Mean-value theorems and extensions of the Elliott–Daboussi theorem on Beurling’s generalized integers (I)

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It has been discovered that, in my paper [3], quoted in the title of this correction, the condition (1.9) on $\psi(x)$ of Theorem 4 and the condition in Remark 3 are incorrect in their current formulation. The mistake is originated in the condition (1.4) on the Chebyshev function $\psi(x)$ of Theorem 2 in [2] due to an incorrect citation from [1].

To derive the correction of the conditions, we begin with Beurling’s theorems. Suppose that the counting function $N(x)$ of Beurling’s generalized integers $\mathcal{N}$ satisfies

$$N(x) = x \sum_{r=1}^m A_r (\log x)^{\rho_r - 1} + O(x \log^{-\gamma} x) \quad (1)$$

with a constant $\gamma > 0$, real constants $A_1, \dots, A_m$ such that $A_m = A > 0$, and constants $\rho_1 < \rho_2 < \dots < \rho_m$ such that $\rho_m = \rho \geq 1$. Then, for $\sigma > 1$,

$$\zeta(s) := \int_{1-}^{\infty} x^{-s} dN(x) = \sum_{j=0}^{\infty} \frac{1}{n_j^s}$$

is the zeta function associated with $\mathcal{N}$. The Euler product formula implies that $\zeta(s)$ has no zeros for $\sigma > 1$. Beurling [1] showed that if $\gamma > 1 + \rho/2$ then the degree $\alpha(t)$ of a zero $1 + it$ of $\zeta(s)$ on the vertical line $\sigma = 1$ is a positive integer and that the total number of such

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zeros does not exceed ρ in the sense that

$$\sum_{t_j > 0} \alpha(t_j) \leq \rho/2.$$

Hence there exist only a finite number of such zeros $1 \pm it_j$, $j = 1, \dots, \ell$. Beurling showed [1, p. 263] also that the counting function $\pi(x)$ of the generalized primes satisfies

$$\Pi(x) \sim v.p. \int_0^x \left(\rho - 2 \sum_{j=1}^{\ell} \alpha(t_j) \cos(t_j \log u) \right) \frac{du}{\log u},$$

where $\Pi(x) = \sum \frac{1}{k} \pi(x^{1/k})$. Hence the Chebyshev function

$$\begin{aligned} \psi(x) &= \int_1^x \log u \, d\Pi(u) = \int_1^x \left(\rho - 2 \sum_{j=1}^{\ell} \alpha(t_j) \cos(t_j \log u) \right) du + o(x) \\ &= \left(\rho - 2 \sum_{j=1}^{\ell} \alpha_j \frac{t_j \sin(t_j \log x) + \cos(t_j \log x)}{1 + t_j^2} \right) x + o(x), \end{aligned}$$

where α_j denote $\alpha(t_j)$.

Therefore the condition

$$\psi(x) = \left(\rho - 2 \sum_{j=1}^{\ell} \alpha_j \frac{t_j \sin(t_j \log x) + \cos(t_j \log x)}{1 + t_j^2} \right) x + O(x(\log x)^{-M}) \quad (2)$$

with $M > M_0$ should replace the one appearing in the condition (1.4) of Theorem 2 of [2] and hence the condition (1.9) of Theorem 4 and the one in Remark 3 of [3].

With the new condition (2), Lemma 3.2 of [2] can be upheld by the argument given in the correction of [2] and no more change in [2, 3] is needed.

References

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