Commuting elements and spaces of homomorphisms

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We correct some cohomology calculations appearing in [1].

In the preprint [2], M. Crabb has pointed out that two of the cohomology calculations in [1], namely Theorems 1.4 and 4.12 are not consistent with the results in his paper. Here we outline how to correct the results in our paper.

Consider the Mayer-Vietoris spectral sequence in integral cohomology associated to the covering $X[3] = X_{1,2} \cup X_{1,3} \cup X_{2,3}$ where $X[3]$ denotes the non-commuting triples in $SU(2)$ and the $X_{i,j}$ correspond to the non-commutativity of the i th and j th coordinates. It will have as its E_1 term

$$E_1^{p,q} \cong \begin{cases} H^q(X_{1,2}, \mathbb{Z}) \oplus H^q(X_{1,3}, \mathbb{Z}) \oplus H^q(X_{2,3}, \mathbb{Z}) & \text{if } p = 0 \\ H^q(X_{1,2} \cap X_{1,3}, \mathbb{Z}) \oplus H^q(X_{1,2} \cap X_{2,3}, \mathbb{Z}) \oplus H^q(X_{1,3} \cap X_{2,3}, \mathbb{Z}) & \text{if } p = 1 \\ H^q(X_{123}, \mathbb{Z}) & \text{if } p = 2 \\ 0 & \text{if } p \geq 3. \end{cases}$$

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The differentials on the E_1 -page are determined as alternating sums of the maps induced by inclusions. The cohomology of the spaces $X_{i,j}$ and their intersections were computed in [1]; there are homotopy equivalences

$$X_{j,k} \simeq SO(3) \times SU(2), \quad X_{j,k} \cap X_{i,k} \simeq SO(3) \times \mathbb{S}^1, \quad X_{123} \simeq SO(3) \times [\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^1].$$

The spectral sequence begins with $d_1^{1,q} : E_1^{i,q} \rightarrow E_1^{i+1,q}$ for $i = 0, 1, 2$. These differentials can be computed explicitly, using our knowledge of these spaces and their intersections. First we must make a digression to consider the fundamental group.

By projecting $X_{1,2} \cap X_{1,3}$ onto its first two coordinates we obtain a split fibration onto the non-commuting pairs $X[2] \simeq SO(3)$ where the fiber is homotopic to $\mathbb{S}^1$. This product can be used to represent the generators of its fundamental group geometrically. Now consider the projection onto the first and third coordinates; again this yields a map onto $X[2]$. However, the previous fiber, homotopic to the circle, is now mapped homotopically non-trivially, as it can be identified with the fiber of the projection of $X[2]$ onto $SU(2) - \{\pm 1\}$ (which is simply further projection onto the first coordinate). Thus, what we have shown is that if $\pi_1(X_{12} \cap X_{13}) \cong \mathbb{Z}w \oplus \mathbb{Z}/2y$ with these generators corresponding to the geometric decomposition above, then the inclusions into X_{12} and X_{13} induce maps $f_1, f_2 : \mathbb{Z}w \oplus \mathbb{Z}/2y \rightarrow \mathbb{Z}/2v$ such that $f_1(w) = 0, f_1(y) = v$ and $f_2(w) = v, f_2(y) = v$. An immediate consequence of this is that $X[3]$ is simply connected, as can be seen by using the Seifert–Van Kampen Theorem for the same decomposition we are considering; indeed in $\pi_1(X[3])$ the homomorphisms induced by the inclusions of the intersections force the identification of the three generators but they also identify $f_1(w) = 0$ with $f_2(w) = v$ and thus the fundamental group is trivial.

Back to the calculation: the key ingredients are that $E_2^{1,i} = 0$ for all $i \geq 0$ and that

$$E_2^{2,i} \cong \begin{cases} 0 & \text{if } i = 0 \\ \mathbb{Z}/2 & \text{if } i = 1 \\ 0 & \text{if } i = 2 \\ \mathbb{Z}/2 & \text{if } i = 3 \\ \mathbb{Z}/2 & \text{if } i = 4 \\ 0 & \text{if } i \geq 5 \end{cases} \quad E_2^{0,i} \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ \mathbb{Z}/2 & \text{if } i = 2 \\ \mathbb{Z}^4 & \text{if } i = 3 \\ 0 & \text{if } i = 4 \\ (\mathbb{Z}/2)^3 & \text{if } i = 5 \\ \mathbb{Z}^6 & \text{if } i = 6 \\ 0 & \text{if } i \geq 7. \end{cases}$$

It turns out that there is a single non-zero differential on the E_2 page, namely $d_2 : E_2^{0,2} \rightarrow E_2^{2,1}$, which is an isomorphism. This differential is forced by the fact that $X[3]$ is simply connected. There are no other non-trivial differentials: the only possible one would be $E_2^{0,5} \rightarrow E_2^{2,4}$; however these terms necessarily appear in the cohomology of $X[3]$ as through duality the term $E_2^{0,5}$ represents the top cohomology of the three copies of the top pieces of $Hom(\mathbb{Z}^2, SU(2))$ which must split off stably from $Hom(\mathbb{Z}^3, SU(2))$. Using this we can obtain a corrected version of Theorem 5.4 in [1]:

$$H_i(X[3], \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, 1, 2 \\ \mathbb{Z}^4 & \text{if } i = 3 \\ (\mathbb{Z}/2)^4 & \text{if } i = 4 \\ \mathbb{Z}/2 & \text{if } i = 5 \\ \mathbb{Z}^3 & \text{if } i = 6 \\ 0 & \text{if } i \geq 7. \end{cases}$$

Applying Poincaré–Lefschetz duality to the pair $(SU(2)^3, \text{Hom}(\mathbb{Z}^3, SU(2)))$ now yields the revised calculation:

Theorem 0.1 *The integral cohomology of the space of (ordered) commuting triples in $SU(2)$ is given by*

$$H^i(\text{Hom}(\mathbb{Z}^3, SU(2)), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ \mathbb{Z}^3 & \text{if } i = 2 \\ \mathbb{Z}^3 \oplus \mathbb{Z}/2 & \text{if } i = 3 \\ (\mathbb{Z}/2)^4 & \text{if } i = 4 \\ \mathbb{Z} & \text{if } i = 5 \\ 0 & \text{if } i \geq 6. \end{cases}$$

This agrees with the results implied in Crabb’s paper, and is thus a corrected version of Theorem 1.4 in [1]. As a consequence of these corrections, our identification of $\text{Hom}(\mathbb{Z}^3, SU(3))/S_3(SU(2))$ with $SU(2) \wedge (\mathbb{S}^6 - SO(3))$ must be modified; as shown by Crabb this piece is in fact homotopy equivalent to the stunted projective space $\mathbb{R}P^5/\mathbb{R}P^3$. The corresponding modification must be made in Examples 6.9 and 6.10.

Similarly there is a mistake in the calculation for the cohomology of $\text{Hom}(\mathbb{Z} \times F_2, SU(2))$ (Proposition 4.11) where F_2 is the free group on two generators. The cohomology of the space $SU(2)^3 - \text{Hom}(\mathbb{Z} \times F_2, SU(2))$ is actually given by

$$H^i(SU(2)^3 - \text{Hom}(\mathbb{Z} \times F_2, SU(2)), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ \mathbb{Z}^3 & \text{if } i = 3 \\ 0 & \text{if } i = 4 \\ \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } i = 5 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 6 \\ 0 & \text{if } i \geq 7. \end{cases}$$

A differential in the spectral sequence used to calculate this was overlooked. Using duality as explained in [1] this leads to the corrected version of Theorem 4.12:

Theorem 0.2 *The integral cohomology of $\text{Hom}(\mathbb{Z} \times F_2, SU(2))$ is given by*

$$H^i(\text{Hom}(\mathbb{Z} \times F_2, SU(2)), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ \mathbb{Z}^2 & \text{if } i = 2 \\ \mathbb{Z}^4 & \text{if } i = 3 \\ (\mathbb{Z}/2)^2 & \text{if } i = 4 \\ \mathbb{Z} & \text{if } i = 5 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 6 \\ 0 & \text{if } i \geq 7. \end{cases}$$

From this analysis it also follows that the corresponding claim (Theorem 4.13) made about the cohomology of $\text{Hom}(\mathbb{Z} \times F_r, SU(2))$ where F_r is the free group on r generators is incorrect; however the correct calculation can be obtained from [2].

We also take this opportunity to correct a typo: in the proof for Proposition 3.1 the correct value for $H_1(\Gamma_2, \mathbb{Z})$ is $\mathbb{Z}/10$, not $\mathbb{Z}/20$.

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References

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