Mean-value theorems for multiplicative functions on Beurling's generalized integers (I)

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It has been discovered that, in my paper [2], quoted in the title of this correction, the condition (1.4) on the Chebyshev function $\psi(x)$ of Theorem 2, the same condition quoted in the abstract, and the condition on the same function $\psi(x)$ in the remark following Corollary 1 are incorrect in their current formulation. The mistake is due to an incorrect citation from [1].

To derive the correction of the conditions, we begin with Beurling's theorems. Suppose that the counting function $N(x)$ of Beurling's generalized integers $\mathcal{N}$ satisfies

$$N(x) = x \sum_{r=1}^m A_r (\log x)^{\rho_r - 1} + O(x \log^{-\gamma} x) \quad (1)$$

with a constant $\gamma > 0$, real constants $A_1, \dots, A_m$ such that $A_m = A > 0$, and constants $\rho_1 < \rho_2 < \dots < \rho_m$ such that $\rho_m = \rho \geq 1$. Then, for $\sigma > 1$,

$$\zeta(s) := \int_{1-}^{\infty} x^{-s} dN(x) = \sum_{j=0}^{\infty} \frac{1}{n_j^s}$$

is the zeta function associated with $\mathcal{N}$. The Euler product formula implies that $\zeta(s)$ has no zeros for $\sigma > 1$. Beurling [1] showed that if $\gamma > 1 + \rho/2$ then the degree $\alpha(t)$ of a zero $1 + it$ of $\zeta(s)$ on the vertical line $\sigma = 1$ is a positive integer and that the total number of such

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zeros does not exceed ρ in the sense that

$$\sum_{t_j > 0} \alpha(t_j) \leq \rho/2.$$

Hence there exist only a finite number of such zeros $1 \pm it_j$, $j = 1, \dots, \ell$. Beurling showed [1, p. 263] also that the counting function $\pi(x)$ of the generalized primes satisfies

$$\Pi(x) \sim v.p. \int_0^x \left(\rho - 2 \sum_{j=1}^{\ell} \alpha(t_j) \cos(t_j \log u) \right) \frac{du}{\log u},$$

where $\Pi(x) = \sum \frac{1}{k} \pi(x^{1/k})$. Hence the Chebyshev function

$$\begin{aligned} \psi(x) &= \int_1^x \log u \, d\Pi(u) = \int_1^x \left(\rho - 2 \sum_{j=1}^{\ell} \alpha(t_j) \cos(t_j \log u) \right) du + o(x) \\ &= \left(\rho - 2 \sum_{j=1}^{\ell} \alpha_j \frac{t_j \sin(t_j \log x) + \cos(t_j \log x)}{1 + t_j^2} \right) x + o(x), \end{aligned}$$

where α_j denote $\alpha(t_j)$.

Therefore the condition

$$\psi(x) = \left(\rho - 2 \sum_{j=1}^{\ell} \alpha_j \frac{t_j \sin(t_j \log x) + \cos(t_j \log x)}{1 + t_j^2} \right) x + O(x(\log x)^{-M}) \quad (2)$$

with $M > M_0$ should replace the one appearing in the abstract, the condition (1.4) of Theorem 2, and the one in the remark following Corollary 1 of [2].

With the new condition (2), Lemma 3.2 of [2] can be upheld by the following argument and no more change in [2] is needed. Actually, in the proof of the lemma we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{j=0}^{\infty} \Lambda(n_j) n_j^{-s} = \int_1^N x^{-s} d\psi(x) + \int_{N+}^{\infty} x^{-s} d\psi(x) = I_1 + I_2,$$

say, with I_1 estimated as before. Then

$$I_2 = I_{21} + I_{22},$$

where

$$\begin{aligned} I_{21} &= \int_{N+}^{\infty} x^{-s} d \left(\left(\rho - 2 \sum_{j=1}^{\ell} \alpha_j \frac{t_j \sin(t_j \log x) + \cos(t_j \log x)}{1 + t_j^2} \right) x \right) \\ &= \rho \frac{N^{-s+1}}{s-1} - \sum_{j=1}^{\ell} \alpha_j \left(\left(\frac{N^{-s+1+it_j}}{(s-1-it_j)(1+it_j)} + \frac{N^{-s+1-it_j}}{(s-1+it_j)(1-it_j)} \right) \right. \\ &\quad \left. + it_j \left(\frac{N^{-s+it_j}}{(s-it_j)(1+it_j)} - \frac{N^{-s-it_j}}{(s+it_j)(1-it_j)} \right) \right) = O(1) \end{aligned}$$

for $1 \leq \sigma$ and t with $|t| \geq t_\ell + 1$ or $|t \pm t_j| > d$, $j = 0, 1, \dots, \ell$, and

$$I_{22} = \int_{N+}^{\infty} x^{-s} d \left(\psi(x) - \left(\rho - 2 \sum_{j=1}^{\ell} \alpha_j \frac{t_j \sin(t_j \log x) + \cos(t_j \log x)}{1 + t_j^2} \right) x \right) \\ \ll 1 + |s| \int_N^{\infty} x^{-\sigma-1} O(x(\log x)^{-M}) dx \ll 1 + \frac{|t|(\log N)^{-M+1}}{M-1}$$

for $1 \leq \sigma \leq 2$ by integration by parts. Letting $\log N = |t|^{\frac{1}{M}}$, the result of the lemma follows.

References

1. Beurling, A.: Analyse de la loi asymptotique de la distribution des nombres premiers généralisés I. Acta Math. **68**, 255–291 (1937)
2. Zhang, W.-B.: Mean-value theorems for multiplicative functions on Beurling's generalized integers (I). Math. Z. **251**, 359–391 (2005). doi:[10.1007/s00209-005-0807-8](https://doi.org/10.1007/s00209-005-0807-8)