On the topological stable rank of non-selfadjoint operator algebras

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Published online: 1 April 2008
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Erratum to: Math. Ann.
DOI 10.1007/s00208-007-0180-5

There is an error in our paper in the proof of Theorem 2.11. The purpose of this note is to make the necessary corrections. We thank Prof. You Qing Ji for pointing out the mistake, which occurs in the last line of the proof: *At this point, we can apply the second half of the proof of Theorem 2.1.* In fact, some non-trivial modifications of that proof are required.

Theorem 2.11 *Suppose that $\mathcal{N}$ is a nest ordered like ω , all of whose atoms are finite dimensional. Let A_n , $n \geq 1$, denote the atoms of $\mathcal{N}$, and let $r_n = \dim A_n$ for $n \geq 1$. Set $R(k) = \max_{1 \leq i \leq k} r_i$ for $k \geq 1$. Suppose that there exists a $\gamma > 0$ and an integer $J > 0$ such that*

$$R((k+1)J) \geq (1+\gamma)R(kJ) \text{ for all } k \geq 1.$$

The online version of the original article can be found under doi:[10.1007/s00208-007-0180-5](https://doi.org/10.1007/s00208-007-0180-5).

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Then

$$\text{ltsr}(\mathcal{T}(\mathcal{N})) = \infty \quad \text{and} \quad \text{rtsr}(\mathcal{T}(\mathcal{N})) = 2.$$

Proof The proof as published remains the same up to the last paragraph. We also define $Q_k = \sum_{i=1}^k F_k$.

We wish to multiply A' and B' on the right by invertible elements of $\mathcal{T}(\mathcal{N})$ to make the block diagonal equal to the identity and the first super-diagonal with respect to the block decomposition equal to 0. To accomplish this, let $\Delta_1(T) = \sum_{k \geq 1} F_k T F_{k+1}$ for $T \in \mathcal{T}(\mathcal{N})$. Then define

$$X = D_a^{-1} \exp(-\Delta_1(A') D_a^{-1}) \quad \text{and} \quad Y = D_b^{-1} \exp(-\Delta_1(B') D_b^{-1}).$$

Then $A'X = I + X_0$ and $B = I + Y_0$ where X_0 and Y_0 lie in the ideal

$$\mathcal{T}'_0(\mathcal{N})^2 = \left\{ T \in \mathcal{T}(\mathcal{N}) : \sum_{k \geq 1} F_k T F_k = 0 = \sum_{k \geq 1} F_k T F_{k+1} \right\}.$$

So we may write $X_0 = \sum_{k \geq 3} Q_{k-2} X_0 F_k$ and $Y_0 = \sum_{k \geq 3} Q_{k-2} Y_0 F_k$.

Arguing as in the proof of Theorem 2.1, we see that

$$\text{rank}(X_0 F_k) + \text{rank}(Y_0 F_k) \leq \frac{1}{2} \text{rank } F_{k-1}.$$

Thus we can choose projections $P_k \leq F_k$ with $\text{rank } P_k \leq \frac{1}{2} \text{rank } F_{k-1}$ so that $X_0 F_k = X_0 P_k$ and $Y_0 F_k = Y_0 P_k$. The problem with a trivial modification of the argument is that we can no longer assert that P_k and the partial isometries constructed in the proof of Theorem 2.1 belong to the nest algebra. However, because we have made some extra room to maneuver, we can construct partial isometries U_k mapping $P_k \mathfrak{H}$ into $P_{k-1}^\perp F_{k-1} \mathfrak{H}$; and these operators evidently lie in the nest algebra. Set $U = \sum_{k \geq 3} U_k$. Moreover

$$X_0 U^* = \sum_{k \geq 3} Q_{k-2} X_0 P_k \sum_{j \geq 3} P_j U_j^* = \sum_{k \geq 3} Q_{k-2} (X_0 P_k U_k^*) F_{k-1}.$$

So this operator also lies in $\mathcal{T}(\mathcal{N})$. Similarly $Y_0 U^*$ belongs to the nest algebra. Moreover the range of U is orthogonal to the range of $P = \sum_{k \geq 3} P_k$, and therefore $X_0 U = Y_0 U = 0$.

Now let $A'' = A' + \delta X_0 U^* X^{-1}$ and $B'' = B' - \delta Y_0 U^* Y^{-1}$, where $\delta > 0$ is chosen to be sufficiently small so that $\|A - A''\| < \varepsilon$ and $\|B - B''\| < \varepsilon$. Observe that

$$\begin{aligned} & A'' X (I - \delta^{-1} U) + B'' Y (I + \delta^{-1} U) \\ &= (I + X_0 + \delta X_0 U^*) (I - \delta^{-1} U) + (I + Y_0 - \delta Y_0 U^*) (I + \delta^{-1} U) \\ &= (I + X_0 + \delta X_0 U^* - \delta^{-1} U - X_0) + (I + Y_0 - \delta Y_0 U^* + \delta^{-1} U - Y_0) \\ &= 2I + \delta (X_0 - Y_0) U^*. \end{aligned}$$

With δ sufficiently small, this is invertible, and hence $[A'' \ B'']$ is right invertible.

Therefore $\text{rtsr}(\mathcal{T}(\mathcal{N})) = 2$. $\square$