Constructing infinitely many smooth structures on $3\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$

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The proofs in [4] are incomplete when $n = 10$ and $n = 12$, as they assume that the numerical Godeaux surface in [1] is simply connected as was claimed in that paper. From [2], we have learned that this surface may not be simply connected after all.

However, we can still construct an infinite family of pairwise nondiffeomorphic irreducible 4-manifolds that are homeomorphic to $3\mathbb{CP}^2\#10\overline{\mathbb{CP}^2}$. One such construction can be given by slightly modifying the construction of exotic $3\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$ in [5]. In the notation of [5], we can find an embedded sphere of self-intersection -16 in the 4-manifold $Z_{K_1, K_2, K_3}\#4\overline{\mathbb{CP}^2}$. This sphere is gotten by resolving the intersection of a pseudo-section of Z_{K_1, K_2, K_3} and a single fishtail fiber (instead of two fishtail fibers as in [5]) and then blowing up at the three positive double points of the pseudo-section and at the double point of the fishtail fiber. Using 12 (-2) -spheres in the I_{16} fiber, we can then extend this (-16) -sphere to the configuration $C_{14,1}$. Let M_{K_1, K_2, K_3} denote the result of the rational blowdown of this configuration of spheres. The proof of Proposition 3.2 in [5] applies almost verbatim to show that M_{K_1, K_2, K_3} is homeomorphic to $3\mathbb{CP}^2\#10\overline{\mathbb{CP}^2}$. By setting $K_1 = K_2 = K_3 = T_n$, the n -twist knot and mimicking the proof of Theorem 3.3 in [5], we can show that $M_n = M_{T_n, T_n, T_n}$ ($n = 1, 2, 3, \dots$) are pairwise nondiffeomorphic irreducible 4-manifolds with inequivalent nonzero Seiberg–Witten invariants. Unfortunately, we do not obtain infinitely many exotic symplectic manifolds for $n = 10$ case, as M_n is nonsymplectic when $n \geq 2$.

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For exotic $3\mathbb{CP}^2 \# 12\overline{\mathbb{CP}^2}$, we can just use the manifold M in Sect. 4 of [3]. (Note that our original approach in [4] was to use the manifold P in Sect. 4 of [3].) In other words, we let $X_{12} = M$, $M_1 = W$, and $M_2 = \overline{R} = E(1)_K \# \overline{\mathbb{CP}^2}$ in Table 1 of [4]. We will be done if we can verify conditions (i)–(iii) on p. 268 of [4] for these new choices. We can argue exactly the same way as in Sect. 3 of [4]. We let $f = T_{2,4} \subset W - (D^2 \times \Sigma_1)$ and condition (i) holds immediately. To satisfy condition (ii), we just need to choose the gluing diffeomorphism ψ so that it maps the generator g_4 to one of the vanishing cycles on a torus fiber of the elliptic fibration $E(1)_K$. Just as in the proof of Lemma 3 in [4] we have

$$\pi_1(W - (T_{2,4} \cup \Sigma_1)) / \langle \pi_1(\Sigma_1^{\parallel}) \rangle = 1.$$

Since the exceptional sphere in $\overline{R}$ intersects $\overline{\Sigma}_2$ transversely once, we have $\pi_1(\overline{R} - (D^2 \times \overline{\Sigma}_2)) = \pi_1(\overline{R}) = 1$. Since

$$X_{12} - T_{2,4} = [W - (T_{2,4} \cup (D^2 \times \Sigma_1))] \cup [\overline{R} - (D^2 \times \overline{\Sigma}_2)],$$

it follows from Seifert–Van Kampen’s theorem that $\pi_1(X_{12} - T_{2,4}) = 1$. Thus condition (iii) holds as well. Now we are free to apply the machinery in [4] and conclude that all of the theorems in [4] still hold when $n = 12$.

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