Erratum

Formality of Donaldson submanifolds

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Regarding the article “Formality of Donaldson submanifolds” published in Mathematische Zeitschrift, vol. 250 (2005), pp. 149–175, Lemma 2.7 was incorrect. This page is considered as the erratum. The following is the correct version of Lemma 2.7.

Lemma 2.7 *Let M be a manifold with minimal model $(\bigwedge V, d)$. Then M is s -formal if and only if there is a map of differential algebras*

$$\varphi : \left(\bigwedge V^{\leq s}, d \right) \longrightarrow (H^*(M), d = 0),$$

such that the map $\varphi^: H^*(\bigwedge V^{\leq s}, d) \longrightarrow H^*(M)$ induced on cohomology is equal to the map $\iota^*: H^*(\bigwedge V^{\leq s}, d) \longrightarrow H^*(\bigwedge V, d) = H^*(M)$ induced by the inclusion $\iota: (\bigwedge V^{\leq s}, d) \longrightarrow (\bigwedge V, d)$.*

In particular, $\varphi^: H^i(\bigwedge V^{\leq s}) \longrightarrow H^i(M)$ is an isomorphism for $i \leq s$, and a monomorphism for $i = s + 1$. So, if M is simply connected, then the dual of the real homotopy vector space $\pi_i(M) \otimes \mathbb{R}$ is isomorphic to $V^i = W^i$ for any $i \leq s$, $(\bigwedge W, d)$ being the minimal model of $(H^*(M), d = 0)$.*

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Proof Since $(\bigwedge V, d)$ is a minimal model of M , we know that there is a morphism $\rho: (\bigwedge V, d) \rightarrow (\Omega M, d)$ inducing an isomorphism ρ^* on cohomology. Thus to prove the *only if* part, it is sufficient to show that there is a map of differential algebras

$$\psi: \left(\bigwedge V^{\leq s}, d \right) \longrightarrow \left(H^*(\bigwedge V), d = 0 \right),$$

such that the map $\psi^*: H^*(\bigwedge V^{\leq s}) \rightarrow H^*(\bigwedge V)$ coincides with the map $\iota^*: H^*(\bigwedge V^{\leq s}) \rightarrow H^*(\bigwedge V)$ induced by the inclusion $\iota: (\bigwedge V^{\leq s}, d) \rightarrow (\bigwedge V, d)$. Then the map φ given by $\varphi = \rho^* \circ \psi$ satisfies the conditions that we need.

We define $\psi(x) = [x]$ for $x \in C^i$ and $\psi(x) = 0$ for $x \in N^i$, $i \leq s$. We extend ψ to an algebra map $\psi: \bigwedge V^{\leq s} \rightarrow H^*(\bigwedge V)$ by multiplicativity. We see that s -formality implies that ψ commutes with the differentials, as follows. Let $x \in \bigwedge V^{\leq s}$. Then, from [1, Eq. (1)], it follows that dx decomposes $dx = a + b$ with $a \in \bigwedge C^{\leq s}$ and $b \in N^{\leq s} \cdot (\bigwedge V^{\leq s})$. Since a is closed, so is $b = dx - a$ and hence exact by s -formality. Therefore $a = dx - b$ is exact as well, and $\psi(dx) = \psi(a) = [a] = 0$. This shows that $\psi: (\bigwedge V^{\leq s}, d) \rightarrow (H^*(\bigwedge V), d = 0)$ is a map of differential algebras.

Moreover, if $x \in \bigwedge V^{\leq s}$ is closed, decompose $x = a + b$ with $a \in \bigwedge C^{\leq s}$, $b \in N^{\leq s} \cdot (\bigwedge V^{\leq s})$. Then $b = x - a$ is closed, hence exact by s -formality. So $\psi^*[x] = [\psi(x)] = \psi(x) = \psi(a) = [a] = [a + b] = [x]$, hence $\psi^* = \iota^*$, as required.

To prove the *if* part, suppose that we have a map $\varphi: (\bigwedge V^{\leq s}, d) \rightarrow (H^*(M), d = 0)$ satisfying $\varphi^* = \iota^*$. We want to find an s -formal model for M , i.e., $\bigwedge V = \bigwedge \hat{V}$ such that $\hat{V}^i = \hat{C}^i \oplus \hat{N}^i$ satisfies the conditions of Definition 2.2. Such a model is the one constructed in the pp. 154–155 of [1]. $\square$

Remark The correction in Lemma 2.7 only affects the proof of Lemma 2.11 of [1]. There, we must proceed as follows. To show that M is s -formal if M_i ($i = 1, 2$) is s -formal, we consider the map φ defined in p. 157 of [1] by $\varphi = \varphi_1 \otimes \varphi_2$. Since M_i is s -formal, Lemma 2.7 implies that the map of differential algebras φ_i is such that the induced map in cohomology φ_i^* equals the map induced by the inclusion $\bigwedge V_i^{\leq s} \hookrightarrow \bigwedge V_i$. Thus φ satisfies the conditions of Lemma 2.7, and hence M is s -formal.

Suppose now that $M = M_1 \times M_2$ is s -formal. Then by Lemma 2.7, there exists a map of differential algebras

$$\varphi: \bigwedge V^{\leq s} \longrightarrow H^*(M),$$

such that $\varphi^*: H^*(\bigwedge V^{\leq s}) \rightarrow H^*(\bigwedge V) \cong H^*(M)$ is the map induced by the inclusion $\bigwedge V^{\leq s} \hookrightarrow \bigwedge V$. Define φ_1 as in p. 157 of [1]. Then, φ_1^* equals the map induced in cohomology by the composition $\bigwedge V_1^{\leq s} \hookrightarrow \bigwedge V^{\leq s} \hookrightarrow \bigwedge V = \bigwedge (V_1 \oplus V_2) \rightarrow \bigwedge V_1$, and this map is the inclusion $\bigwedge V_1^{\leq s} \hookrightarrow \bigwedge V_1$. Therefore, φ_1 satisfies the conditions of Lemma 2.7.

Reference

1. Fernández, M., Muñoz, V.: Formality of Donaldson submanifolds. *Math. Zeit.* **250**, 149–175 (2005)