Erratum to: Smooth extensions of functions on separable Banach spaces

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We fill a gap in the proof of Theorem 1 in the original paper by providing a sharper version of Lemma 1 in that paper.

We will use the same notation as in that paper. In the second part of the proof of Theorem 1 [where one wants to show that if one requires f to be Lipschitz on Y then one can choose the approximate extension g to be Lipschitz with $\text{Lip}(g) \leq C\text{Lip}(f)$], we need to use the fact that there is a function $\delta_k : X \rightarrow \mathbb{R}$ such that

- (1) $|T_k(x) - F(x) - \delta_k(x)| < 2^{-i-2}\varepsilon M_k^{-1}$ for all $x \in B_{2s_k}$
- (2) $\|\delta'_k(y)\|_{X^*} < \varepsilon/8$ for all $y \in B_{2s_k} \cap Y$, and
- (3) $\text{Lip}(\delta_k) \leq C_0\text{Lip}(T_k - F) \leq 2C_0\text{Lip}(F)$.

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In that proof we claimed that the existence of a function δ_k satisfying (1) and (2) was guaranteed by Lemma 1. In fact the version of Lemma 1 in the paper gives the existence of a function δ_k satisfying (2), but property (1) only holds on $B_{2s_k} \cap Y$ for this function [and it is no use trying to apply Lemma 1 to $T_k - F$ on $Y = X$ because that way we would obtain another function δ_k satisfying (1) and (3), but not (2)].

As well, the equation

$$\text{Lip}(\Delta_i) \leq \max \left\{ M\text{Lip}(f) + \frac{\varepsilon}{8}, C_0\text{Lip}(f) \right\},$$

on page 292 of the paper is wrong, as it should have read

$$\text{Lip}(\Delta_i) \leq M\text{Lip}(f) + 3C_0\text{Lip}(f).$$

This is a minor mistake since the latter is true as long as δ_k satisfies property (3) above.

Next we state and prove a sharp version of Lemma 1 of the original paper which allows one to find functions δ_k satisfying all three properties above, thus filling the gap in the proof of Theorem 1. The proof of this sharp version of Lemma 1 can be easily deduced from the version stated in the paper, which we give below.

Lemma 1 *Let X be a separable Banach space with a C^1 smooth norm. There exists a constant $C_0 \geq 1$ such that, for every subspace $Y \subseteq X$, every Lipschitz function $f : X \rightarrow \mathbb{R}$, and every $\varepsilon > 0$, there exists a C^1 -smooth function $K : X \rightarrow \mathbb{R}$ such that*

- (1) $|f(x) - K(x)| < \varepsilon$ for all $x \in X$,
- (2) $\text{Lip}(K) \leq C_0\text{Lip}(f)$, and
- (3) $\|K'(y)\|_{X^*} \leq C_0\text{Lip}(f|_Y)$ for all $y \in Y$ (in particular the Lipschitz constant of the restriction of K to Y is of the order of the Lipschitz constant of the restriction of f to Y).

Proof Let us denote $\eta = \text{Lip}(f|_Y)$ and $L = \text{Lip}(f)$. Obviously $\eta \leq L$.

Consider the function $\varphi : X \rightarrow \mathbb{R}$ defined by

$$\varphi(x) = \inf\{f(y) + \eta\|x - y\| : y \in Y\},$$

which is an η -Lipschitz extension of the restriction $f|_Y$ to the subspace Y . By the version of Lemma 1 in the paper applied to φ with $Y = X$, there exists a $C\eta$ -Lipschitz, C^1 function $g : X \rightarrow \mathbb{R}$ such that $|g - \varphi| \leq \varepsilon/3$ on X . Similarly there exists a CL -Lipschitz, C^1 function $h : X \rightarrow \mathbb{R}$ such that $|h - f| \leq \varepsilon$ on X . Here C is the constant (not depending on f or ε) provided by the old statement of Lemma 1 in the paper.

Consider the quotient space X/Y . It is standard that $(X/Y)^*$ is isometric to the closed subspace $Y^\perp$ of X^* (see e.g., [2, Proposition 2.7]), and since X^* is separable then so is $(X/Y)^*$. Since in a space with a separable dual every equivalent norm can be uniformly approximated in the unit ball by C^1 smooth equivalent norms (see [1]), X/Y has an equivalent C^1 smooth norm (which we will also denote $\|\cdot\|$) with the property that

$$\text{dist}(x, Y) \leq \|q(x)\| \leq 2\text{dist}(x, Y) \quad \text{for all } x \in X.$$

In particular the function $x \mapsto \|q(x)\|$ is 2-Lipschitz on X , as is easily checked.

Let $r = \varepsilon/6L > 0$, and let $\alpha : \mathbb{R} \rightarrow [0, 1]$ be a C^1 function such that $\alpha(t) = 1$ for $t \leq r$, $\alpha(t) = 0$ for $t \geq 2r$, and $\text{Lip}(\alpha) \leq 2/r = 12L/\varepsilon$. Define $\beta = 1 - \alpha$. Then $\{\alpha, \beta\}$ form a $12L/\varepsilon$ -Lipschitz, C^1 smooth partition of unity on $\mathbb{R}$ with $\text{supp}(\alpha) \subset (-\infty, 2r)$ and $\text{supp}(\beta) \subset (r, +\infty)$.

Define a function $K : X \rightarrow \mathbb{R}$ by

$$K(x) = g(x)\alpha(\|q(x)\|) + h(x)\beta(\|q(x)\|).$$

Note that if $\alpha(\|q(x)\|) \neq 0$ then $\|q(x)\| < 2r$, hence also $\text{dist}(x, Y) < 2r$, and there exists $y_x \in Y$ with $\|x - y_x\| < 2r$. Then, since g is η -Lipschitz and f is L -Lipschitz, $|g - \varphi| \leq \varepsilon/3$ and $\varphi|_Y = f|_Y$, we have

$$\begin{aligned} |g(x) - g(y_x)| &\leq \eta\|x - y_x\| \leq \eta 2r \leq \varepsilon/3, \\ |g(y_x) - f(y_x)| &\leq \varepsilon/3, \\ |f(y_x) - f(x)| &\leq L\|x - y_x\| \leq L 2r \leq \varepsilon/3, \end{aligned}$$

from which we get

$$|g(x) - f(x)| \leq \varepsilon \text{ whenever } \alpha(\|q(x)\|) \neq 0.$$

And we always have $|h - f| \leq \varepsilon$. Therefore we can estimate

$$\begin{aligned} |K(x) - f(x)| &= |(g(x) - f(x))\alpha(\|q(x)\|) + (h(x) - f(x))\beta(\|q(x)\|)| \\ &\leq |g(x) - f(x)|\alpha(\|q(x)\|) + |h(x) - f(x)|\beta(\|q(x)\|) \\ &\leq \varepsilon(\alpha(\|q(x)\|) + \beta(\|q(x)\|)) = \varepsilon, \end{aligned}$$

which shows (1).

As for the Lipschitz constant of K , assuming $\alpha(\|q(x)\|) \neq 0$, we have

$$\begin{aligned} |K(x) - K(z)| &= |g(x)\alpha(\|q(x)\|) + h(x)\beta(\|q(x)\|) - g(z)\alpha(\|q(z)\|) - h(z)\beta(\|q(z)\|)| \\ &= |(g(x) - f(x))(\alpha(\|q(x)\|) - \alpha(\|q(z)\|)) \\ &\quad + (h(x) - f(x))(\beta(\|q(x)\|) - \beta(\|q(z)\|)) \\ &\quad + (g(x) - g(z))\alpha(\|q(z)\|) + (h(x) - h(z))\beta(\|q(z)\|)| \\ &\leq \sup\{|f(u) - g(u)| : \alpha(\|q(u)\|) \neq 0\} 2\text{Lip}(\alpha)\|x - z\| \\ &\quad + \sup\{|h(v) - f(v)| : v \in X\} 2\text{Lip}(\beta)\|x - z\| \\ &\quad + \text{Lip}(g)\|x - z\|\alpha(\|q(z)\|) + \text{Lip}(h)\|x - z\|\beta(\|q(z)\|) \\ &\leq 2\varepsilon\text{Lip}(\alpha)\|x - z\| + 2\varepsilon\text{Lip}(\beta)\|x - z\| \\ &\quad + \max\{\text{Lip}(g), \text{Lip}(h)\}(\alpha(\|q(z)\|) + \beta(\|q(z)\|))\|x - z\| \\ &\leq 2\varepsilon\left(\frac{12L}{\varepsilon} + \frac{12L}{\varepsilon}\right)\|x - z\| + CL\|x - z\| = (48L + CL)\|x - z\|. \end{aligned}$$

The same estimation holds in the case $\alpha(\|q(z)\|) \neq 0$ (just replace x with z and use the symmetry of the norm). And in the case $\alpha(\|q(x)\|) = 0 = \alpha(\|q(z)\|)$ we have $|K(x) - K(z)| = |h(x) - h(z)| \leq CL\|x - z\|$. In any case we see that

$$|K(x) - K(z)| \leq (48L + CL)\|x - z\| \quad \text{for all } x, z \in X,$$

which shows (2) with $C_0 := 48 + C$.

Finally, since $K = g$ on the open neighborhood $\{x \in X : \text{dist}(x, Y) < r/2\}$ of Y , and $\text{Lip}(g) \leq C\eta$, it is clear that

$$\|K'(y)\|_{X^*} = \|g'(y)\|_{X^*} \leq C\eta \leq C_0\eta \quad \text{for all } y \in Y.$$

□

We finally note that for the proof of Theorems 4 and 5 of the original paper, where $Y \subset X$ is assumed to be (merely) a closed subset, that one can again employ the technique of the above Lemma 1 by simply replacing the function $x \rightarrow \|q(x)\|$ with a uniform approximation of the distance function to Y by a Lipschitz, C^1 smooth map.

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