphi(x_1)), \dots, \pi(\varphi(x_n))$ in $\overline{B}$, which generates $\mathfrak{m}_A\overline{B}$, does not generate $\mathfrak{m}_A\overline{B}$ minimally. Thus one of the $\pi(\varphi(x_i))$, say $\pi(\varphi(x_1))$, is a linear combination of the others and we have in $\overline{B}$ a relation $\pi(\varphi(x_1) + \alpha_2\varphi(x_2) + \cdots + \alpha_n\varphi(x_n)) = 0$ for some $\alpha_i \in B$. Now the element $y = \varphi(x_1) + \alpha_2\varphi(x_2) + \cdots + \alpha_n\varphi(x_n)$ is in $\mathfrak{m}_A B \backslash \mathfrak{m}_B\mathfrak{m}_A B$ and in $\operatorname{Ker}\pi = (u)$. □

Lemma 5.7 *Let $\varphi : A \to B$ be a morphism of Artin local rings. Assume that there is a minimal generator u of $\mathfrak{m}_B$ and a minimal generator x of $\mathfrak{m}_A$ such that $\varphi(x)$ is a multiple of u and does not belong to $\mathfrak{m}_B\mathfrak{m}_A B$. Let us denote by*

$$\overline{\varphi} : \overline{A} = A/(x) \to B/(u) = \overline{B}$$

the induced morphism of Artin local rings.

(i) *If φ is a complete intersection morphism then so is $\overline{\varphi}$.*
(ii) *If $\mathfrak{m}_A^2 = 0$, then $\mathfrak{m}_{\overline{A}}^2 = 0$.*
(iii) *The embedding dimensions of $\overline{A}$ and $\overline{B}$ are respectively* $\operatorname{edim}(A) - 1$ *and* $\operatorname{edim}(B) - 1$. *In particular, if A and B have same embedding dimension, then so have $\overline{A}$ and $\overline{B}$.*
(iv) *If $\mathfrak{m}_A B/\mathfrak{m}_A^2 B$ is free of rank* $\operatorname{edim}(A)$ *over $B/\mathfrak{m}_A B$, then $\mathfrak{m}_{\overline{A}}\overline{B}/\mathfrak{m}_{\overline{A}}^2\overline{B}$ is free of rank* $\operatorname{edim}(\overline{A})$ *over $\overline{B}/\mathfrak{m}_{\overline{A}}\overline{B}$. (The interest of this property is that it implies that* $\dim_{\kappa(B)} \mathfrak{m}_A B/\mathfrak{m}_B\mathfrak{m}_A B = \operatorname{edim}(A)$.)
(v) *If M is a φ-1-flat B-module, then $\overline{M} := M/uM$ is a $\overline{\varphi}$-1-flat $\overline{B}$-module.*
(vi) *If $B/\mathfrak{m}_A B$ is Gorenstein and if $\overline{\varphi}$ is nice, then φ is nice.*

Proof (ii) and (iii) are obvious. (i) is a consequence of the implication $(i) \Rightarrow (ii)$ of 5.6, taking $\varphi(x)$ for the element y.

(iv) Let $n = \operatorname{edim}(A)$ and let $(x_1, \dots, x_n)$ be a minimal system of generators of $\mathfrak{m}_A$, with $x_1 = x$. We will prove that the images of the elements $(\varphi(x_2), \dots, \varphi(x_n))$ in $\mathfrak{m}_{\overline{A}}\overline{B}/\mathfrak{m}_{\overline{A}}^2\overline{B}$ form a basis of this module over $\overline{B}/\mathfrak{m}_{\overline{A}}\overline{B}$. We only have to prove that they are linearly independant. So let $\alpha_2, \dots, \alpha_n$ be elements of B such that the image of $\alpha_2\varphi(x_2) + \cdots + \alpha_n\varphi(x_n)$ in the module $\mathfrak{m}_{\overline{A}}\overline{B}/\mathfrak{m}_{\overline{A}}^2\overline{B}$ is zero, and let us prove that for all i, $\overline{\alpha_i}$ belongs to $\mathfrak{m}_{\overline{A}}\overline{B}$, *i.e.* that α_i belongs to $\mathfrak{m}_A B + (u)$. By assumption, we have

$$\sum_{i=2}^{n} \alpha_i\varphi(x_i) \in (u) + \mathfrak{m}_A^2 B.$$

We multiply this by an element $d \in B$ such that $du = \varphi(x)$ and obtain

$$\sum_{i=2}^{n} d\alpha_i\varphi(x_i) \in (\varphi(x)) + \mathfrak{m}_A^2 B.$$

But, with the assumption in (iv), the images of the $\varphi(x_i)$'s in $\mathfrak{m}_A B/\mathfrak{m}_A^2 B$ form a basis of this free module over $B/\mathfrak{m}_A B$, so that $d\alpha_i \in \mathfrak{m}_A B$ for all i. Hence we can write:

$$d\alpha_i = \sum_{j=1}^{n} \beta_{ij}\varphi(x_j)$$

with $\beta_{ij} \in B$. Multiplying by u, we get:

$$\varphi(x)\alpha_i = \sum_{j=1}^{n} \beta_{ij}u\varphi(x_j)$$
$$\varphi(x)(\alpha_i - u\beta_{i1}) = \sum_{j=2}^{n} \beta_{ij}u\varphi(x_j)$$

We remember that the images of the $\varphi(x_i)$'s in $\mathfrak{m}_A B/\mathfrak{m}_A^2 B$ form a basis of this free $B/\mathfrak{m}_A B$-module, and we conclude that $\alpha_i - u\beta_{i1} \in \mathfrak{m}_A B$.

(v) Let M be a φ-1-flat B-module and let us view M as an A-module *via* φ. Let $(x, x_2, \ldots, x_n)$ be a minimal system of generators of $\mathfrak{m}_A$. Then $(\overline{x_2}, \ldots, \overline{x_n})$ is a minimal system of generators of $\mathfrak{m}_{\overline{A}}$. Let

$$\overline{x_2 m_2} + \cdots + \overline{x_n m_n} = 0$$

be a relation in $\overline{M}$ viewed as an $\overline{A}$-module *via* $\overline{\varphi}$. We then have

$$x_2 m_2 + \cdots + x_n m_n = u m_1$$

in M. Since M is φ-1-flat, since $\varphi(x) = du$ for some $d \in B$, this implies that m_1 belongs to $\mathfrak{m}_A M$, *i.e.* $m_1 = x_1 m_1' + \cdots + x_n m_n'$. We get:

$$x_2(m_2 - u m_2') + \cdots + x_n(m_n - u m_n') = u x_1 m_1'$$

thus $m_i - u m_i' \in \mathfrak{m}_A M$ for $i \geq 2$ using 1-flatness again. This proves that $\overline{m_i}$ belongs to $\mathfrak{m}_{\overline{A}}\overline{M}$.

(vi) Let M be a φ-1-flat B-module. Let $v \in B$ and $m \in M$ such that $vm \in \mathfrak{m}_A M$ and $v \notin \mathfrak{m}_A B$. We want to prove that $m \in \mathfrak{m}_B M$. Plainly we may assume that v modulo $\mathfrak{m}_A B$ generates the socle of $B/\mathfrak{m}_A B$. We write $\varphi(x) = ud$, with $d \in B$. The element d is not in $\mathfrak{m}_A B$ (otherwise $\varphi(x)$ would belong to $\mathfrak{m}_B \mathfrak{m}_A B$). Thus $(v) \subset (d)$ in $B/\mathfrak{m}_A B$ and there is a $b \in B$ and a $\lambda \in \mathfrak{m}_A B$ such that $v = db + \lambda$. Now $dbm \in \mathfrak{m}_A M$, thus

$$dbm = udm_1 + \varphi(x_2)m_2 + \cdots + \varphi(x_n)m_n.$$
$$ud(bm - um_1) = \varphi(x_2)um_2 + \cdots + \varphi(x_n)um_n$$

Since M is φ-1-flat, this implies that $bm - um_1$ belongs to $\mathfrak{m}_A M$. Hence the element $\overline{b}\overline{m}$ of $\overline{M}$ belongs to $\mathfrak{m}_{\overline{A}}\overline{M}$. But $\overline{b}$ does not belong to $\mathfrak{m}_{\overline{A}}\overline{B}$ (otherwise db would belong to $\mathfrak{m}_A B$ since du does). Moreover we have seen that $\overline{M}$ is $\overline{\varphi}$-1-flat so that $\overline{M}/\mathfrak{m}_{\overline{A}}\overline{M}$ is a weakly torsion-free $\overline{B}/\mathfrak{m}_{\overline{A}}\overline{B}$-module since $\overline{\varphi}$ is nice. Thus $\overline{m}$ belongs to $\mathfrak{m}_{\overline{B}}\overline{M}$. This easily implies that $m \in \mathfrak{m}_B M$. □

Lemma 5.8 *Let $\varphi : A \to B$ be a morphism of Artin local rings. Assume that $\mathfrak{m}_A^2 = 0$. Let $(x_1, \ldots, x_n)$ be a minimal system of generators of $\mathfrak{m}_A$, and $(y_1, \ldots, y_n)$ a system of generators of $\varphi(\mathfrak{m}_A)B$ (not necessarily minimal). In case A does not contain a field, we assume moreover that $x_1 = p.1_A$ and $y_1 = p.1_B$, where p is the characteristic of $\kappa(A)$. Then:*

(i) *There is a morphism* $\psi : A \to B$ *such that* $\psi(x_i) = y_i$.
(ii) *For such a morphism, the ideals* $\varphi(\mathfrak{m}_A)B$ *and* $\psi(\mathfrak{m}_A)B$ *are equal.*

Remark 5.9 If A does not contain a field, then the characteristic p of $\kappa(A)$ is different from zero and we have $p.1_A \neq 0$ [4, chap. 0, 19.6.3], so that $p.1_A$ is a minimal generator of $\mathfrak{m}_A$ since $\mathfrak{m}_A^2 = 0$. Thus the additional hypothesis $x_1 = p$ and $y_1 = p$ is not a restriction on φ but only on the choice of the minimal system of generators of $\mathfrak{m}_A$.

Proof (i) If A contains a field, then it contains $\kappa(A)$ and the map $T_i \mapsto x_i$ defines an isomorphism from

$$\frac{\kappa(A)[|T_1, \dots, T_n|]}{(T_1, \dots, T_n)^2}$$

to A (we use $\mathfrak{m}_A^2 = 0$). *Via* this isomorphism, and since $(\mathfrak{m}_A B)^2 = 0$, we can obviously define ψ by $T_i \mapsto y_i$.

If A does not contain a field, let W_A be a Cohen ring, the residue field of which is $\kappa(A)$. W_A is a discrete valuation ring, and its maximal ideal is generated by p. Since $\mathfrak{m}_A^2 = 0$, we have an isomorphism

$$A \simeq \frac{W_A[|T_2, \dots, T_n|]}{(p, T_2, \dots, T_n)^2}$$

identifying T_i with x_i $(2 \leq i \leq n)$ and we can again define the map ψ sending T_i to y_i.

(ii) Obvious by construction. □

The following lemma prepares the proof of the Conjecture 1.1 in embedding dimension 2.

Lemma 5.10 *Let* $\varphi : A \to B$ *be a morphism of Artin local rings of embedding dimension* 2. *Assume that*

$$\dim_{\kappa(B)} \frac{\mathfrak{m}_A B}{\mathfrak{m}_B \mathfrak{m}_A B} = 2.$$

Then for every minimal generator u *of* $\mathfrak{m}_B$*, there is an element* y *in* $\mathfrak{m}_A B \backslash \mathfrak{m}_B \mathfrak{m}_A B$ *such that* u *divides* y *in* B.

Proof Let (u, v) be a minimal system of generators of $\mathfrak{m}_B$ and (x, y) a minimal system of generators of $\mathfrak{m}_A B$ (*i.e.* a basis of $\frac{\mathfrak{m}_A B}{\mathfrak{m}_B \mathfrak{m}_A B}$). We can write

$$x = \alpha v^n + u b_1$$
$$y = \beta v^m + u b_2$$

with $b_1, b_2 \in B$ and $\alpha, \beta \in B^\times$. We can assume $m \geq n$. Then $y - \beta\alpha^{-1} v^{m-n} x$ is a multiple of u, and is an element of $\mathfrak{m}_A B \backslash \mathfrak{m}_B \mathfrak{m}_A B$. □

Theorem 5.11 *Let* $\varphi : A \to B$ *be a Gorenstein morphism of Artin local rings of embedding dimension two. Then* φ *is nice.*

Proof Using 4.9 and 4.6, we can assume that

$$\dim_{\kappa(B)} \frac{\mathfrak{m}_A B}{\mathfrak{m}_B \mathfrak{m}_A B} = 2$$

and that $\mathfrak{m}_A^2 = 0$. Let u be a minimal generator of $\mathfrak{m}_B$ and y given by the previous lemma. Using Lemma 5.8 there is a morphism $\psi : A \to B$ such that $\psi(x) = y$ for a minimal generator x of $\mathfrak{m}_A$ and $\psi(\mathfrak{m}_A)B = \varphi(\mathfrak{m}_A)B$. Using Lemma 4.11, we may replace φ with ψ so that we may assume $\varphi(x) = y$. Now owing to Theorem 5.1, the induced morphism $\overline{\varphi} : A/(x) \to B/(u)$ is nice. Hence φ is nice because of Lemma 5.7 (vi). □

Remark 5.12 The same argument proves theorem 1.6 (ii).

Theorem 5.13 *Let $\varphi : A \to B$ be a morphism of Artin local rings with same embedding dimension. Write $B_0 = B/\mathfrak{m}_A B$ and assume that for some minimal system of generators $u = (u_1, \dots, u_n)$ of $\mathfrak{m}_{B_0}$, there is an upper triangular u-Wiebe matrix for B_0. Then φ is nice.*

Proof Note that, since there is a u-Wiebe matrix for B_0, φ is a complete intersection morphism (5.3). Proceeding by induction on $n = \text{edim}(A)$ (5.7), and using 5.1 for the initial step, we may assume (and we do) that the theorem is true in embedding dimension $n-1$. We can also assume (4.9, 4.6) that

$$\dim_{\kappa(B)} \frac{\mathfrak{m}_A B}{\mathfrak{m}_B \mathfrak{m}_A B} = \text{edim}(A)$$

and that $\mathfrak{m}_A^2 = 0$. Let u' be a lifting of u_1 in B. Owing to Lemma 5.6, there is an element y in $\mathfrak{m}_A B \backslash \mathfrak{m}_B \mathfrak{m}_A B$ such that u' divides y in B. Using Lemma 5.8 there is a morphism $\psi : A \to B$ sending a minimal generator x of $\mathfrak{m}_A$ on y and such that $\psi(\mathfrak{m}_A)B = \varphi(\mathfrak{m}_A)B$. We may replace φ by ψ (4.11) so that we may assume $\varphi(x) = y$. Now the induced morphism $\overline{\varphi} : A/(x) \to B/(u)$ is nice (use Lemma 5.6 and the induction hypothesis). Hence φ is nice because of Lemma 5.7. □

Remark 5.14 The question of existence of such upper triangular Wiebe matrices has been considered in [11].

6 Examples and further developments

6.1 Examples

Example 6.1.1 Let us consider the morphism

$$\varphi : A = \frac{(\mathbb{Z}/p^2\mathbb{Z})[x_1, x_2, x_3]}{(x_1^2, x_2^2, x_3^2)} \longrightarrow B = \frac{(\mathbb{Z}/p^2\mathbb{Z})[t, u, v, w]}{(t^2 - p, \alpha_1^2, \alpha_2^2, \alpha_3^2)}$$

where φ is defined by $\varphi(x_i) = \alpha_i$ and

$$\alpha_1 = u^2 + tw^2$$

$$\alpha_2 = v^2 + t^3 + u^3$$

$$\alpha_3 = w^3 + vw$$

The rings A and B are Artin local with embedding dimension 4. We let the reader check that the (image in B_0 of the) matrix

$$\begin{pmatrix} t & w^2 & t^2 & 0 \\ 0 & u & u^2 & 0 \\ 0 & 0 & v & w \\ 0 & 0 & 0 & w^2 \end{pmatrix}$$

is a (t, u, v, w)-Wiebe matrix for the ring $B_0 = B/\mathfrak{m}_A B \simeq \frac{\mathbb{F}_p[t,u,v,w]}{(t^2,\alpha_1,\alpha_2,\alpha_3)}$. (Compute a Gröbner basis and check that $tuvw^2$ is not zero in B_0.) Hence we deduce from Theorem 5.13 that any A-flat B-module is B-flat.

Example 6.1.2 Let us consider

$$p_1 = u^2 + vw$$
$$p_2 = v^2 + uw$$
$$p_3 = w^2 + uv$$

and the morphism $\varphi : A = \frac{\mathbb{C}[[x_1,x_2,x_3]]}{(x_1,x_2,x_3)^2} \to B = \frac{\mathbb{C}[[u,v,w]]}{(p_1,p_2,p_3)^2}$ defined by $\varphi(x_i) = p_i$. We let the reader check that φ is flat (*e.g.* $1, w, w^2, w^3, v, vw, v^2, u$ is an A-basis of B). Hence $B/\mathfrak{m}_A B$ is a complete intersection owing to 2.5. Moreover, the element $u - v$ is a minimal generator of $\mathfrak{m}_B$, and divides the element $p_1 - p_2$ of $\mathfrak{m}_A B \backslash \mathfrak{m}_B \mathfrak{m}_A B$. Thus using 1.6 (ii) we see that every A-flat B-module is B-flat.

Example 6.1.3 Now let us try with

$$p_1 = u^3 + vw$$
$$p_2 = v^2 + uw^2$$
$$p_3 = w^3 + u^2 v$$

We can check (with the help of a computer) that the condition given in 1.6 (ii) is not true for φ. Hence de Smit's question remains open in this case.

6.2 Further developments

For $A \to B$ a flat morphism of local rings, Lech [8] and Hironaka [7] asked whenever we have inequality of i-th sum transforms of the Hilbert series of A and of B (see [6]). This question is a generalization of the inequality of the embedding dimensions (edim $(A) \leq$ edim (B)). We could discuss a weaker version of de Smit's conjecture using these transforms of Hilbert series. In view of 1.6 it is also tempting to ask whether a morphism of Artin local rings with the same embedding dimension is always nice.

De Smit's question has origin in Wiles' proof of Fermat's theorem ([13]). Wiles' proof needs an isomorphism between an Hecke algebra and a deformation ring. De Smit mentionned his conjecture in order to make easier one argument in this direction ([9]). But such a criterion will not only have applications in deformation theory but also in many other fields. For example let R be a regular ring of characteristic p and $F : R \to R$ the Frobenius. Consider $F^* R$ the $(R-R)$ bimodule with additive group R and left and right scalar multiplication given by $arb = arF(b)$ for $a, b \in R$ and $r \in F^* R$. Then it is well known that $R \to F^* R$ is a flat couple ([3] Sect. 8.2). Then such a flatness criterium could also have some applications in the theory of φ-module of Fontaine in order to study R and $F^* R$-modules.

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