The Cucker–Smale Equation: Singular Communication Weight, Measure-Valued Solutions and Weak-Atomic Uniqueness

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Abstract

The Cucker–Smale flocking model belongs to a wide class of kinetic models that describe a collective motion of interacting particles that exhibit some specific tendency, e.g. to aggregate, flock or disperse. This paper examines the kinetic Cucker–Smale equation with a singular communication weight. Given a compactly supported measure as an initial datum we construct a global in time weak measure-valued solution in the space $C_{weak}(0, \infty; \mathcal{M})$. The solution is defined as a mean-field limit of the empirical distributions of particles, the dynamics of which is governed by the Cucker–Smale particle system. The studied communication weight is $\psi(s) = |s|^{-\alpha}$ with $\alpha \in (0, \frac{1}{2})$. This range of singularity admits the sticking of characteristics/trajectories. The second result concerns the weak-atomic uniqueness property stating that a weak solution initiated by a finite sum of atoms, i.e. Dirac deltas in the form $m_i \delta_{x_i} \otimes \delta_{v_i}$, preserves its atomic structure. Hence these coincide with unique solutions to the system of ODEs associated with the Cucker–Smale particle system.

1. Introduction

Flocking, swarming, aggregation—there is a multitude of actual real-life phenomena that from the mathematical point of view can be interpreted as one of these concepts. The mathematical description of collective dynamics of self-propelled agents with nonlocal interaction originates from one of the basic equations of the

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kinetic theory—Vlasov’s equation from 1938. Recently it was noted that such models provide a way to describe a wide range of phenomena that involve interacting agents with a tendency to aggregate their certain qualities. This approach proved to be useful and the language of aggregation now appears not only in the models of groups of animals but also in the description of seemingly unrelated phenomena such as the emergence of common languages in primitive societies, the distribution of goods or reaching a consensus among individuals [3, 26, 28, 35]. The main class of kinetic equations associated with aggregation models reads as follows:

$$\partial_t f + v \cdot \nabla f + \operatorname{div}_v[(k * f)f] = 0, \quad (1)$$

where $f = f(x, v, t)$ is usually interpreted as the density of those particles that at time t have position x and velocity v . The function k is the kernel of the potential governing the motion of particles. It is responsible for the non-local interaction between particles and depending on it the particles may exhibit various tendencies like flocking, aggregation or dispersal. The common properties of kernels k required in most models include Lipschitz continuity and boundedness. In such a case the particle system associated with (1) is well posed, the characteristic method can be performed for (1) and one can pass from the particle system to the kinetic equation by the mean-field limit procedure. Our goal is to consider singular k that is neither Lipschitz continuous nor bounded and to refine the mean-field limit to be applicable in such a scenario. Singularity of the kernel admits the possibility of the sticking of characteristics, which causes a loss of backward uniqueness in time for (1). The present paper studies the case of the Cucker–Smale (CS) flocking model.

In [14] from 2007, Cucker and Smale introduced a model for the flocking of birds associated with the following system of ODEs:

$$\begin{cases} \frac{d}{dt}x_i = v_i, \\ \frac{d}{dt}v_i = \sum_{j=1}^N m_j(v_j - v_i)\psi(|x_j - x_i|), \end{cases} \quad (2)$$

where N is the number of the particles while $x_i(t)$, $v_i(t)$ and m_i denote the position and velocity of i th particle at time t and its mass, respectively. Function $\psi : [0, \infty) \rightarrow [0, \infty)$, usually referred to as *the communication weight*, is non-negative and nonincreasing and can be vaguely interpreted as the perception of particles. The communication weight plays the crucial role in our investigations. It characterizes mathematical difficulties and determines possible physical interpretations of the studied model.

As $N \rightarrow \infty$ the particle system is replaced by the following Vlasov-type equation:

$$\begin{aligned} \partial_t f + v \cdot \nabla f + \operatorname{div}_v[F(f)f] &= 0, \quad x \in \mathbb{R}^d, \quad v \in \mathbb{R}^d, \\ F(f)(x, v, t) &:= \int_{\mathbb{R}^{2d}} \psi(|y - x|)(w - v)f(y, w, t)dw dy, \end{aligned} \quad (3)$$

which can be written as (1) with $k(x, v) = v\psi(|x|)$. As mentioned before we are considering (3) with a singular kernel

$$\psi(s) = \begin{cases} s^{-\alpha} & \text{for } s > 0, \\ \infty & \text{for } s = 0, \end{cases} \quad \alpha > 0. \quad (4)$$

Before we proceed with a more detailed statement of our goals let us briefly introduce the current state of the art for models of flocking and the motivations behind studying such models with singular kernels. The literature on aggregation models associated with Vlasov-type equations of the form (1) is rich, thus we mention only a few examples of the most popular branches of the field. Here we find analysis of time asymptotics (see e.g. [21]) and pattern formation (see e.g. [20,34]) or analysis of the models with additional forces that simulate various natural factors (see e.g. [10,17]—deterministic forces or [13]—stochastic forces). The other variations of the model include forcing particles to avoid collisions (see e.g. [11]) or to aggregate under the leadership of certain individuals (see e.g. [12]). A well-rounded analysis of a model that includes effects of attraction, repulsion and alignment is presented in [4]. The story of the CS model should probably begin with [36] by VICSEK et al., where a model of flocking with nonlocal interactions was introduced and it is widely recognized to be up to some degree an inspiration for [14]. Since 2007 the CS model with a regular communication weight of the form

$$\psi_{cs}(s) = \frac{K}{(1 + s^2)^{\frac{\beta}{2}}}, \quad \beta \geq 0, \quad K > 0 \quad (5)$$

was extensively studied in the directions similar to those of more general aggregation models (i.e. collision avoiding, flocking under leadership, asymptotics and pattern formation as well as additional deterministic or stochastic forces—see [2,9,19,22,29,32]). Particularly interesting from our point of view is the case of passage from the particle system (2) to the kinetic equation (3), which in case of the regular communication weight was done, for example, in [23] or [24]. For a more general overview of the passage from microscopic to mesoscopic and macroscopic descriptions in aggregation models of the form (1) we refer to [5,15,16].

In the paper [23] from 2009 the authors considered the CS model with the singular weight (4) obtaining asymptotics for the particle system, but even the basic question of the existence of solutions remained open till later years. It turned out that system (2) possesses drastically different qualitative properties depending on whether $\alpha \in (0, 1)$ or $\alpha \in [1, \infty)$. More precisely, in [1], the authors observed that for $\alpha \geq 1$ the trajectories of the particles exhibit a tendency to avoid collisions, which they used to prove conditional existence and uniqueness of smooth solutions to the particle system. On the other hand in [30] the author proved the existence of so called *piecewise weak* solutions to the particle system with $\alpha \in (0, 1)$ and gave an example of solution that experienced not only collisions of the trajectories but also sticking (i.e. two different trajectories could start to coincide at some point). This dichotomy is an effect of integrability (or of the lack of thereof) of ψ in a neighborhood of 0. It is also the reason why the approach to the CS model should vary depending on α . One of the latest contributions to this topic is [6], where the

authors showed local in-time well-posedness for the kinetic equation (3) with a singular communication weight (4) and with an optional nonlinear dependence on the velocity in the definition of $F(f)$. They also presented a thorough analysis of the asymptotics for this model. The other more recent addition is [31], where the author proved the existence and uniqueness of $W^{1,1}$ strong solutions to the particle system (2) with a singular weight (4) and $\alpha \in (0, \frac{1}{2})$.

Our present results are strongly dependent on the regularity of solutions to the particle system, hence we assume throughout the paper, after [31], that $\alpha \in (0, \frac{1}{2})$. However, let us mention that these results (in particular Proposition 31) can be generalized to models of flocking with nonlinear dependence on the velocity in the alignment force term $F(f)$. Such models are considered for example in [6], where the force term takes the form

$$F_\beta(f) := \int_{\mathbb{R}^{2d}} \psi(|x - y|) \nabla_v \phi(v - w) f(y, w, t) dy dw,$$

with $\phi(s) \sim s^\beta$ for $\beta > \frac{3-d}{2}$. In particular, ϕ with large β reduces the impact of the singularity of ψ . This approach allows one to push the singularity of the communication weight ψ up to 1. However, this interesting extension is outside of the scope of this paper.

As a final remark we suggest a possible application of this type of mathematical model. The phenomenon of sticking of trajectories creates the possibility of the creation of Dirac measure solutions from regular distributions, as in the case of [8] for the equations of attraction/repulsion. Here we think about the formation of polymers from a solution of monomers. Still this qualitative nature of the CS model is an open question. However it would give a nice description of polymerization. Due to the possibility of the sticking of trajectories, the kinetic structure of the considered model is more complex than in [25] where the authors considered singular kinetic models that preserved L^∞ norm of the distribution of the particles.

The propose of the model with regular weight is to suppress the distant interactions between particles. However from the modeling point of view it is often convenient to also amplify the local interactions, which was done for example in [27], by introducing a different nonsymmetric CS-type model known as the Mosch-Tadmor model. Singular communication weight in the CS model can also be viewed as a less effective yet easier to analyze way of emphasizing the local interactions between particles.

1.1. Main Goal—The CS Model with a Singular Communication Weight

We aim to solve the issue of well posedness for (3) with the singular weight (4) and initial data from the class of Radon measures. The goal is twofold:

- Prove existence and analyze continuous dependence on the initial data. The existence is obtained by approximating measure solutions to (3) by solutions to particle system (2) using the mean-field limit, similarly to [23]. The key obstacle is a lack of sufficient information about the continuous dependence for solutions to particle system (2), thus we are not allowed to apply the standard approach. To the best of our knowledge the most that can be assumed is solvability of (2) in the

$W^{1,1}$ —class that has been proved in [31]. Therefore by results from [31] we restrict our considerations to $\alpha \in (0, \frac{1}{2})$ and modify the mean-field limit procedure to that regularity.

– Prove the weak-atomic uniqueness property to system (3). This means that any weak solution is unique and corresponds to a solution to the particle system (2) provided it is initiated from a finite sum of Dirac’s deltas $m_i \delta_{x_i(t)} \otimes \delta_{v_i(t)}$. Thus any atomic solution is preserved by kinetic equation (3), and since it is generated by particle system (2), it is unique. This result is a step in the direction of stability of solutions to (3). We elaborate further on the difficulties in obtaining stability in Remark 32.

The paper is organized as follows. In Section 2 we provide the preliminary definitions and tools required throughout the paper, in particular we introduce the weak formulation for (3). In Section 3 we state the main result along with the overview of the proof. In Section 4 we present the proof of the existence part, while in Section 5 we establish the weak-atomic uniqueness of the solutions. The paper is closed with “Appendix A”, where one can find more technical/tedious elements of proofs and a simple proof of uniqueness to particle system (2).

2. Preliminaries and Notation

In this section we present the basic toolset and definitions of considered problems. Let $\Omega \subset \mathbb{R}^d$ be an arbitrary domain with $d \in \mathbb{N}$. By $W^{k,p}(\Omega)$ we denote the Sobolev’s space of the functions with up to k th weak derivative belonging to space $L^p(\Omega)$, while by $C(\Omega)$ and $C^1(\Omega)$ we denote the space of continuous and continuously differentiable functions, respectively. Hereinafter, $B((x_0, v_0), R) = B_{x,v}((x_0, v_0), R)$ denotes a ball in $\mathbb{R}^{2d}$ centered in (x_0, v_0) with radius R . On the other hand, $B_x(x_0, R)$ and $B_v(v_0, R)$ denote balls in $\mathbb{R}^d$ with radius R centered in x_0 and v_0 , respectively. For any positive a , by $aB_v(v_0, R)$ we understand a homothetic transformation of $B_v(v_0, R)$, i.e., $B_v(v_0 a, Ra)$. Throughout the paper letter C denotes a generic positive constant that may change in the same inequality and usually depends on other constants that are of less importance from the point of view of estimates.

2.1. Bounded-Lipschitz Distance

The standard tool used in the studies of Vlasov-type models are Wasserstein metrics. They provide convenient topologies on the space of Radon measures and are often used in the research on CS model. The general definition of Wasserstein distances is relatively complex, however Wasserstein-1 (or Kantorovich–Rubinstein) distance is, in the sense explained in [33] p. 26, equivalent to the easily defined bounded-Lipschitz distance. Due to such convenient representation we use only the bounded-Lipschitz distance but it is worth noting that in similar cases (e.g. [6]) other Wasserstein metrics can be applied.

Definition 21. (Bounded-Lipschitz distance) For any probabilistic measures μ and ν we define

$$d(\mu, \nu) := \sup_g \left| \int_{\Omega} g d\mu - \int_{\Omega} g d\nu \right|,$$

where the supremum is taken over all bounded and Lipschitz continuous functions g , such that $\|g\|_{\infty} \leq 1$ and $Lip(g) \leq 1$.

In the above definition $\|g\|_{\infty}$ and $Lip(g)$ represent the L^{∞} norm and Lipschitz constant of g . We also need to distinguish between spaces of measures with different topologies, i.e. we denote $\mathcal{M} = \mathcal{M}(\Omega) = (\mathcal{M}, TV)$ as the space of finite Radon measures defined on Ω with *total variation* topology and we denote $(\mathcal{M}, d)$ as the space of finite Radon measures defined on Ω with bounded-Lipschitz distance topology. The importance of the space $(\mathcal{M}, d)$ comes from the prime difference between the bounded-Lipschitz distance and the total variation. Namely, for $x_1 \neq x_2$, $TV(\delta_{x_1} - \delta_{x_2}) = 2$, while $d(\delta_{x_1}, \delta_{x_2}) \leq 2|x_1 - x_2|$. In particular, if $x_n \rightarrow x$ in Ω then $\delta_{x_n} \rightarrow \delta_x$ in d , which is not the case in TV .

In our considerations a crucial role is played by

$$\mathcal{M}_+ := \left\{ \mu \in \mathcal{M} : \mu \text{ is nonnegative} \right\},$$

both with TV and $d(\cdot, \cdot)$ topology. If Ω is a compact subset of $\mathbb{R}^d$, then $\mathcal{M}$ is isomorphic to $(C_b(\Omega))^*$.¹ There is a convenient relation between the weak * topology in $(C_b(\Omega))^*$ and the topology generated by the bounded-Lipschitz distance on $\mathcal{M}$ that we present below.

Definition 22. We say that a sequence $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$ is *tight* if for all $\epsilon > 0$ there exists a compact $K_{\epsilon} \subset \subset \Omega$ such that for all $n \in \mathbb{N}$ we have

$$|\mu_n|(\Omega \setminus K_{\epsilon}) < \epsilon,$$

where $|\mu|$ is the total variation measure of μ .

Proposition 21. Suppose that Ω is compact and let $\{\mu_n\}_{n \in \mathbb{N}}$ be a tight sequence in $\mathcal{M}$ and let $\mu \in \mathcal{M}$. Then $\mu_n \rightarrow \mu$ weakly * if and only if $d(\mu_n, \mu) \rightarrow 0$ and $\{\mu_n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{M}$.

Proof. The proof of Proposition 21 is a modification of the proof of Theorem 2.7 from [18]. $\square$

Remark 21. Since in our considerations Ω is compact, our situation is more straightforward than in a general case (e.g. in [18]). In such a case the so called weak convergence of measures (otherwise known as the narrow convergence) is equivalent to the weak * convergence in $(C_b(\Omega))^*$.

¹ This nice property does not hold in general. For example if $\Omega = \mathbb{R}^d$ then $(C_b(\Omega))^*$ is isomorphic to the space of regular bounded finitely additive measures, while $(C_0(\Omega))^*$ is isomorphic to $\mathcal{M}$.

In our considerations we deal only with nonnegative measures, which equipped with the bounded-Lipschitz distance are a complete metric space.

Proposition 22. *The space $(\mathcal{M}_+, d)$ is a complete metric space.*

The following corollary is the very reason for which the bounded-Lipschitz distance is applied. It serves as a topology with pointwise sequential compactness for measure-valued functions. What we mean is that if $f_n : [0, T] \mapsto (\mathcal{M}, d)$ and f_n are uniformly bounded in $L^\infty(0, T; (\mathcal{M}, TV))$, then for each $t \in [0, T]$ the sequence $f_n(t)$ is relatively compact in $(\mathcal{M}, d)$, which is one of the assumptions of the Arzela–Ascoli theorem.

Corollary 21. *Let $\{\mu_n\}_{n \in \mathbb{N}}$ be a sequence bounded in $(\mathcal{M}_+, TV)$ with supports contained in some given ball. Then there exists a $(\mathcal{M}_+, d)$ -convergent subsequence $\{\mu_{n_k}\}$.*

Proof. Since the supports of μ_n are uniformly bounded then there exist compact set $K \subset \subset \Omega \subset \subset \mathbb{R}^d$ such that

$$\bigcup_n \text{supp } \mu_n \subset K.$$

Thus we may treat $\{\mu_n\}_{n \in \mathbb{N}}$ as a sequence of measures defined on a compact Ω . Then $(\mathcal{M}, TV)$ is isomorphic to $(C_b(\Omega))^*$, which is a separable normed vector space then by Banach-Alaoglu theorem the set $\{\mu_n : n = 1, 2, \dots\}$ is sequentially weakly $*$ compact in $(C_b(\Omega))^*$. Therefore there exists a measure $\mu \in (C_b(\Omega))^*$ such that up to a subsequence μ_n converges to μ weakly $*$ in $(C_b(\Omega))^*$. Proposition 21 implies that the weak $*$ $(C_b(\Omega))^*$ convergence is equivalent to the convergence in $d(\cdot, \cdot)$ if only $\{\mu_n\}_{n \in \mathbb{N}}$ is tight (which is true since μ_n vanish on $\Omega \setminus K$). Thus μ_n converges to μ in $d(\cdot, \cdot)$. Finally, since by Proposition 22 $(\mathcal{M}_+, d)$ is a complete space, we conclude that actually $\mu \in (\mathcal{M}_+, d)$ and the proof are finished. $\square$

Lastly we present a useful lemma related to the bounded-Lipschitz distance.

Lemma 21. *Let $d(\cdot, \cdot)$ be the bounded-Lipschitz distance. Then for any $\mu, \nu \in \mathcal{M}$ and any bounded and Lipschitz continuous function g , we have*

$$\left| \int_{\Omega} g d\mu - \int_{\Omega} g d\nu \right| \leq \max\{\|g\|_{\infty}, Lip(g)\} d(\mu, \nu).$$

Proof. The proof of this lemma belongs to the standard theory and can be found, for example, in [23]. $\square$

2.2. Measure-Valued Solutions to the Kinetic Equation

We introduce the following weak formulation for (3):

Definition 23. Let $T > 0$. We say that f is a weak solution to (3) with the initial data $f_0 \in \mathcal{M}_+$, such that $\text{supp } f_0 \subset B(\mathcal{R}_0)$ with $\mathcal{R}_0 > 0$ if:

1. $f \in L^\infty(0, T; \mathcal{M}_+)$ and $\partial_t f \in L^p(0, T; (C_b^1(B(\mathcal{R})))^*)$ for some $p > 1$;

2. $\text{supp } f(t) \subset B(\mathcal{R})$ for $t \in (0, T]$ for some positive constant $\mathcal{R}$;
3. The following identity holds:

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^{2d}} f[\partial_t \phi + v \nabla \phi] dx dv dt + \int_0^T \int_{\mathbb{R}^{2d}} F(f) f \nabla_v \phi dx dv dt \\ &= - \int_{\mathbb{R}^{2d}} f_0 \phi(\cdot, \cdot, 0) dx dv \end{aligned} \quad (6)$$

for all $\phi \in \mathcal{G}$, where

$$\mathcal{G} := \left\{ \phi \in C^1([0, T] \times \mathbb{R}^{2d}) : \partial_t \phi, \nabla \phi, \nabla_v \phi \text{ are bounded} \right. \\ \left. \text{and Lipschitz continuous and } \phi \text{ has a compact support in } t \right\};$$

4. The function $g(x, y, v, w, t) := (w - v)\psi(|x - y|)$ is integrable with respect to the measure $f(x, v, t) \otimes f(y, w, t) dx dv dy dw$, i.e. term $F(f)$ is defined as a measure with respect to the measure $f dx dv$. In particular, by Fubini's theorem, the integral

$$\int_0^T \int_{\mathbb{R}^{2d}} F(f) f \nabla_v \phi dx dv dt = \int_0^T \int_{\mathbb{R}^{4d}} g \nabla_v \phi f \otimes f dx dv dy dw dt$$

is bounded and the term $\text{div}_v[F(f)f]$ is well defined as a distribution;

5. For each pair of concentric balls $B((x_0, v_0), r) \subset B((x_0, v_0), R) \subset \mathbb{R}^{2d}$, the following statement holds: if

$$\text{supp } f_0 \cap B((x_0, v_0), R) \subset B((x_0, v_0), r), \quad (7)$$

then there exists $T^* \in [0, T]$, such that

$$\text{supp } f(t) \cap B\left((x_0, v_0), \frac{3R+r}{4}\right) \subset B\left((x_0, v_0), \frac{r+R}{2}\right) \quad (8)$$

for all $t \in [0, T^*]$.

Remark 22. There is a natural question of the correspondence between solutions to (3) in the sense of Definition 23 and solutions to (2). The answer to this question is to some extent positive, which we explain below. Let

$$f_0(x, v) := \sum_{i=1}^N m_i \delta_{x_{i,0}}(x) \otimes \delta_{v_{i,0}}(v) \quad (9)$$

with $\sum_{i=1}^N m_i = 1$. Then f_0 defines an initial data $x_0 = (x_{1,0}, \dots, x_{N,0})$, $v_0 = (v_{1,0}, \dots, v_{N,0})$ for the system of ODE's (2). For this system let (x, v) be a sufficiently smooth solution.² Then the function

$$f(x, v, t) := \sum_{i=1}^N m_i \delta_{x_i(t)}(x) \otimes \delta_{v_i(t)}(v) \quad (10)$$

² By "sufficiently smooth" we mean for instance that $(x, v) \in W^{1,1}([0, T])$, which is a reasonable assumption in view of Theorem 32.

is a solution of (3) in the sense of Definition 23 with the initial data f_0 . Indeed, if we plug f defined in (10) into (6), by a simple use of a chain rule, we obtain

$$\begin{aligned}
 & \int_0^T \sum_{i=1}^N m_i \left((\partial_t \phi)(x_i, v_i, t) + v_i (\nabla \phi)(x_i, v_i, t) \right) \\
 & + \sum_{i,j=1}^N m_i m_j \psi(|x_i - x_j|) (v_j - v_i) (\nabla_v \phi)(x_i, v_i, t) \, dt \\
 & = \int_0^T \sum_{i=1}^N m_i \frac{d}{dt} \phi(x_i(t), v_i(t), t) \, dt = - \sum_{i=1}^N m_i \phi(x_{i,0}, v_{i,0}, t) \\
 & = - \int_{\mathbb{R}^{2d}} f_0 \phi(\cdot, \cdot, 0) \, dx \, dv
 \end{aligned}$$

for all $\phi \in \mathcal{G}$. The converse assertion that a solution to (3) in the sense of Definition 23 corresponds to a solution of (2) is also true provided that the initial data are of the form (9). However, the proof is much more involved and it is in fact the second part of the paper.

Definition 24. We say that f is an atomic solution if it is of form (10).

Definition 25. In the case of solutions of particle system (2) we say that the i th and j th particles *collide* at the time t if and only if $x_i(t) = x_j(t)$, and we say that they *stick together* at the time t if and only if $x_i(t) = x_j(t)$ and $v_i(t) = v_j(t)$.

Remark 23. Throughout the paper whenever we consider a solution of system (2) we assume that the number of particles is constant in time. However if in our proofs (particularly in Section 5) any two particles stick together, then we tend to treat them as a single particle with a bigger mass, thus reducing the total number of particles in the system. This is justified by the fact that according to [31] solutions to (2) with $\alpha \in (0, \frac{1}{2})$ are unique (see Theorem 32).

Remark 24. Point 5 of Definition 23 requires some explanation. Its purpose is to establish a local control over the propagation of the support of f . Basically if we can divide the support of f_0 into two parts of distance $R - r$, then in some small time interval $[0, T^*]$ the distances between those parts is no lesser than $\frac{R-r}{4}$.

Remark 25. In Section 5 we frequently test our weak solution by various test functions that at the first glance may seem not admissible. In particular we test with functions with derivatives in x and v not necessarily Lipschitz continuous. This is however correct since by a simple density argument we may test (6) by C^1 functions. Moreover we are allowed to test (6) by functions that are not compactly supported in time. In such case we get a version of (6) with both endpoints of the time interval,

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^{2d}} f[\partial_t \phi + v \nabla \phi] dx dv dt + \int_0^T \int_{\mathbb{R}^{2d}} F(f) f \nabla_v \phi dx dv dt \\
&= \int_{\mathbb{R}^{2d}} f(t) \phi(\cdot, \cdot, t) dx dv - \int_{\mathbb{R}^{2d}} f_0 \phi(\cdot, \cdot, 0) dx dv.
\end{aligned} \tag{11}$$

The justification of the above equation is standard and can be found in the proof of Proposition 31,(v) in “Appendix A”.

3. Main Result

The main result of the paper is the following:

Theorem 31. *Let $0 < \alpha < \frac{1}{2}$. For any compactly supported initial data $f_0 \in \mathcal{M}$ and any $T > 0$, Cucker–Smale’s flocking model (3) admits at least one solution in the sense of Definition 23. Moreover if f_0 is of the form (9) then f is atomic of form (10), hence it is a unique measure-valued solution to (3).*

The part of the proof concerning the issue of existence follows from the analysis of approximation by atomic solutions originating from sums of Dirac’s deltas, which correspond in the sense of Remark 22 to solutions of (2). The main idea behind this approach is twofold. First, we have better regularity of solutions of (2) for $\alpha < \frac{1}{2}$. This was connected to results from [31], where we proved that for $0 < \alpha < \frac{1}{2}$, system (2) admits a unique $W^{1,1}([0, T])$ solution (x, v) , which by Remark 22 corresponds to a solution of (3) in the sense of Definition 23. Since, in fact, $\alpha \in (0, \alpha_0)$ for some $\alpha_0 < \frac{1}{2}$, we can prove that (x, v) is bounded in $W^{1,p}([0, T])$ for some $p > 1$. Such a bound provides equicontinuity of sequences of solutions of (2), which allows us to extract a convergent subsequence. The second element of the proof is to change the way of looking at the alignment force term

$$\int_0^T \int_{\mathbb{R}^{2d}} F(f_n) f_n \nabla_v \phi dx dv dt, \tag{12}$$

where if $f_n \rightharpoonup f$ then it is not clear whether $F(f_n) f_n \rightharpoonup F(f) f$. It is useful to look at (12) as

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^{4d}} \psi(|x - y|)(w - v) \nabla_v \phi d\mu_n dt \\
& \text{for } d\mu_n := f_n(x, v, t) \otimes f_n(y, w, t) dx dv dy dw.
\end{aligned}$$

The uniqueness part of Theorem 31 is explained and proved in Section 5. The main idea is to analyze possible support of measure-valued solutions for initial atomic configurations. It is related to point 5 of Definition 6 of weak solutions to (3). We emphasize that we derive the propagation of the support described in point 5 from the behavior of the approximate solutions. The question of whether these properties are exhibited by a wider class of possible weak solutions is outside the scope of this paper; such generalization would require an essentially different approach to the construction of solutions.

Remark 31. Throughout the remaining part of the paper we assume, without a loss of generality, that the total mass of f_0 equals to 1.

Let us give an overview of the proof of existence. Suppose that f_0 is a given, compactly supported measure belonging to $\mathcal{M}$ and assume without a loss of generality that

$$\text{supp } f_0 \subset B(R_0), \quad (13)$$

where $B(R_0)$ is a ball centered at 0 with radius R_0 . For such f_0 we take $f_{0,\epsilon} \in \mathcal{M}$ of the form

$$f_{0,\epsilon} = \sum_{i=1}^N m_i \delta_{x_{0,i}^\epsilon} \otimes \delta_{v_{0,i}^\epsilon}, \quad (14)$$

which corresponds to the initial data $(x_{0,\epsilon}, v_{0,\epsilon})$ to a particle system (2). Moreover we assume that

$$d(f_{0,\epsilon}, f_0) \xrightarrow{\epsilon \rightarrow 0} 0,$$

and that the support of $f_{0,\epsilon}$ is contained in $B(2R_0)$. The existence of such approximation is standard (we refer for example to the beginning of Section 6.1 in [23] for the details). Now suppose that $(x_\epsilon^n, v_\epsilon^n)$ is a solution to (2) with the communication weight

$$\psi_n(s) := \min\{\psi(s), n\}, \quad (15)$$

subjected to the initial data $(x_{0,\epsilon}, v_{0,\epsilon})$, which by Remark 22 means that

$$f_\epsilon^n = \sum_{i=1}^N m_i \delta_{x_{\epsilon,i}^n} \otimes \delta_{v_{\epsilon,i}^n} \quad (16)$$

is a solution of (3) with the initial data $f_{0,\epsilon}$. Our goal is to converge with ϵ to 0 and with n to ∞ to obtain a solution f of equation (3) subjected to the initial data f_0 .

The proof of existence can be summarized in the following steps:

- Step 1. Given $T > 0$, for each ϵ and n , we prove existence of a solution f_ϵ^n corresponding to the initial data $f_{0,\epsilon}$ and satisfying various regularity properties;
- Step 2. We take a sequence $f_n = f_\epsilon^n$ for $\epsilon = \frac{1}{n}$. Due to the conservation of mass and the regularity proved in step 1 we extract a subsequence f_{n_k} converging in $L^\infty(0, T; (\mathcal{M}_+, d))$ to some $f \in L^\infty(0, T; \mathcal{M}_+)$;
- Step 3. We converge with each term in the weak formulation for f_{n_k} to the respective term in the weak formulation for f . This can be easily done for each term except the alignment force term, i.e. the term

$$\int_0^T \int_{\mathbb{R}^{2d}} F_{n_k}(f_{n_k}) f_{n_k} \nabla_v \phi dx dv dt; \quad (17)$$

Step 4. Analysis of the alignment force term is not straightforward. Formally we multiply an L_p function by a measure, thus the result is not well defined at the very first sight. We replace (17) with an n_k -independently regular modification of the form

$$\int_0^T \int_{\mathbb{R}^{2d}} F_m(f_{n_k}) f_{n_k} \nabla_v \phi dx dv dt.$$

The error between the alignment force term and its substitute will be controlled in terms of m and uniformly with respect to n_k ;

Step 5. For such subsequence we converge with the substitute alignment force term to

$$\int_0^T \int_{\mathbb{R}^{2d}} F_m(f) f \nabla_v \phi dx dv dt;$$

Step 6. We are then left with converging with the substitute alignment force term to the original alignment force term, i.e. with $m \rightarrow \infty$. We show that $F(f)$ is a measure with respect to the measure $df = f dx dv$.

Step 7. We finish the proof by making sure that each and every point of Definition 23 is satisfied by our candidate for the solution.

Remark 32. Before we proceed further, let us compare our strategy to the mean-field limit used for the CS model with regular weight like in [23]. For Lipschitz continuous ψ , through standard argumentation, one can easily show well-posedness for the particle system (2). This includes Lipschitz continuous dependence of solutions with respect to the perturbations of the initial data. Such stability can in turn be translated into Lipschitz continuous dependence of measure-valued solutions with respect to perturbations of initial measures. Thus inequality

$$d(f_1, f_2) \leq C(T)d(f_{0,1}, f_{0,2}) \quad (18)$$

can be obtained, where f_1 and f_2 are two solutions of the type (10), with initial data $f_{0,1}$ and $f_{0,2}$ of the form (9). Then stability ensures the possibility of extracting a convergent subsequence as presented after Remark 13. If one aims to apply such reasoning to the case with singular weight it turns out that the constant $C(T)$ from (18) depends on the Lipschitz constant of ψ on $[\delta, \infty)$, where δ is the lower bound of the distances between particles. However even for the initial data, the distance between particles δ converges to 0 as $\epsilon \rightarrow 0$. Thus the Lipschitz constant of ψ on $[\delta, \infty)$ converges to infinity and so does $C(T)$. There are ways to overcome this difficulty at least locally in time like in [6], but for a global in time result stability of the type (18) does not seem to work since particles may eventually collide causing a blowup of $C(T)$. For the discussion on the possibility of collisions between particles in the singular case of $\alpha \in (0, 1)$ we refer to [30], while in case of $\alpha \geq 1$ we refer to [7].

Let us state some various properties of the approximative solutions f_ϵ^n . It is in fact the first step of the proof (as presented above), but since it is self-contained and quite lengthy we will present it in a form of separate proposition, the proof of which can be found in “Appendix A”.

Proposition 31. *Given $T > 0$. Let $f_{0,\epsilon}$ be of the form (9). Then for each $n = 1, 2, \dots$, there exists a unique solution f_ϵ^n to kinetic equation (3) that corresponds³ to a smooth and classical solution (x^n, v^n) of particle system (2). Moreover there exist n and ϵ independent constants $M > 0$ and $p > 1$, such that the following conditions are satisfied:*

- (i) *For all $t \in [0, T]$ and all n and ϵ the total mass of f_ϵ^n i.e. the value $\int_{\mathbb{R}^{2d}} f_\epsilon^n dx dv$ is equal to 1;*
- (ii) *The support of f_ϵ^n is contained in a ball $B(\mathcal{R})$, where $\mathcal{R} := 2R_0(T + 1)$;*
- (iii) *We have*

$$\int_0^T \sum_{i=1}^{N_n} m_i |\dot{v}_{i,\epsilon}^n|^p dt + \int_0^T \sum_{i,j=1}^{N_n} m_i m_j \psi_n^p(|x_{i,\epsilon}^n - x_{j,\epsilon}^n|) |v_{i,\epsilon}^n - v_{j,\epsilon}^n|^p dt \leq M(\mathcal{R});$$

- (iv) *We have*

$$\int_0^T \sum_{i,j=1}^{N_n} m_i m_j \psi_n^p(|x_{i,\epsilon}^n - x_{j,\epsilon}^n|) |v_{i,\epsilon}^n - v_{j,\epsilon}^n| dt \leq M(\mathcal{R});$$

- (v) *For each Lipschitz continuous and bounded $g : \mathbb{R}^{2d} \rightarrow \mathbb{R}$, we have*

$$\left\| \frac{d}{dt} \int_{\mathbb{R}^{2d}} g f_\epsilon^n dx dv \right\|_{L^p([0,T])} \leq M_g(\text{Lip}(g), \mathcal{R}).$$

Remark 33. Point (iii) of Proposition 31 implies in particular that the sequence $(x_\epsilon^n, v_\epsilon^n)$ is uniformly bounded in $W^{1,p}([0, T])$. We mention this to keep the continuity with the idea of the proof presented at the beginning of this section.

Remark 34. It is worthwhile noting that, since by (iii) from Proposition 31 the derivative of velocity $\dot{v}$ is uniformly integrable, then

$$|v_i^n(t) - v_i^n(0)| \leq \int_0^t |\dot{v}_i^n| ds \leq \omega(t) \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

Moreover the function ω is independent of i and n .

Proposition 31 is similar and was inspired by Theorem 3.1 from [31] that we present below for the reader's convenience.

Theorem 32. *Let $\alpha \in (0, \frac{1}{2})$ be given. Then for all $T > 0$ and arbitrary initial data there exists a unique solution (x, v) to (2) with communication weight given by (4). Moreover $v \in W^{1,1}([0, T])$, and thus $x \in W^{2,1}([0, T])$.*

The proof of the above theorem can be found in [31]. Moreover its existence part follows directly from Theorem 31 and Remark 22. Furthermore our argumentation from Section 5 can be used to simplify the uniqueness part of the proof of Theorem 32. We present the simplified proof in “Appendix A” for the sake of completeness.

³ See Remark 22.

4. Proof of Theorem 31 (Existence)

In this section we follow the steps presented in the previous section and prove the existence part of Theorem 31.

Step 1. From the very beginning we fix $T > 0$. Proposition 31 and Remark 22 ensure the existence of f_ϵ^n with properties (i)–(v) from Proposition 31. We solve particle system (2) with initial data (14) in the time interval $[0, T]$ under the assumption that the communication weight is in form (15). By Proposition 31 we are ensured that for some $p > 1$,

$$\|f_\epsilon^n\|_{L^\infty(0,T;\mathcal{M})} = 1, \quad \|F_n(f_\epsilon^n)f_\epsilon^n\|_{L^p(0,T;\mathcal{M})} \leq M(T).$$

Step 2. We take $\epsilon = \frac{1}{n}$ and denote $f_n := f_{\frac{1}{n}}^n$. Since f_n is of the form (16) it is clear that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f_n dx dv = \sum_{i=1}^{N_n} m_{i,n} = 1.$$

For each n the function f_n may be treated as a mapping from $[0, T]$ into the metric space $(\mathcal{M}_+, d)$. For the purpose of showing that f_n has a convergent subsequence we use the Arzela–Ascoli theorem. We make sure f_n is a bounded and equicontinuous sequence of functions with a relatively compact pointwise sequences $f_n(t)$. Uniform boundedness of f_n is implied by the conservation of mass, while the relative compactness of $f_n(t)$ follows from the uniform boundedness of $f_n(t)$ in TV topology and Corollary 21. Finally in order to prove equicontinuity of f_n we take arbitrary $s, t \in [0, T]$ and arbitrary Lipschitz continuous bounded function g with $Lip(g) \leq 1$ and $\|g\|_\infty \leq 1$ and use estimation (v) from Proposition 31 to write

$$\left| \int_{\mathbb{R}^{2d}} g(f_n(s) - f_n(t)) dx dv \right| = \left| \int_t^s \frac{d}{dr} \int_{\mathbb{R}^{2d}} g f_n dx dv dr \right| =: \omega(|s - t|). \quad (19)$$

Point (v) of Proposition 31 states that functions $t \mapsto \frac{d}{dt} \int_{\mathbb{R}^{2d}} g f_n(t) dx dv$ are uniformly bounded in $L^p([0, T])$ for some $p > 1$, which in particular means that they are uniformly integrable. On the other hand it implies that the function ω is a good modulus of uniform continuity for the left-hand side of (19). Now since this estimation does not depend on the choice of g (only on the choice of $Lip(g)$), it is also valid for the supremum over all g , which implies that

$$d(f_n(s), f_n(t)) \leq \omega(|s - t|).$$

The above inequality proves that the sequence of functions $t \mapsto f_n(t)$ is equicontinuous as a mapping from $[0, T]$ to $(\mathcal{M}, d)$ (recall the bounded–Lipschitz distance defined in (21)). Thus the sequence f_n satisfies the assumptions of Arzela–Ascoli theorem. Therefore there exists $f \in L^\infty(0, T; \mathcal{M}_+)$, such that up to a subsequence

$$\|d(f_n, f)\|_\infty \rightarrow 0.$$

By (ii) from Proposition 31 it is implied that the support of f is included in $B(\mathcal{R})$.

Step 3. After a brief look at the weak formulation for f_n i.e. (6), we understand that since $f_n \rightarrow f$ in $L^\infty(0, T; (\mathcal{M}_+, d))$, then in particular for $\phi \in \mathcal{G}$, we have

$$\int_0^T \int_{\mathbb{R}^{2d}} f_n [\partial_t \phi + v \nabla \phi] dx dv dt \rightarrow \int_0^T \int_{\mathbb{R}^{2d}} f [\partial_t \phi + v \nabla \phi] dx dv dt,$$

and

$$\int_{\mathbb{R}^{2d}} f_{0, \frac{1}{n}} \phi(\cdot, \cdot, 0) dx dv \rightarrow \int_{\mathbb{R}^{2d}} f_0 \phi(\cdot, \cdot, 0) dx dv,$$

and the only problem is with the second term on the left-hand side of (6), i.e. the alignment force term

$$\int_0^T \int_{\mathbb{R}^{2d}} F_n(f_n) f_n \nabla_v \phi dx dv dt. \quad (20)$$

Step 4. To deal with the problem of convergence with the alignment force term we replace it in the following manner:

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^{2d}} f_n [\partial_t \phi + v \nabla \phi] dx dv dt + \int_0^T \int_{\mathbb{R}^{2d}} F_m(f_n) f_n \nabla_v \phi dx dv dt \\ = - \int_{\mathbb{R}^{2d}} f_{0, \frac{1}{n}} \phi(\cdot, \cdot, 0) dx dv + \mathcal{J}, \end{aligned}$$

where

$$\mathcal{J} := \int_0^T \int_{\mathbb{R}^{2d}} (F_m(f_n) - F_n(f_n)) f_n \nabla_v \phi dx dv dt$$

for

$$F_m(f_n)(x, v, t) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi_m(|x - y|)(w - v) f_n(y, w, t) dy dw.$$

However, as mentioned at the beginning of Section 3, instead of looking at (20) as an integral of a product of $F_n(f_n)$ with f_n , we are going to see it as an integral of

$$g_n(x, y, w, v) := \psi_n(|x - y|)(w - v) \nabla_v \phi(x, v, t) \quad (21)$$

with respect to the measure

$$d\mu_n(x, y, w, v, t) := f_n(x, v, t) \otimes f_n(y, w, t) dx dv dy dw.$$

By Fubini's theorem we have

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^{2d}} F_n(f_n) f_n \nabla_v \phi dx dv dt = \\ = \int_0^T \int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} \psi_n(|x - y|)(w - v) f(y, w, t) dy dw \right) \end{aligned}$$

$$\begin{aligned} & \nabla_v \phi(x, v, t) f(x, v, t) dx dv dt \\ &= \int_0^T \int_{\mathbb{R}^{4d}} g_n d\mu_n dt \end{aligned}$$

and a similar identity holds for $\int_0^T \int_{\mathbb{R}^{2d}} F_m(f_n) f_n \nabla_v \phi dx dv dt$. Therefore

$$\mathcal{J} = \int_0^T \int_{\mathbb{R}^{4d}} (g_m - g_n) d\mu_n dt.$$

Moreover we have

$$g_m - g_n = 0$$

in the set $\{(x, y, w, v) : |x - y| > \max\{m^{-\frac{1}{\alpha}}, n^{-\frac{1}{\alpha}}\}\}$, which provided that $n > m$, implies that

$$|g_m - g_n| \leq |g_n| \chi_{\{(x, y, w, v) : |x - y| \leq m^{-\frac{1}{\alpha}}\}}. \quad (22)$$

Therefore for

$$\begin{aligned} A(m, n) &:= \left\{ t : \int_{B(m, n)} |w - v| d\mu_n > m^{-\frac{1}{2}} \right\}, \\ B(m, n) &:= \left\{ (x, y, w, v) : |x - y| \leq m^{-\frac{1}{\alpha}} \right\} \end{aligned}$$

we have

$$|\mathcal{J}| \leq C \left(\int_{A(m, n)} \int_{B(m, n)} |g_n| d\mu_n dt + \int_{(A(m, n))^c} \int_{B(m, n)} |g_n| d\mu_n dt \right) =: I + II.$$

Now if $|x - y| \leq m^{-\frac{1}{\alpha}}$ then $\psi_n(|x - y|) \geq \min\{m, n\} = m$ and for all $t \in A(m, n)$ we have

$$\begin{aligned} \mathcal{L}_n(t) &:= \int_{\mathbb{R}^{4d}} \psi_n(|x - y|) |w - v| d\mu_n \\ &\geq \int_{B(m, n)} \psi_n(|x - y|) |w - v| d\mu_n \geq m \cdot \int_{B(m, n)} |w - v| d\mu_n > m^{\frac{1}{2}}. \end{aligned}$$

Furthermore, integrating with respect to $d\mu_n$ reveals that

$$\mathcal{L}_n(t) = \sum_{i, j=1}^N \psi(|x_i^n(t) - x_j^n(t)|) |v_i^n(t) - v_j^n(t)|,$$

⁴ Which we may assume since we are going to converge with $n \rightarrow \infty$ for each fixed m .

which by Proposition 31, (iv) implies that the sequence $\mathcal{L}_n$ is uniformly bounded in $L^p([0, T])$ for some $p > 1$ and thus—it is uniformly integrable which further implies that

$$I \leq C \|\nabla_v \phi\|_\infty \int_{\{t: \mathcal{L}_n(t) > m^{\frac{1}{2}}\}} \mathcal{L}_n(t) dt \leq C(m) \|\nabla_v \phi\|_\infty \xrightarrow{m \rightarrow \infty} 0, \quad (23)$$

since $|\mathcal{L}_n(t) > m^{\frac{1}{2}}| \leq \frac{\|\mathcal{L}_n\|_{L^1}}{m^{\frac{1}{2}}} \rightarrow 0$ as $m \rightarrow \infty$.

To estimate II we introduce the set $B_t(m, n)$ of those pairs (i, j) such that $|x_i^n(t) - x_j^n(t)| \leq m^{-\frac{1}{\alpha}}$. Then by Hölder's inequality with exponent $q = \frac{1}{\theta}$, for some arbitrarily small $\theta > 0$, we have

$$\begin{aligned} II &\leq \|\nabla_v \phi\|_\infty \int_{(A(m, n))^c} \sum_{i, j \in B_t(m, n)} m_{i, n} m_{j, n} \psi_n(|x_i^n - x_j^n|) |v_i^n - v_j^n| dt \\ &= \|\nabla_v \phi\|_\infty \int_{(A(m, n))^c} \sum_{i, j \in B_t(m, n)} (m_{i, n} m_{j, n})^{1-\theta} \psi_n(|x_i^n - x_j^n|) |v_i^n - v_j^n|^{1-\theta} \\ &\quad \cdot (m_{i, n} m_{j, n})^\theta |v_i^n - v_j^n|^\theta dt \end{aligned} \quad (24)$$

$$\begin{aligned} &\leq \|\nabla_v \phi\|_\infty \left(\int_{(A(m, n))^c} \sum_{i, j \in B_t(m, n)} m_{i, n} m_{j, n} \psi_n^{\frac{1}{1-\theta}}(|x_i^n - x_j^n|) |v_i^n - v_j^n| dt \right)^{1-\theta} \\ &\quad \cdot \left(\int_{(A(m, n))^c} \sum_{i, j \in B_t(m, n)} m_{i, n} m_{j, n} |v_i^n - v_j^n| dt \right)^\theta \\ &\leq \|\nabla_v \phi\|_\infty \left(\int_0^T \sum_{i, j=1}^{N_n} m_{i, n} m_{j, n} \psi_n^{\frac{1}{1-\theta}}(|x_i^n - x_j^n|) |v_i^n - v_j^n| dt \right)^{1-\theta} \\ &\quad \cdot \left(\int_{(A(m, n))^c} \int_{B(m, n)} |w - v| d\mu_n dt \right)^\theta \\ &\leq \|\nabla_v \phi\|_\infty \left(\int_0^T \sum_{i, j=1}^{N_n} m_{i, n} m_{j, n} \psi_n^{\frac{1}{1-\theta}}(|x_i^n - x_j^n|) |v_i^n - v_j^n| dt \right)^{1-\theta} \cdot (T m^{-\frac{1}{2}})^\theta. \end{aligned} \quad (25)$$

(26)

By Proposition 31, (iv) the first multiplicand on the right-hand side of (26) is uniformly bounded, which implies that

$$II \leq C \|\nabla_v \phi\|_\infty (T m^{-\frac{1}{2}})^\theta \xrightarrow{m \rightarrow \infty} 0. \quad (27)$$

Estimations (23) and (27) imply that

$$|\mathcal{J}| \leq C(m) \|\nabla_v \phi\|_\infty$$

for some n -independent positive constant $C(m)$ such that $C(m) \rightarrow 0$ as $m \rightarrow \infty$.

Step 5. Our next goal is to ensure that the convergence

$$\int_0^T \int_{\mathbb{R}^{2d}} F_m(f_n) f_n \nabla_v \phi dx dv dt \rightarrow \int_0^T \int_{\mathbb{R}^{2d}} F_m(f) f \nabla_v \phi dx dv dt \quad (28)$$

holds for each m and each $\phi \in \mathcal{G}$. Let us fix $\phi \in \mathcal{G}$ and $m = 1, 2, \dots$. For g_m defined in (21), we have

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{R}^{2d}} F_m(f_n) f_n \nabla_v \phi dx dv dt - \int_0^T \int_{\mathbb{R}^{2d}} F_m(f) f \nabla_v \phi dx dv dt \right| \\ &= \left| \int_0^T \int_{\mathbb{R}^{4d}} g_m(d\mu_n - d\mu) dt \right| \\ &\leq \left| \int_0^T \int_{\mathbb{R}^{4d}} g_m[d(f_n \otimes f_n) - d(f_n \otimes f)] dt \right| \\ &+ \left| \int_0^T \int_{\mathbb{R}^{4d}} g_m[d(f_n \otimes f) - d(f \otimes f)] dt \right| =: I + II. \end{aligned} \quad (29)$$

Furthermore, again by Fubini's theorem, we have

$$I = \left| \int_0^T \int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} g_m(d f_n - d f) \right) d f_n dt \right|,$$

and since for each x, v the function $(y, w) \mapsto g_m(x, y, v, w)$ is Lipschitz continuous and bounded with $Lip(g_m) + \|g_m\|_\infty \leq C_1$ for some $C_1 = C_1(m, \|\nabla_v \phi\|_\infty, Lip(\nabla_v \phi))$ then by Lemma 21 we have

$$I \leq C_1 \int_0^T \int_{\mathbb{R}^{2d}} d(f_n, f) d f_n \leq C_1 T \|d(f_n, f)\|_\infty \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Similarly, we also have that $II \rightarrow 0$ with $n \rightarrow \infty$. This concludes the proof of convergence (28).

Step 6. At this point after converging with n to infinity we are left with the weak formulation for f that reads as follows:

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^{2d}} f[\partial_t \phi + v \nabla \phi] dx dv dt + \int_0^T \int_{\mathbb{R}^{2d}} F_m(f) f \nabla_v \phi dx dv dt \\ &= - \int_{\mathbb{R}^{2d}} f_0 \phi(\cdot, \cdot, 0) dx dv + \mathcal{J}(m) \end{aligned}$$

for all $m = 1, 2, \dots$ and all $\phi \in \mathcal{G}$ with

$$\mathcal{J}(m) \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Therefore it suffices to show that

$$\int_0^T \int_{\mathbb{R}^{2d}} F_m(f) f \nabla_v \phi dx dv dt \rightarrow \int_0^T \int_{\mathbb{R}^{2d}} F(f) f \nabla_v \phi dx dv dt. \quad (30)$$

By Fubini's theorem, for

$$\begin{aligned} d\mu &= (f \otimes f)(x, v, y, w, t) dx dv dy dw, \\ g_m &= \psi_m(|x - y|)(w - v) \nabla_v \phi, \quad g = \psi(|x - y|)(w - v) \nabla_v \phi, \end{aligned}$$

we have

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^{2d}} F_m(f) f \nabla_v \phi dx dv dt &= \int_0^T \int_{\mathbb{R}^{2d}} g_m d\mu dt, \\ \int_0^T \int_{\mathbb{R}^{2d}} F(f) f \nabla_v \phi dx dv dt &= \int_0^T \int_{\mathbb{R}^{2d}} g d\mu dt, \end{aligned} \quad (31)$$

provided that the integral on the right-hand side of (31) is well defined. Therefore to show (30) it suffices to prove that $g_m \rightarrow g$ in L^1 with respect to measure $d\mu$. To prove this we first show that $g_m \rightarrow g$ a.e. with respect to the measure $d\mu$. Clearly the convergence holds on

$$A := \{(x, v, y, w, t) : x \neq y\} \cup \{(x, v, y, w, t) : x = y, v = w\},$$

and it suffices to show that the set $A^c = \{(x, v, y, w, t) : x = y, v \neq w\}$ is of measure $d\mu$ zero. We have $\psi_m \equiv m$ on A^c and thus

$$\begin{aligned} I_m &:= \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu dt = \int_0^T \int_{\mathbb{R}^{4d}} \psi_m(|x - y|) |w - v| |\nabla_v \phi| d\mu dt \\ &\geq \int_{A^c} \psi_m(|x - y|) |w - v| |\nabla_v \phi| d\mu dt = \int_{A^c} m |w - v| |\nabla_v \phi| d\mu dt \\ &= m \int_{A^c} |w - v| |\nabla_v \phi| d\mu dt. \end{aligned}$$

Thus either

$$I_m \rightarrow \infty \quad \text{or} \quad \int_{A^c} |w - v| |\nabla_v \phi| d\mu = 0. \quad (32)$$

The proofs of Step 4 and Step 5 remain true if we substitute g_m and g_n with $|g_m|$ and $|g_n|$ respectively.⁵ Therefore the respective convergences also hold for $|g_m|$ and $|g_n|$, yielding

$$\left| \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu_n dt - \int_0^T \int_{\mathbb{R}^{4d}} |g_n| d\mu_n dt \right| \leq C(m) \|\nabla_v \phi\|_\infty \xrightarrow{m \rightarrow \infty} 0 \quad (33)$$

and

$$\left| \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu_n dt - \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu dt \right| \xrightarrow{n \rightarrow \infty} 0. \quad (34)$$

⁵ Indeed, since $||g_m| - |g_n|| \leq |g_m - g_n|$ we may replace in (22) g_m and g_n with $|g_m|$ and $|g_n|$ and proceed with the proof in the same way as in Step 4. On the other hand in Step 5, the convergence of I and II from (29) was a result of that $\|d(f_n, f)\|_\infty \rightarrow 0$ and that g_m is a Lipschitz continuous function, which remains true for $|g_m|$.

Moreover for each m and n , we have

$$I_m \leq \left| \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu dt - \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu_n dt \right| \\ + \left| \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu_n dt - \int_0^T \int_{\mathbb{R}^{4d}} |g_n| d\mu_n dt \right| + \int_0^T \int_{\mathbb{R}^{4d}} |g_n| d\mu_n dt.$$

Now, (34) implies that for each m we may choose n big enough so that

$$\left| \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu dt - \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu_n dt \right| \leq 1.$$

Furthermore, by (33), for such n we have

$$\left| \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu_n dt - \int_0^T \int_{\mathbb{R}^{4d}} |g_n| d\mu_n dt \right| \leq |\mathcal{J}(m)|,$$

and finally by estimation (iii) from Proposition 31 we have

$$\int_0^T \int_{\mathbb{R}^{4d}} |g_n| d\mu_n dt \leq \|\nabla_v \phi\|_\infty \int_0^T \int_{\mathbb{R}^{4d}} \psi_n(|x - y|) |w - v| d\mu_n dt \\ = \|\nabla_v \phi\|_\infty \int_0^T \sum_{i,j=1}^{N_n} m_i^n m_j^n \psi_n(|x_i^n - x_j^n|) |v_j^n - v_i^n| dt \leq M,$$

and thus

$$I_m \leq 1 + |\mathcal{J}(m)| + M \leq C_2 \quad (35)$$

for some positive constant C_2 . Therefore (32) and (35) imply that $\int_{A^c} |w - v| |\nabla_v \phi| d\mu = 0$ and since the function $|w - v|$ is positive on A^c , then by a standard density argument A^c is of measure μ zero and we have proved that

$$\psi_m(|x - y|)(w - v) \nabla_v \phi \rightarrow \psi(|x - y|)(w - v) \nabla_v \phi \quad \mu\text{-a.e.}, \\ \psi_m(|x - y|) |w - v| |\nabla_v \phi| \rightarrow \psi(|x - y|) |w - v| |\nabla_v \phi| \quad \mu\text{-a.e.}$$

Moreover, by Fatou's lemma,

$$\int_0^T \int_{\mathbb{R}^{2d}} \psi(|x - y|) |w - v| |\nabla_v \phi| d\mu dt \\ \leq \liminf_{m \rightarrow \infty} \int_0^T \int_{\mathbb{R}^{2d}} \psi_m(|x - y|) |w - v| |\nabla_v \phi| d\mu dt \\ = \liminf_{m \rightarrow \infty} I_m \leq C_2. \quad (36)$$

Therefore the function $(x, y, v, w, t) \mapsto \psi(|x - y|) |w - v| |\nabla_v \phi|$ belongs to $L^1(d\mu)$. This function is a proper dominating function for $\psi_m(|x - y|)(w - v) \nabla_v \phi$ and by the dominated convergence theorem we have (30) and the proof of step 6 is finished.

Step 7. Let us now wrap up the proof and compare Definition 23 with what we were able to prove about f . We took the arbitrary initial data $f_0 \in \mathcal{M}_+$ and proved the existence of $f \in L^\infty(0, T; \mathcal{M}_+)$. Moreover in step 2 using estimates (ii) and (v) from Proposition 31 we proved that actually $\text{supp } f \subset B(\mathcal{R})$ and (point 1 of Definition 23). Point 2 of Definition 23 is an immediate consequence of (ii) from Proposition 31, while point 3 was the main focus of all the steps of the proof and it was finally proved in step 6. Point 4 of Definition 23 follows from (36) and Fubini's theorem. As a consequence of the weak formulation for f we conclude that also $\partial_t f \in L^p(0, T; (C^1(B(\mathcal{R})))^*)$. We are left with point 5 of Definition 23. Suppose that $B(R)$ and $B(r)$ are two concentric balls, such that (7) is satisfied. Then the construction of $f_{0,n}$ ensures that

$$\text{supp } f_{0,n} \cap B\left(R - \frac{1}{n}\right) \subset B\left(r + \frac{1}{n}\right),$$

and for sufficiently large n we have $r + \frac{1}{n} < r + \frac{R-r}{8} < R - \frac{R-r}{8}$. Translating this according to (16) we write that in the set $\mathcal{I}$ of those i that $(x_{0,i}^n, v_{0,i}^n) \in B(R - \frac{R-r}{8})$ we actually have $(x_{0,i}^n, v_{0,i}^n) \in B(r + \frac{R-r}{8})$. By (ii) and (iii) from Proposition 31 (and in particular by Remark 34), for each $i \in \mathcal{I}$ and for each sufficiently big n , we have the n independent bounds:

$$|x_i^n(t)| \leq |x_{0,i}^n| + t\mathcal{R} \xrightarrow{t \rightarrow 0} |x_{0,i}^n|, \quad |v_i^n(t)| \leq |v_{0,i}^n| + \omega(t) \xrightarrow{t \rightarrow 0} |v_{0,i}^n|.$$

The above bounds, for sufficiently small t , imply that $(x_i^n(t), v_i^n(t)) \in B(r + \frac{R-r}{6})$ as long as $i \in \mathcal{I}$. Similarly for $i \notin \mathcal{I}$ in a sufficiently small neighborhood of $t = 0$, we have $(x_i^n(t), v_i^n(t)) \notin B(R - \frac{R-r}{6})$. Therefore

$$\text{supp } f_n(t) \cap B\left(R - \frac{R-r}{6}\right) \subset B\left(r + \frac{R-r}{6}\right)$$

for sufficiently large n and sufficiently small t . Thus we may pass to the limit with $n \rightarrow \infty$ to obtain (8). This finishes the proof of the existence part of Theorem 31.

5. Proof of Theorem 31 (Weak-Atomic Uniqueness)

In what follows we aim to prove that if initial configuration f_0 is an atomic measure, i.e. it satisfies (9), then solution f in the sense of Definition 23 is of the form (10), and it is unique. We will base the proof on a very careful analysis of the local propagation of the support of f that comes from point 5 of Definition 23. What we basically need is that any amount of the mass f that is separated from the rest of the mass remains separated at least for some time. It is required to refine this property by adding a control over the shape in which the support in the x and v coordinates propagates. The difficulty comes from the fact that in the case of the particle system the position x_i of the i th particle changes with its own unique velocity v_i , however in the case of the kinetic equation, characteristics are not well defined.

Step 1. By point 1 in Definition 23 it is sufficient to prove the proposition only in an arbitrarily small neighborhood of $t = 0$. Let f_0 be of the form (9). Our first task is to restrict f_0 to small balls with one particle (say i th particle) in $\mathbb{R}^{2d}$. Then we will use the local propagation of the support to prove that the mass that initially formed the i th particle remains atomic in some right-sided neighborhood of $t = 0$. Since

$$f_0 = \sum_{i=1}^N m_i \delta_{x_{0,i}} \otimes \delta_{v_{0,i}} \quad (37)$$

for a number of atoms N , we have a finite number of initial positions and velocities of the particles $(x_{0,i}, v_{0,i})$ for $i = 1, \dots, N$, which implies that there exists $R_1 > 0$ such that for all $r_0 < R_1$, we have

$$f_0|_{B_i(r_0)} = m_i \delta_{x_{0,i}} \otimes \delta_{v_{0,i}} \quad (38)$$

for $B_i(R) := B_{x,v}((x_{0,i}, v_{0,i}), r_0)$.

At this point let us concentrate on one atom, we fix i . We aim to show that there exists T^* such that

$$f^D := f|_{B_i(r_0/4)} = m_i \delta_{x_i(t)} \otimes \delta_{v_i(t)} \quad (39)$$

in $[0, T^*]$ for some $\mathbb{R}^d$ valued functions x_i and v_i . We emphasize that r_0 and $T^*(r_0)$ can be chosen to be arbitrarily small. Identity (38) implies that for any $0 < r < r_0$, we have

$$\text{supp } f_0 \cap B_i(r_0) \subset B_i(r),$$

which by point 5 of Definition 23 ensures that there exists T^* such that

$$\text{dist}\{\text{supp } f^D(t), \text{supp } f^C(t)\} > \frac{r_0}{8} \quad (40)$$

for all $t \in [0, T^*]$, where $f^C(t) := f(t) - f^D(t)$. Then one can find a smooth function $\eta : \mathbb{R}^{2d} \times [0, T_*] \rightarrow [0, 1]$ such that $\eta \equiv 1$ over the support of f^D and $\eta \equiv 0$ over the support of f^C . We have then $f^D \eta = f^D$. All these properties allow to state the following equation satisfied by f^D on $[0, T^*]$:

$$\partial_t f^D + v \cdot \nabla_x f^D + \text{div}_v[(F(f^C) + F(f^D))f^D] = 0. \quad (41)$$

This equation is satisfied in the same sense that (6) from Definition 23. To prove that f^D is indeed of form (39) we introduce

$$\begin{cases} \frac{d}{dt} x_a(t) = v_a(t) \\ \frac{d}{dt} v_a(t) = \int_{\mathbb{R}^{2d}} \psi(|x_a(t) - y|)(w - v_a(t)) f^C \, dy dw, \end{cases} \quad (42)$$

with the initial data $(x_a(0), v_a(0)) = (x_{0,i}, v_{0,i})$. Condition (40) ensures that the right-hand side of (42)₂ is smooth and thus (42) has exactly one smooth solution in $[0, T^*]$. Our goal is to show that f^D is supported on the curve $(x_a(t), v_a(t))$ and

that in fact (39) holds with $(x_i(t), v_i(t)) \equiv (x_a, v_a)$. Since this feature will hold for all atoms, the whole f will then be atomic.

Step 2. In the next step we characterize possible evolution of the support of the weak solution to (41).

Lemma 51. *Let f be a weak solution to (3) in the sense of Definition 23. Assume further that f has the structure of $f = f^D + f^C$ and fulfills the weak formulation of (41), and*

$$\text{supp } f_0^D = (x_0, v_0)$$

for some given (x_0, v_0) . Then for any $R > 0$ there exists T^* , such that

$$\text{supp } f^D(t) \subset (x_0, v_0) + (t B_x(v_0, \epsilon)) \times B_v(0, R)$$

for all $t \in [0, T^*]$, with $\epsilon := \sqrt{2R(R + |v_0|)}$, which can be arbitrarily small depending on smallness of R .

To prove Lemma 51 it is required to show the following result:

Lemma 52. *Let f^D be a weak solution to (41) in the sense of Definition 23. Assume further that there exists T^* , such that*

$$\text{supp } f^D(t) \subset B((x_0, v_0), R) \quad (43)$$

for some given (x_0, v_0) and $R > 0$ and all $t \in [0, T^*]$. Then

$$\text{supp } f^D(t) \subset \text{supp } f_0^D + t B_x(v_0, R) \times B_v(0, R). \quad (44)$$

This means that the support in the x -coordinates propagates in a cone defined by the ball $B_x(v_0, R)$ in direction v_0 .

Proof of Lemma 52. Without a loss of generality, we assume that $(x_0, v_0) = (0, 0)$. The boundedness of the support in the v -coordinates is trivial and thus we focus on the support in the x -coordinates. Suppose that $x_1 \in \mathbb{R}^d$ and $\rho > 0$ are such that

$$\text{supp } f_0^D \cap B_x(x_1, \rho) \times \mathbb{R}^d = \emptyset \quad \text{and let} \quad \phi(x, t) := ((\rho - Rt)^2 - |x - x_1|^2)_+.$$

Hence

$$\text{supp } \phi(\cdot, t) = \{|x - x_1| \leq |\rho - Rt|\}. \quad (45)$$

We test (41) by ϕ^2 and integrate over the time interval $[0, T^*]$, obtaining

$$\begin{aligned} & \int_{\mathbb{R}^{2d}} f^D(T^*) \phi(T^*)^2 dx dv + 4 \int_0^{T^*} \int_{\mathbb{R}^{2d}} f^D \phi[(\rho - Rt)R - (x - x_1)v] dx dv dt = \\ & = \int_{\mathbb{R}^{2d}} f_0^D \phi(0)^2 dx dv = 0. \end{aligned}$$

Since the first term on the left-hand side of the above equality is nonnegative, we have

$$\int_0^{T^*} \int_{\mathbb{R}^{2d}} f^D \phi[(\rho - Rt)R - (x - x_1)v] dx dv dt \leq 0,$$

but for the interior of the support of ϕ , we have $\rho - Rt > |x - x_1|$ and by (43) $R > |v|$. This implies that

$$0 < (\rho - Rt)R - (x - x_1)v, \quad \text{and hence } f\phi \equiv 0.$$

This way we proved that in the complement of the support in x of $f(t)$ lay all the balls centered outside of $\text{supp } f_0$ and with a radius equals to $\rho - Rt$, which implies (44). $\square$

Proof of Lemma 51. We base the proof on Lemma 52. First we establish proper R and T^* . Since f_0^D is concentrated in one point (x_0, v_0) , then for arbitrarily small ρ ,

$$\text{supp } f_0^D \subset B((x_0, v_0), \rho).$$

Now, Definition 23 point 5 ensures that there exist $R(\rho)$ and $T^*(\rho)$ such that

$$\text{supp } f^D(t) \subset B((x_0, v_0), R)$$

in $[0, T^*]$ and R can be chosen arbitrarily small (then also T^* is small but still positive). We fix such R and T^* and note that we may apply Lemma 52 on $[0, T^*]$. Without a loss of generality we assume that $x_0 = 0$ and test (41) with the function ϕ^2 , where

$$\phi(x, t) := ((x - v_0 t)^2 - (t\epsilon)^2)_+$$

and

$$\text{supp } \phi(\cdot, t) = \{x \in \mathbb{R}^d : |x - v_0 t| \geq t\epsilon\}. \quad (46)$$

By (11), we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^{2d}} f^D(t) \phi^2(t) dx dv \\ &\quad - 4 \int_0^t \int_{\mathbb{R}^{2d}} f^D \phi[-v_0(x - v_0 t) - t\epsilon^2 + v(x - v_0 t)] dx dv dt \\ &\geq 4 \int_0^t \int_{\mathbb{R}^{2d}} f^D \phi[t\epsilon^2 - (v - v_0)(x - v_0 t)] dx dv dt. \end{aligned} \quad (47)$$

On the support of f^D , we have $|v - v_0| \leq R$ and by Lemma 52 it holds that

$$|x - v_0 t| \leq |x - \underbrace{x_0}_{=0}| + |v_0|t \leq t(|v_0| + R) + t|v_0| \leq t(2|v_0| + R).$$

Hence, in view of definition of ϵ , we conclude

$$(v - v_0)(x - v_0 t) \leq (2|v_0| + R)Rt < t\epsilon^2.$$

Therefore the integrand on the right-hand side of (47) is nonnegative, which means that it has to be equal to 0, which further implies that

$$f^D \phi \equiv 0 \quad \text{in } [0, T^*].$$

By the definition of ϕ it follows that $f^D(t)$ vanishes outside of the cone balls $tB_x(v_0, \epsilon) \times \mathbb{R}^d$. The lemma is proved. $\square$

Step 3. In this part we show that f initiated by a state of (37) stays indeed atomic for all time.

Proposition 51. *Let f be a solution to (41) in the sense of Definition 23. Then if f_0 is of form (9) then f is an atomic solution (of form (10)) and it is unique.*

Proof. We show separately for each of atoms that each initial particle generates a mono-atomic solution (at least locally in time). Finiteness of the number of atoms allows us to conclude that the whole solutions is atomic. Hence, we study (41) with a mono-atomic initial data located in (x_0, v_0) .

We test (41) by $(v - v_a(t))^2$ getting

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^{2d}} f^D (v - v_a(t))^2 dx dv &= -2 \int_{\mathbb{R}^{2d}} f^D (v - v_a(t)) \dot{v}_a(t) dx dv \\ &\quad + 2 \int_{\mathbb{R}^{2d}} F(f^C) f^D (v - v_a(t)) dx dv + 2 \int_{\mathbb{R}^{2d}} F(f^D) f^D (v - v_a(t)) dx dv \\ &= -2I + 2II + 2III. \end{aligned} \tag{48}$$

First we deal with III . By symmetry of $f^D \otimes f^D$ with respect to (x, v) and (y, w) , we have

$$\begin{aligned} III &= \int_{\mathbb{R}^{4d}} \psi(|x - y|)(w - v) f^D f^D (v - v_a(t)) dx dv dy dw \\ &= \int_{\mathbb{R}^{4d}} \psi(|x - y|)(v - w) f^D f^D (w - v_a(t)) dx dv dy dw \\ &= \frac{1}{2} \int_{\mathbb{R}^{4d}} \psi(|x - y|)(w - v) f^D f^D (v - w) dx dv dy dw \\ &= -\frac{1}{2} \int_{\mathbb{R}^{4d}} \psi(|x - y|)(w - v)^2 f^D f^D dx dv dy dw \leq 0. \end{aligned}$$

Next let us take a closer look at II . By the definition of $F(f^C)$

$$\begin{aligned} II &= \int_{\mathbb{R}^{4d}} \psi(|x - y|)(w - v) f^D f^C (v - v_a(t)) dx dv dy dw \\ &= \int_{\mathbb{R}^{4d}} \psi(|x - y|)(w - v_a(t) + v_a(t) - v) f^D f^C (v - v_a(t)) dx dv dy dw \\ &= \int_{\mathbb{R}^{4d}} \psi(|x - y|)(w - v_a(t)) f^D f^C (v - v_a(t)) dx dv dy dw \end{aligned}$$

$$\begin{aligned}
& - \underbrace{\int_{\mathbb{R}^{4d}} \psi(|x-y|) f^D f^C (v-v_a(t))^2 dx dv dy dw}_{\leq 0} \\
& \leq \int_{\mathbb{R}^{4d}} \psi(|x-y|) (w-v_a(t)) f^D f^C (v-v_a(t)) dx dv dy dw =: II_2.
\end{aligned}$$

Now we compare II_2 with I :

$$|II_2 - I| = \left| \int_{\mathbb{R}^{4d}} (\psi(|x_a(t)-y|) - \psi(|x-y|)) \right. \quad (49)$$

$$\begin{aligned}
& \left. (w-v_a(t)) f^D f^C (v-v_a(t)) dx dv dy dw \right| \\
& \leq \int_{\mathbb{R}^{4d}} |\psi(|x_a(t)-y|) - \psi(|x-y|)| |w - v_a(t)| f^D f^C |v-v_a(t)| dx dv dy dw. \quad (50)
\end{aligned}$$

$$-v_a(t)| f^D f^C |v-v_a(t)| dx dv dy dw. \quad (51)$$

The main problem with estimating the right-hand side of the above inequality lays in the estimation of

$$|\psi(|x_a(t)-y|) - \psi(|x-y|)|.$$

This is the place where the separation of supports explained by Lemma 51 comes into play. Both $(x_a(t), v_a(t))$ and (x, v) are in the support of f^D , while (y, w) is in the support of f^C . Thus (40) implies that either

$$|x-y| > \frac{r_0}{8} \quad \text{and} \quad |x_a(t)-y| > \frac{r_0}{8} \quad (52)$$

or

$$|v-w| > \frac{r_0}{8} \quad \text{and} \quad |v_a(t)-w| > \frac{r_0}{8}. \quad (53)$$

We handle the above two cases separately.

In case (52) it is clear that

$$|\psi(|x_a(t)-y|) - \psi(|x-y|)| \leq L|x-x_a(t)| = Lt^{\frac{1}{2}} \frac{|x-x_a(t)|}{t^{\frac{1}{2}}} \quad (54)$$

for some constant $L = L(r_0) > 0$, since ψ is smooth outside of any neighborhood of 0.

In case of (53) we are actually in a situation when at $t = 0$ multiple particles are situated in the same spot with different velocities i.e. f^C is divided into two parts f^{C_1} and f^{C_2} . The first part submits to the same bounds as (52) while for the second, f^{C_2} , we have

$$f^{C_2}(0) = \sum_j m_j \delta_{x_{0,i}} \otimes \delta_{v_{0,j}} =: \sum_j f_j^{C_2}(0).$$

Thus, initially f^{C_2} is concentrated in the same position as f^D but with different velocities. In this case we apply Lemma 51 multiple times (once for f^D and multiple

times for each $f_j^{C_2}$. Even though Lemma 51 is written for solutions of (6) we may still apply it for f^D and each of $f_j^{C_2}$, since the proof does not involve directly the dependence on v . Therefore, by Lemma 51, we have

$$\begin{aligned}\text{supp } f^D(t) &\subset (x_{0,i}, v_{0,i}) + tB_x(v_{0,i}, \epsilon) \quad \text{and} \\ \text{supp } f_j^{C_2}(t) &\subset (x_{0,i}, v_{0,j}) + tB_x(v_{0,j}, \epsilon).\end{aligned}$$

At this point we fix $R > 0$ and T^* from Lemma 51, so that ϵ is small enough that

$$B_x(v_{0,i}, \epsilon) \cap B_x(v_{0,j}, \epsilon) = \emptyset \quad \text{for } i \neq j.$$

Moreover

$$\text{dist}(B_x(v_{0,i}, \epsilon), B_x(v_{0,j}, \epsilon)) > C(r_0) > 0.$$

Again, we used that the number of all atoms is finite. If so, then we also have

$$|x - y| > tC(R) \quad \text{and} \quad |x_a(t) - y| > tC(R)$$

for $x \in \text{supp } f^D$ and $y \in \text{supp } f^{C_2}$. Therefore in the cases $(\psi(|s|) = s^{-\alpha}$ and $\psi'(|s|) \sim s^{-1-\alpha})$, we have

$$\begin{aligned}|\psi(|x_a(t) - y|) - \psi(|x - y|)| &\leq C(R)t^{-1-\alpha}|x - x_a(t)| \\ &= C(R)t^{-\frac{1}{2}-\alpha} \frac{|x - x_a(t)|}{t^{\frac{1}{2}}}.\end{aligned}\tag{55}$$

We combine inequalities (49), (54)⁶ and (55) with the global bounds on the support of f obtaining

$$|II_2 - I| \leq A(t) \int_{\mathbb{R}^{2d}} t^{-\frac{1}{2}} |x - x_a(t)| |v - v_a(t)| f^D dx dv$$

for $A := Lt^{\frac{1}{2}} + C(R)t^{-\frac{1}{2}-\alpha}$, which thanks to the fact that $\alpha < \frac{1}{2}$ is integrable with respect to t over $[0, T^*]$. Taking into the account our estimations of I , II and III we come back to (48) and claim that

$$\begin{aligned}\frac{d}{dt} \int_{\mathbb{R}^{2d}} f^D |v - v_a(t)|^2 dx dv &\leq A(t) \int_{\mathbb{R}^{2d}} t^{-\frac{1}{2}} |x - x_a(t)| |v - v_a(t)| f^D dx dv \\ &\leq A(t) \left(\int_{\mathbb{R}^{2d}} f^D t^{-1} |x - x_a(t)|^2 dx dv + f^D |v - v_a(t)|^2 \right).\end{aligned}\tag{56}$$

To finish the proof there is a need to estimate the first integrand on the right-hand side of (56). We test⁷ (41) with $|x - x_a(t)|^2 t^{-1}$ getting

$$\frac{d}{dt} \int_{\mathbb{R}^{2d}} t^{-1} f^D |x - x_a(t)|^2 dx dv + \int_{\mathbb{R}^{2d}} t^{-2} f^D |x - x_a(t)|^2 dx dv$$

⁶ Here is the entire estimation in case (52) and the estimation of f^{C_1} in case (53).

⁷ Even though $|x - x_a(t)|^2 t^{-1}$ is not a good test function for (41), we can approximate the singularity at $t = 0$ by modification $(t + l)^{-1}$ and then let $l \rightarrow 0$.

$$\begin{aligned} &\leq 2 \int_{\mathbb{R}^{2d}} t^{-1} f^D(x - x_a(t)) \dot{x}_a(t) dx dv \\ &\quad - 2 \int_{\mathbb{R}^{2d}} t^{-1} f^D(x - x_a(t)) v dx dv. \end{aligned} \quad (9)$$

We then apply Young's inequality with $\delta > 0$ to obtain

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^{2d}} t^{-1} f^D |x - x_a(t)|^2 dx dv + \int_{\mathbb{R}^{2d}} t^{-2} f^D |x - x_a(t)|^2 dx dv \\ &\leq 2 \int_{\mathbb{R}^{2d}} t^{-1} f^D |x - x_a(t)| |v - v_a(t)| dx dv \\ &\leq \delta \int_{\mathbb{R}^{2d}} t^{-2} f^D |x - x_a(t)|^2 dx dv \end{aligned} \quad (57)$$

$$+ C \int_{\mathbb{R}^{2d}} f^D |v - v_a(t)|^2 dx dv. \quad (58)$$

Finally we fix a suitable $\delta > 0$ and combine inequalities (56) and (57), which leaves us with

$$\begin{aligned} &\frac{d}{dt} \left(\int_{\mathbb{R}^{2d}} (t^{-1} f^D |x - x_a(t)| + f^D |v - v_a(t)|^2) dx dv \right) \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^{2d}} t^{-2} f^D |x - x_a(t)|^2 dx dv \\ &\leq A(t) \int_{\mathbb{R}^{2d}} (t^{-1} f^D |x - x_a(t)|^2 + f^D |v - v_a(t)|^2) dx dv, \end{aligned}$$

which by Gronwall's lemma and the fact that $A(t) \sim t^{-1/2-\alpha}$ is integrable in a neighborhood of $t = 0$ (restriction $\alpha \in (0, \frac{1}{2})$ is again used here) implies

$$\int_{\mathbb{R}^{2d}} (t^{-1} f^D |x - x_a(t)|^2 + f^D |v - v_a(t)|^2) dx dv \equiv 0 \text{ on } [0, T^*].$$

Thus on $[0, T^*]$ we have $x \equiv x_a$ and $v \equiv v_a$ on the support of f^D , which is exactly equivalent to (39).

We have proved that f^D is mono-atomic. Repeating the procedure for all atoms (the number is finite) we conclude that f is atomic on a time interval $[0, T^*]$ with possibly smaller, but positive $T^* > 0$. This procedure works until the first moment of sticking of an ensemble of particles.

As a final remark we explain the case of the sticking some particles in a finite time, say T_1 . The above considerations prove uniqueness and atomic structure of the solutions for the time interval $(0, T_1)$ without the sticking of particles, and we want to reach T_1 . The regularity of the weak solution guarantees that $\partial_t f \in L^p(0, T; (C^1(B(R)))^*)$ with $T > T_1$, hence trajectories of atoms are $W^{1,p}$ vector functions and they are uniquely extended up to T_1 . Moreover, time regularity and (11) exclude other possibilities of evolution of the studied measure-valued solution. Thus, we can reinitiate our analysis from time T_1 with atomic initial state $f(T_1)$, reaching the given T . Note that the number of moments of sticking is finite, since N is finite (see Remark 23). As a conclusion, since the solution exists globally in time it must be atomic on $[0, T]$. Theorem 31 is proved. $\square$

A Appendix

Proof of Proposition 31. The existence and uniqueness part as well as points (i) and (ii) are no different than in the case of regular weight and we will not prove them here. Their proofs can be found in the literature (see for instance [23] or [30]). Thus it remains to prove (iii)–(v).

First, assuming for notational simplicity that $(x^n, v^n, N_n, m_i^n) = (x, v, N, m_i)$ let us prove a particularly useful estimate. Let $1 < p < q$ be given numbers satisfying additional conditions that will be specified later. For each $n = 1, 2, \dots$, velocity v^n (denoted by v) is absolutely continuous on $[0, T]$ and thus by (2)₂, we have

$$\begin{aligned}
 m_i \int_0^T |\dot{v}_i|^p dt &= m_i \int_0^T \left| \sum_{j=1}^N m_j (v_j - v_i) \psi_n(|x_i - x_j|) \right|^p dt \\
 &\leq \sum_{j=1}^N m_i m_j \int_0^T |v_j - v_i|^p \psi_n^p(|x_i - x_j|) dt \\
 &= \sum_{j=1}^N \int_0^T (m_i m_j)^{\frac{p}{q}} |v_j - v_i|^{p \cdot \frac{p}{q}} \psi_n^p(|x_i - x_j|) \cdot (m_i m_j |v_j - v_i|^p)^{(1-\frac{p}{q})} dt \\
 &\leq \sum_{j=1}^N m_i m_j \int_0^T |v_j - v_i|^p \psi_n^q(|x_i - x_j|) dt + \sum_{j=1}^N m_i m_j \int_0^T |v_j - v_i|^p dt
 \end{aligned} \tag{59}$$

$$\begin{aligned}
 &\leq \epsilon \sum_{j=1}^N m_i m_j \underbrace{\int_0^T |v_j - v_i|^2 \psi_n^{\frac{2q}{p}}(|x_i - x_j|) dt}_{=: A} + C(\epsilon) T m_i \\
 &\quad + \sum_{j=1}^N m_i m_j \int_0^T |v_j - v_i|^p dt.
 \end{aligned} \tag{60}$$

Inequality (59) is obtained by Young's inequality with exponent $\frac{q}{p}$ while (60) follows by Young's inequality with exponent $\frac{2}{p}$. In both of the above inequalities we also use the assumption that $\sum_{i=1}^N m_i = 1$.

Furthermore recalling that $\psi_n^{\frac{2q}{p}}(s) \leq \psi^{\frac{2q}{p}}(s) = |s|^{-\lambda}$, where $\lambda := \frac{2q\alpha}{p}$, integral A can be estimated as follows:

$$\begin{aligned}
 A &\leq \sum_{k=1}^d \int_0^T (v_j^k - v_i^k) \cdot (v_j^k - v_i^k) |x_i^k - x_j^k|^{-\lambda} dt \\
 &= \sum_{k=1}^d \int_0^T (v_j^k - v_i^k) \cdot \left((x_j^k - x_i^k) |x_i^k - x_j^k|^{-\lambda} \right)' dt \\
 &= -\frac{1}{1-\lambda} \sum_{k=1}^d \int_0^T (\dot{v}_j^k - \dot{v}_i^k) \cdot (x_j^k - x_i^k) |x_i^k - x_j^k|^{-\lambda} dt
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{1-\lambda} \sum_{k=1}^d (v_j^k - v_i^k) \cdot (x_j^k - x_i^k) |x_i^k - x_j^k|^{-\lambda} \Big|_0^T \\
& \leq C_\lambda \int_0^T |\dot{v}_i| |x_i - x_j|^{1-\lambda} dt + C_\lambda \int_0^T |\dot{v}_j| |x_i - x_j|^{1-\lambda} dt \\
& \quad + 2C_\lambda \sup_{t \in [0, T]} |v_j - v_i| |x_i - x_j|^{1-\lambda}.
\end{aligned}$$

However, the above estimation is valid only if $\lambda < 1$, which means that $\frac{q}{p} \cdot 2\alpha < 1$ and such condition can be easily satisfied if $\alpha < \frac{1}{2}$ and $1 < p < q$ are small enough. By point (ii) we have $|v| \leq \mathcal{R}$ and $|x| \leq \mathcal{R}$. This leads to the concluding estimation of A , which reads:

$$A \leq C(\mathcal{R})^{1-\lambda} \int_0^T |\dot{v}_i| dt + C(\mathcal{R})^{1-\lambda} \int_0^T |\dot{v}_j| dt + C(\mathcal{R})^{2-\lambda}. \quad (61)$$

Now we will apply the above calculation (particularly estimations (60) and (61)) in the effort to prove (iii) and (iv). For (iii) let us assume that $p = q = 1$.⁸ We sum (60) over $i = 1, \dots, N$ to get

$$\sum_{i=1}^N m_i \int_0^T |\dot{v}_i| dt \leq \epsilon \sum_{i,j=1}^N m_i m_j A + C(\epsilon)T + 2\mathcal{R}T$$

and plug in (61) to obtain

$$\sum_{i=1}^N m_i \int_0^T |\dot{v}_i| dt \leq 2\epsilon C(\mathcal{R})^{1-\lambda} \sum_{i=1}^N m_i \int_0^T |\dot{v}_i| dt + \epsilon C(\mathcal{R})^{2-\lambda} + C(\epsilon)T + 2\mathcal{R}T,$$

which after fixing sufficiently small ϵ and rearranging yields

$$\sum_{i=1}^N m_i \int_0^T |\dot{v}_i| dt \leq C(\mathcal{R})^{2-\lambda} + CT + C\mathcal{R}T, \quad (62)$$

which proves (iii) for $p = 1$. Then for $1 < p = q$ using (60), (61) and (62), we have

$$\begin{aligned}
\sum_{i=1}^N m_i \int_0^T |\dot{v}_i|^p dt & \leq 2C(\mathcal{R})^{1-\lambda} \sum_{i=1}^N m_i \int_0^T |\dot{v}_i| dt + C(\mathcal{R})^{2-\lambda} \\
& + CT + C\mathcal{R}^p T \leq C(\mathcal{R}, p, T, \lambda)
\end{aligned} \quad (63)$$

and (iii) is proved for some sufficiently small $p > 1$. In order to prove (iv) we take $1 = p < q$ in (60), which leads us to a very similar result to (63) and to the end of the proof of (iv).

⁸ Note that (59) remains true also for $p = q = 1$.

Let us prove (v). Fix $n = 1, 2, \dots$ and a bounded, Lipschitz continuous function $g = g(x, v)$. Then according to Definition 23, for $t \in [0, T)$, $\epsilon > 0$ and

$$\chi_{\epsilon,t}(s) := \begin{cases} 1 & \text{for } 0 \leq s \leq t - \epsilon \\ -\frac{1}{2\epsilon}(s - t - \epsilon) & \text{for } t - \epsilon < s \leq t + \epsilon \\ 0 & \text{for } t_\epsilon < s \end{cases}$$

the function $\phi(s, x, v) := \chi_{\epsilon,t}(s)g(x, v) \in \mathcal{G}$ is a good test function in the weak formulation for each f_n . Thus we plug ϕ into (6) obtaining

$$\begin{aligned} & \frac{1}{2\epsilon} \int_{t-\epsilon}^{t+\epsilon} \int_{\mathbb{R}^{2d}} f_n g dx dv dt = \\ & = - \int_0^T \int_{\mathbb{R}^{2d}} f_n \chi_{\epsilon,t} v \nabla g dx dv dt - \int_0^T \int_{\mathbb{R}^{2d}} F_n(f_n) f_n \chi_{\epsilon,t} \nabla_v g dx dv dt \\ & \quad - \int_{\mathbb{R}^{2d}} f_0 g dx dv. \end{aligned}$$

Since $t \mapsto \int_{\mathbb{R}^{2d}} f_n g dx dv$, $t \mapsto \int_{\mathbb{R}^{2d}} f_n \chi_{\epsilon,t} v \nabla g dx dv$ and $t \mapsto \int_{\mathbb{R}^{2d}} F_n(f_n) f_n \chi_{\epsilon,t} \nabla_v g dx dv$ are integrable functions (for fixed n and g), then converging with $\epsilon \rightarrow 0$ leads to the following equation holding for a.a $t \in [0, T)$:

$$\begin{aligned} \int_{\mathbb{R}^{2d}} f_n(t) g dx dv dt &= \int_0^t \int_{\mathbb{R}^{2d}} f_n v \nabla g dx dv dt + \int_0^t \int_{\mathbb{R}^{2d}} F_n(f_n) f_n \nabla_v g dx dv dt \\ &\quad - \int_{\mathbb{R}^{2d}} f_0 g dx dv = \int_0^t G(t) dt - \int_{\mathbb{R}^{2d}} f_0 g dx dv, \end{aligned}$$

where

$$\begin{aligned} G(t) &:= \int_{\mathbb{R}^{2d}} f_n(t) v \nabla g dx dv + \int_{\mathbb{R}^{2d}} F_n(f_n)(t) f_n(t) \nabla_v g dx dv \\ &= \sum_{i=1}^N m_i v_i^n(t) \nabla g(x_i^n(t), v_i^n(t)) + \sum_{i,j=1}^N m_i m_j (v_j^n(t) \\ &\quad - v_i^n(t)) \psi(|x_i^n(t) - x_j^n(t)|) \nabla_v g(x_i^n(t), v_i^n(t)). \end{aligned}$$

By virtue of points (ii) and (iii) of this proposition, we have

$$\begin{aligned} \int_0^T |G(t)|^p dt &\leq \int_0^T \left| \sum_{i=1}^N m_i v_i^n(t) (\nabla g)(x_i^n(t), v_i^n(t)) \right|^p dt \\ &\quad + \int_0^T \left| \sum_{i,j=1}^N m_i m_j \psi_n(|x_i^n(t) - x_j^n(t)|) (v_j^n(t) - v_i^n(t)) (\nabla_v g)(x_i^n(t), v_i^n(t)) \right|^p dt \\ &\leq Lip(g)^p T(\mathcal{R})^p + Lip(g)^p M(\mathcal{R}) =: M_g(Lip(g), \mathcal{R}) \end{aligned}$$

which finishes the proof of (v). $\square$

Now we aim to give a sketch of the proof of uniqueness to the particle system (2). Note that point (ii) of Proposition 31 implies that solutions to ODEs (2) are of $W^{2,p}$ regularity for some $p > 1$.

Proof of the uniqueness part of Theorem 32. Consider two solutions to the system (2) with the same initial data; name them (x_i, v_i) and $(\bar{x}_i, \bar{v}_i)$. These fulfill the following systems:

$$\begin{aligned} \frac{d}{dt}x_i &= v_i, & \frac{d}{dt}\bar{x}_i &= \bar{v}_i, \\ \frac{d}{dt}v_i &= \sum_j m_j(v_j - v_i)\psi(|x_i - x_j|), & \frac{d}{dt}\bar{v}_i &= \sum_j m_j(\bar{v}_j - \bar{v}_i)\psi(|\bar{x}_i - \bar{x}_j|). \end{aligned} \quad (64)$$

Taking

$$\delta x_i = x_i - \bar{x}_i, \quad \delta v_i = v_i - \bar{v}_i, \quad (65)$$

we find

$$\begin{aligned} \frac{d}{dt}\delta x_i &= \delta v_i, \\ \frac{d}{dt}\delta v_i &= \sum_j m_j(\delta v_j - \delta v_i)\psi(|\bar{x}_i - \bar{x}_j|) \\ &\quad - \sum_j m_j(v_j - v_i)(\psi(|\bar{x}_i - \bar{x}_j|) - \psi(|x_i - x_j|)). \end{aligned} \quad (66)$$

Now we use fine properties of solutions to (2). The solutions to (2) are atomic, so in particular they fulfill Definition 23. The most important features concern propagation of the support. In the language of the ODE solutions, they control the change of trajectories $x_i(t)$ and $v_i(t)$. In particular we have

$$|\bar{x}_i(t) - \bar{x}_j(t)| \geq \min\{C, Ct\}.$$

There are two options: either $x_{0,i} \neq x_{0,j}$, in which case the difference is bounded from below by a constant (on a short time interval), or $x_{0,i} = x_{0,j}$ and $v_{0,i} \neq v_{0,j}$. Then the particles travel in different cones and by Lemmas (51) and (52) we find that the difference is bounded by t . Hence at least for a short time

$$\psi(|\bar{x}_i(t) - \bar{x}_j(t)|) \leq C + Ct^{-\alpha}.$$

Next we examine $\psi(|\bar{x}_i - \bar{x}_j|) - \psi(|x_i - x_j|)$. Here again we have two cases. The first one holds as $x_{0,i} \neq x_{0,j}$. Then for a short time interval,

$$|\psi(|\bar{x}_i(t) - \bar{x}_j(t)|) - \psi(|x_i - x_j|)| \leq C(|\delta x_i(t)| + |\delta x_j(t)|).$$

In the second case $x_{0,i} = x_{0,j}$ and $v_{0,i} \neq v_{0,j}$. Then by Lemma 51 we have $\text{supp } x_i(t), \bar{x}_i(t) \in x_{0,i} + tB(v_{0,i}, \epsilon)$ and $\text{supp } x_j(t), \bar{x}_j(t) \in x_{0,i} + tB(v_{0,j}, \epsilon)$, then

$$|x_i(t) - x_j(t)|, |\bar{x}_i(t) - \bar{x}_j(t)| \geq Ct$$

for $t \in [0, T_*]$. Hence $(\psi'(|s|) \sim s^{-1-\alpha})$, and

$$|\psi(|\bar{x}_i(t) - \bar{x}_j(t)|) - \psi(|x_i - x_j|)| \leq Ct^{-1-\alpha}(|\delta x_i(t)| + |\delta x_j(t)|).$$

Next, trivially from (66), we find that

$$\sup_{\tau \leq t} |\delta x_i(\tau)| \leq t \sup_{\tau \leq t} |\delta v(\tau)|.$$

Hence we obtain

$$\begin{aligned} \left| \frac{d}{dt} \delta v_i(t) \right| &\leq \sum_j m_j C t^{-\alpha} (\sup_{\tau \leq t} |\delta v_i(\tau)| + \sup_{\tau \leq t} |\delta v_j(\tau)|) \\ &\quad + \sum_j m_j C t^{-1-\alpha} (\sup_{\tau \leq t} |\delta x_i(\tau)| + \sup_{\tau \leq t} |\delta x_j(\tau)|). \end{aligned}$$

Finally,

$$\sum_i m_i \sup_{t \leq T} |\delta v_i(t)| \leq C \int_0^T (t^{-\alpha} + t^{-1-\alpha} t) dt \left(\sum_i m_i \sup_{\tau \leq t} |\delta v_i(\tau)| \right).$$

Taking $T \leq T_*$ so small that $C \int_0^T (t^{-\alpha} + t^{-1-\alpha} t) dt < \frac{1}{2}$, we obtain

$$\sum_i m_i \sup_{t \leq T} |\delta v_i(t)| \equiv 0.$$

The solutions are unique. The above procedure works until the time of the first sticking of a group of particles, say at time T_1 . Then the regularity in time ($v \in W^{1,p}(0, T)$) implies that all solutions are uniquely extended till time T_1 . Then we can restart our procedure with initial configuration $x_i(T_1)$, $v_i(T_1)$, taking into account that sticking particles create a new one with an appropriate mass. $\square$

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