Remark on the anisotropic prescribed mean curvature equation on arbitrary domains

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Abstract In this article we consider the Dirichlet problem for hypersurfaces of anisotropic prescribed mean curvature $H = H(x, u, N)$ depending on $x \in \Omega \subset \mathbb{R}^n$, the height u of the hypersurface $M = \text{graph } u$ over Ω and the unit normal N to M at (x, u) . We give a condition relating H and the mean curvature of $\partial\Omega$ that guarantees the existence of smooth solutions even for not necessarily convex domains.

Keywords Dirichlet problem · Anisotropic prescribed mean curvature · Boundary gradient estimate

Mathematics Subject Classification (2000) 35J25 · 53A10

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and M a hypersurface over Ω given as the graph of $u : \Omega \rightarrow \mathbb{R}$. The Dirichlet problem of prescribed anisotropic mean curvature is given by

$$\frac{1}{n} \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = H(x, u, N(Du)) \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega.$$

The left-hand side of the first equation is the average of the principal curvatures of M , i.e. the mean curvature. $H = H(x, z, N)$ is a given function depending on the point $(x, u) \in M$ and the direction of the upper normal to M at that point, given by

$$N(Du) = \frac{1}{\sqrt{1 + |Du|^2}} \begin{pmatrix} -Du \\ 1 \end{pmatrix}.$$

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Early results of Serrin [4] for $H = H(x)$ are summarised in [3], where existence was proved using a fixed point argument and Hölder estimates for the gradient, reducing the proof to C^1 -a priori estimates.

In the general case $H = H(x, z, N)$ either the condition

$$\int_{\Omega} \sup_{z \in \mathbb{R}, N \in S^n} |H(x, z, N)|^n dx < \omega_n \quad (1)$$

or the assumption $D_z H \geq 0$ together with

$$\left| \int_{\Omega} H(x, 0, N(D\eta)) \eta dx \right| \leq \frac{1-\varepsilon}{n} \int_{\Omega} |D\eta| dx \quad \forall \eta \in C_0^1(\Omega) \quad (2)$$

for some $\varepsilon > 0$ ensure a bound on u . The result is well known in the case that (1) holds and can be established by applying the method of Stampacchia [5] otherwise. A condition comparable to (2) was first imposed by Giaquinta [2].

To derive a gradient estimate it is favourable to impose for some $\beta > 0$ the structure condition

$$(sc) \begin{cases} H_1, H_2 \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^{n+1}) \cap C^{1,\beta}(\Omega \times \mathbb{R} \times \mathbb{R}^{n+1}), \\ H(x, z, N) = H_1(x, z, N) + H_2(x, z, N)N_{n+1}, \quad D_z H_1 \geq 0 \text{ on } \Omega \times \mathbb{R} \times S_+^n. \end{cases}$$

This condition was first used by Bergner [1]. He estimated Du in the interior in terms of the boundary gradient using a maximum principle. The boundary gradient estimate he obtained was only valid in convex domains.

The main result of this paper is Theorem 1, where we prove a boundary gradient estimate assuming an a priori bound for u , (sc) and the Serrin type condition

$$H_{\partial\Omega}(y) \geq \frac{n}{n-1} |H_1(y, \varphi(y), \gamma(y))| \quad \forall y \in \partial\Omega, \quad (3)$$

where γ is the inward pointing unit normal of $\partial\Omega$ considered as a vector in $\mathbb{R}^{n+1}$. As a consequence we obtain existence of smooth solutions in Corollary 1 even for not necessarily convex domains.

2 Results

For a domain $\Omega \subset \mathbb{R}^n$ with C^2 -boundary we define the boundary strip

$$\Gamma_1 := \{x \in \overline{\Omega} \mid d(x) := \text{dist}(x, \partial\Omega) < d_1\}$$

with $d_1 \leq 1$ such that $d \in C^2(\overline{\Gamma}_1)$ (cf. [3, Lemma 14.16]). Using $a := \sup_{\Gamma_1} |\varphi|$, $m := \sup_{\Omega} |u|$ and

$$\begin{aligned} h_i &:= \sup_{\Gamma_1 \times [-m, m] \times S^n} |H_i(x, z, N)| \quad \text{for } i = 1, 2, \\ h_{1,X} &:= \sup_{\Gamma_1 \times [-a, a] \times S^n} |D_X H_1(x, z, N)| \quad \text{for } X = x, z, N, \end{aligned}$$

we obtain the following result:

Theorem 1 (Boundary gradient estimate) *Let $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 -boundary. Suppose that $\varphi \in C^2(\overline{\Omega})$. Let $u \in C^{2,\alpha}(\overline{\Omega})$ be a solution of the Dirichlet problem of anisotropic prescribed mean curvature with an a priori bound m . If H satisfies (sc) and (3), then we have*

$$\sup_{\partial\Omega} |Du| \leq \sup_{\partial\Omega} |D\varphi| + \frac{1}{d_1} \left(1 + 2\|\varphi\|_{C^1(\overline{\Gamma}_1)} \right) e^{(1+C)(a+m)}$$

with $C = C \left(n, \Omega, \|d\|_{C^2(\overline{\Gamma}_1)}, \|\varphi\|_{C^2(\overline{\Gamma}_1)}, h_1, h_2, h_{1,x}, h_{1,z}, h_{1,N} \right)$.

Remark 1 In the case $H = H(x, z)$, condition (3) reduces to the Serrin condition which is known to be sharp.

Corollary 1 (Existence and Uniqueness) *Let $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$ -boundary and $\varphi \in C^{2,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. Suppose that H satisfies (sc), (1) and (3). Then the Dirichlet problem for the equation of anisotropic prescribed mean curvature has a solution $u \in C^{2,\alpha}(\overline{\Omega})$. In the case $D_z H \geq 0$ condition (1) can be replaced by (2). Furthermore if $D_z H \geq 0$ the solution is unique.*

Remark 2 To generalise the result of Corollary 1 to boundary values $\varphi \in C^0(\partial\Omega)$ one can approximate φ uniformly by functions $\varphi_m \in C^{2,\alpha}(\overline{\Omega})$. Demanding $D_z H \geq 0$ and applying a comparison principle, the hereby obtained solutions $u_m \in C^{2,\alpha}(\overline{\Omega})$ converge uniformly to some function $u \in C^0(\overline{\Omega})$ with $u = \varphi$ on $\partial\Omega$. Interior gradient estimates which follow from [6], Theorem 1.1 (one can easily see that they even hold in the case $H_{1,z} \geq 0$), together with interior Hölder and Schauder estimates imply by Arzelà-Ascoli that $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ and $Qu = 0$.

Remark 3 Translating solutions $u(x, t) = u_0(x) + t$ of mean curvature flow have the property that u_0 satisfies our Dirichlet problem with $H_1 \equiv 0$ and $H_2 \equiv 1/n$. Hence Theorem 1 yields a boundary gradient estimate for such solutions in any mean convex domain. The necessary bound on u_0 can be shown explicitly, such that neither (1) nor (2) is needed.

Proof of Corollary 1 The conditions on Ω , φ and H allow us to apply [3], Theorem 13.8 which reduces the existence proof to the derivation of C^1 -a priori bounds for solutions $u \in C^{2,\alpha}(\overline{\Omega})$. Condition (1) on H together with [3], Theorem 10.5 yields the a priori estimate of u . In the case $D_z H \geq 0$ this estimate can be obtained from (2) using the method of Stampacchia [5]. The estimate of Du in the interior can be established using the structure condition (sc) and a maximum principle for the gradient as in [1]. Finally the boundary gradient estimate follows from Theorem 1, (sc) and condition (3). Uniqueness is obtained from $D_z H \geq 0$ by applying a comparison principle. $\square$

Proof of Theorem 1 We will use the elliptic operator $Qv := a^{ij}(Dv)D_{ij}v + b(x, v, Dv)$ with

$$\begin{aligned} a^{ij}(p) &:= (1 + |p|^2)\delta_{ij} - p_i p_j, \\ b(x, z, p) &:= -n(1 + |p|^2)(|p - D\varphi|H_1(x, z, N(p)) + b_0(x, p)), \\ b_0(x, p) &:= ((1 + |p|^2)^{1/2} - |p - D\varphi|)H_1(x, u(x), N(p)) + H_2(x, u(x), N(p)). \end{aligned}$$

Note that solutions u of the prescribed mean curvature equation satisfy $Qu = 0$ and that by $D_z H_1 \geq 0$, $b(x, z, p)$ is decreasing in z . We will prove that for some function ψ , which will

be determined at the end, $w^\pm := \varphi \pm \psi \circ d$ are global upper and lower barriers for u and Q in Γ_1 . Therefore we need to show that the following holds:

$$w^\pm = u \quad \text{on } \partial\Omega, \quad w^- \leq u \leq w^+ \quad \text{on } \partial\Gamma_1 \setminus \partial\Omega \quad \pm Qw^\pm < 0 \quad \text{in } \Gamma_1 \setminus \partial\Omega. \quad (*)$$

Assuming $\psi'(d) \geq 1$, $\psi''(d) \leq 0$ and using [3], Lemma 14.17 we first get

$$\begin{aligned} & \pm a^{ij}(Dw^\pm)D_{ij}w^\pm \\ & \leq -\psi'(d)(1 + |Dw^\pm|^2)(n-1)H_{\partial\Omega}(y) + n^2 \sup_{\Gamma_1} |D\varphi|^2 \sup_{\Gamma_1} |D^2 d| \psi'(d)^2 \\ & \quad + \psi''(d) + 3n^2 \left(1 + \sup_{\Gamma_1} |D\varphi|\right)^2 \sup_{\Gamma_1} |D^2 \varphi| \psi'(d)^2 \\ & =: -\psi'(d)(1 + |Dw^\pm|^2)(n-1)H_{\partial\Omega}(y) + c_1 \psi'(d)^2 + \psi''(d), \end{aligned}$$

where $y \in \partial\Omega$, $d(x) = |x - y|$ and $H_{\partial\Omega}$ denotes the mean curvature of $\partial\Omega$. Using this estimate in combination with

$$\begin{aligned} n|b_0(x, Dw^\pm)|(1 + |Dw^\pm|^2) & \leq 2n(h_1 + h_2) \left(1 + \sup_{\Gamma_1} |D\varphi|\right)^3 \psi'(d)^2 =: c_2 \psi'(d)^2, \\ n\psi'(d)(1 + |Dw^\pm|^2) & \leq 2n \left(1 + \sup_{\Gamma_1} |D\varphi|\right)^2 \psi'(d)^3 =: c_3 \psi'(d)^3 \end{aligned}$$

and the monotonicity of $H_1(x, z, N)$ in z we obtain for $\psi(d) \geq 0$:

$$\begin{aligned} \pm Q(w^\pm) & \leq -\psi'(d)(1 + |Dw^\pm|^2)(n-1)H_{\partial\Omega}(y) + c_1 \psi'(d)^2 + \psi''(d) \\ & \quad \mp n(1 + |Dw^\pm|^2)|Dw^\pm - D\varphi|H_1(x, w^\pm, N(Dw^\pm)) \\ & \quad \mp n(1 + |Dw^\pm|^2)b_0(x, Dw^\pm) \\ & \leq -\psi'(d)(1 + |Dw^\pm|^2) \left[(n-1)H_{\partial\Omega}(y) \pm nH_1(y, \varphi(y), \mp \gamma(y)) \right] \\ & \quad + c_3 \psi'(d)^3 |H_1(y, \varphi(y), \mp \gamma(y)) - H_1(x, \varphi(x), N(Dw^\pm(x)))| \\ & \quad + (c_1 + c_2) \psi'(d)^2 + \psi''(d). \end{aligned}$$

The first term can be estimated using (1). For the second term we get from the condition $\psi'(d) \geq 2\|\varphi\|_{C^1(\overline{\Gamma}_1)}$ and with the help of

$$|N(Dw^\pm(x)) \pm \gamma(y)| \leq 4 \left(1 + \sup_{\Gamma_1} |D\varphi|\right) \psi'(d(x))^{-1}$$

the estimate

$$\begin{aligned} & |H_1(y, \varphi(y), \mp \gamma(y)) - H_1(x, \varphi(x), N(Dw^\pm(x)))| \\ & \leq c(\Omega) \left(h_{1,x} + h_{1,z} \|\varphi\|_{C^1(\overline{\Gamma}_1)} \right) d(x) + 4c(\Omega)h_{1,N} \left(1 + \|\varphi\|_{C^1(\overline{\Gamma}_1)}\right) \psi'(d(x))^{-1} \\ & =: r d + s \psi'(d)^{-1}. \end{aligned}$$

Therefore if $\psi'(d)d \leq 1$ we have finally

$$\pm Q(w^\pm) \leq (c_1 + c_2 + c_3(r+s)) \psi'(d)^2 + \psi''(d) =: C \psi'(d)^2 + \psi''(d).$$

Choosing

$$\psi : [0, \infty) \rightarrow \mathbb{R} : x \mapsto \psi(x) := \frac{\ln(1 + \lambda x)}{1 + C}, \quad \lambda := \frac{e^{(1+C)(a+m)} - 1}{d_0}$$

we see that ψ satisfies $\psi(d) \geq 0$, $\psi'(d)d \leq 1$, $\psi''(d) < 0$ and also $\psi'(d) \geq 1 + 2\|\varphi\|_{C^1(\bar{\Gamma}_1)}$ for

$$d_0 := \frac{e^{(1+C)(a+m)} - 1}{(1+C) \left(1 + 2\|\varphi\|_{C^1(\bar{\Gamma}_0)}\right) e^{(1+C)(a+m)}} d_1.$$

From the definition of ψ and the calculations above we see that $w^\pm$ satisfy (*). From (*) we can derive that the following estimate holds:

$$\sup_{\partial\Omega} |Du| \leq \sup_{\partial\Omega} |D\varphi| + \psi'(0).$$

Thus the desired result follows. $\square$

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