Nonlocal Nonlinear Schrödinger Equations in $\mathbf{R}^3$

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Abstract

This paper studies a class of nonlocal nonlinear Schrödinger equations in $\mathbf{R}^3$, which occurs in the infinite ion acoustic speed limit of the Zakharov system with magnetic fields in a cold plasma. The magnetic fields induce some nonlocal effects in these nonlinear Schrödinger systems, and the main goal of this paper is to understand these effects. The key is to establish some a priori estimates on the nonlocal terms generated by the magnetic field, through which we obtain various conclusions including finite time blow-ups, sharp threshold of global existence and instability of standing waves for these equations.

1. Introduction

This paper concerns the equations that arise in the infinite ion acoustic speed limit ($c_0 \rightarrow \infty$) of the magnetic Zakharov system in a cold plasma in $\mathbf{R}^3$:

$$\begin{aligned} i\mathbf{E}_t + \Delta\mathbf{E} - n\mathbf{E} + i(\mathbf{E} \wedge \mathbf{B}) &= 0, \\ \frac{1}{c_0^2}n_{tt} - \Delta(n + |\mathbf{E}|^2) &= 0, \\ \Delta\mathbf{B} - i\eta\nabla \times \nabla \times (\mathbf{E} \wedge \overline{\mathbf{E}}) + \delta\mathbf{B} &= 0, \end{aligned} \tag{ZSM}$$

where $\mathbf{E}(t, x)$ is a complex vector-valued function from $\mathbf{R}^+ \times \mathbf{R}^3$ into $\mathbf{C}^3$, $n(t, x)$ is a real function from $\mathbf{R}^+ \times \mathbf{R}^3$ into $\mathbf{R}$, $\mathbf{B}(t, x)$ is a real vector-valued function from $\mathbf{R}^+ \times \mathbf{R}^3$ into $\mathbf{R}^3$, η and δ are two constants with $\eta > 0$ and $\delta \leq 0$, $\wedge$ denotes the exterior product of vector-valued functions and $\overline{\mathbf{E}}$ denotes the complex conjugate of $\mathbf{E}$.

Here, the Zakharov system (ZSM) describes the spontaneous generation of a magnetic field in a cold plasma. $\mathbf{E}$ denotes a slowly varying complex amplitude of the high-frequency electric field, $\mathbf{B}$ the self-generated magnetic field, n the fluctuation of the electron density from its equilibrium, c_0 is a parameter which tends to

$+\infty$ in the subsonic limit [6, 11, 27, 28]. In this limit, the Zakharov system (ZSM) formally reduces to the following vector nonlinear Schrödinger equations with magnetic field:

$$\begin{aligned} i\mathbf{E}_t + \Delta\mathbf{E} + |\mathbf{E}|^2\mathbf{E} + i(\mathbf{E} \wedge \mathbf{B}) &= 0, \\ \Delta\mathbf{B} - i\eta\nabla \times \nabla \times (\mathbf{E} \wedge \bar{\mathbf{E}}) + \delta\mathbf{B} &= 0. \end{aligned} \quad (\text{SB-1})$$

Using Fourier transforms, we can solve the second equation in (SB-1): for $\mathbf{E} \in H^1(\mathbf{R}^3)$, $\mathbf{B}(\mathbf{E}) \in L^2(\mathbf{R}^3)$ and

$$\mathbf{B}(\mathbf{E}) = \mathcal{F}^{-1} \left[\frac{i\eta}{|\xi|^2 - \delta} (\xi \wedge (\xi \wedge \mathcal{F}(\mathbf{E} \wedge \bar{\mathbf{E}}))) \right], \quad (\text{SB-2})$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the Fourier inverse transform, respectively (see [13, 16–18]). Due to rotational invariance of (SB-1), we let $\mathbf{E} = (E_1, E_2, 0)$ and $\xi = (\xi_1, \xi_2, 0)$ for simplicity. Then (SB-1) becomes the following nonlinear Schrödinger system with nonlocal terms:

$$\begin{aligned} i\partial_t E_1 + \Delta E_1 + (|E_1|^2 + |E_2|^2)E_1 \\ + E_2 \mathcal{F}^{-1} \left[\frac{\eta|\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \bar{E}_2 - \bar{E}_1 E_2) \right] &= 0, \quad t \in \mathbf{R}^+, x \in \mathbf{R}^3, \end{aligned} \quad (1.1)$$

$$\begin{aligned} i\partial_t E_2 + \Delta E_2 + (|E_1|^2 + |E_2|^2)E_2 \\ + E_1 \mathcal{F}^{-1} \left[\frac{\eta|\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\bar{E}_1 E_2 - E_1 \bar{E}_2) \right] &= 0, \quad t \in \mathbf{R}^+, x \in \mathbf{R}^3, \end{aligned} \quad (1.2)$$

along with the initial data

$$E_1(0, x) = E_{10}(x), \quad E_2(0, x) = E_{20}(x), \quad x \in \mathbf{R}^3. \quad (1.3)$$

Our main interest is to study the influence of magnetic fields on the solutions of (1.1)–(1.2). For this purpose, we first consider the case in which $E_1(t, x)$, $E_2(t, x)$ have the same frequency ω , with $\omega > 0$. Let $E_1(t, x) = e^{i\omega t}u(x)$, $E_2(t, x) = e^{i\omega t}v(x)$. Here $(u(x), v(x))$ is a pair of complex valued functions which satisfies the following equations:

$$\begin{aligned} -wu + \Delta u + (|u|^2 + |v|^2)u \\ + v \mathcal{F}^{-1} \left[\frac{\eta|\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \right] - v \mathcal{F}^{-1} \left[\frac{\eta|\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\bar{u}v) \right] &= 0, \quad x \in \mathbf{R}^3, \end{aligned} \quad (1.4)$$

$$\begin{aligned} -wv + \Delta v + (|u|^2 + |v|^2)v \\ + u \mathcal{F}^{-1} \left[\frac{\eta|\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\bar{u}v) \right] - u \mathcal{F}^{-1} \left[\frac{\eta|\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \right] &= 0, \quad x \in \mathbf{R}^3. \end{aligned} \quad (1.5)$$

It is easy to check that the nonlocal operator $\mathcal{F}^{-1} \left[\frac{\eta|\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \bar{E}_2 - \bar{E}_1 E_2) \right]$ in (1.1)–(1.2) has the symbol $\sigma(\xi) = \frac{\eta|\xi|^2}{|\xi|^2 - \delta}$ (see [8, 20, 21]). In [7], a simplified version was considered.

If one neglects the magnetic field, the Equations (SB-1) become the cubic focusing vector nonlinear Schrödinger equation:

$$i\mathbf{E}_t + \Delta\mathbf{E} + |\mathbf{E}|^2\mathbf{E} = 0, \quad t \geq 0, \quad x \in \mathbf{R}^N, \quad (\text{S})$$

which is a classical nonlinear model in quantum mechanics. GINIBRE and VELO [10] established the local existence of the Cauchy problem for Equation (S) in the energy class $H^1(\mathbf{R}^N)$ when $\mathbf{E}(t, x)$ is a scalar-valued function from $[0, +\infty) \times \mathbf{R}^N$ into $\mathbf{C}$. GLASSEY [9] and OGAWA-TSUTSUMI [22, 23] proved that the solutions to the Cauchy problem for Equation (S) blow up in a finite time for some initial data, especially for a class of sufficiently large data. When $\mathbf{E}(t, x)$ is a two- or three-dimensional vector-valued function, MCKINSTRIE in [15] obtained a similar finite time blow-up result for the solution to the Cauchy problem.

Returning to the nonlocal nonlinear Schrödinger equations (1.1)–(1.2), to our best knowledge, there is no rigorous proof of existence of singular solutions for (1.1)–(1.2), although physical reasonings suggest solutions may collapse in a finite time [11]. Moreover, it would be interesting to consider the sharp threshold for the global existence of a Cauchy problem (see [29, 30]) for such nonlinear Schrödinger equations with nonlocal effects, as with equations (1.1)–(1.2). As in many previous works [1, 2, 7, 14, 22, 25–27], we obtain finite time blow-ups, a sharp threshold of global existence and orbital instability of standing waves for (1.1)–(1.2) in $\mathbf{R}^3$. Due to a hiding symmetry, the nonlocal (magnetic field) effect preserves the underlying variational structure. This is a very useful fact that leads to conservation laws of energy and mass of the system. On the other hand, the nonlocal term leads to various complications in applications of virial identities (which have played a fundamental role in many of classical papers on the subject). We have introduced several new techniques to improve some of the earlier ideas. In particular, the proof of existence of ground states (minimal energy standing waves) contains some very general methods that may be useful for other related problems, as well. Our analysis also provides some preliminary understandings of the effect of a self-generated magnetic field in a cold plasma.

For simplicity, we shall denote various positive constants by C throughout this paper.

2. Preliminaries

2.1. Local Well-Posedness

From (SB-1)–(SB-2), the Cauchy problem (1.1)–(1.3) is equivalent to

$$\begin{cases} i\mathbf{E}_t + \Delta\mathbf{E} + |\mathbf{E}|^2\mathbf{E} + i(\mathbf{E} \wedge \mathbf{B}(\mathbf{E})) = 0, \\ \mathbf{B}(\mathbf{E}) = \mathcal{F}^{-1} \left[\frac{i\eta}{|\xi|^2 - \delta} (\xi \wedge (\xi \wedge \mathcal{F}(\mathbf{E} \wedge \bar{\mathbf{E}}))) \right], \\ \mathbf{E}(0, x) = \mathbf{E}_0(x), \quad x \in \mathbf{R}^3. \end{cases} \quad (\text{VSE})$$

It can also be written as the integral equation

$$\mathbf{E}(t) = U(t)\mathbf{E}_0 + i \int_0^t U(t-t') [|\mathbf{E}|^2\mathbf{E} + i(\mathbf{E} \wedge \mathbf{B}(\mathbf{E}))](t') dt',$$

where $U(t) = \exp[it\Delta]$ is the unitary semigroup generated by the free Schrödinger equation $i\mathbf{E}_t + \Delta\mathbf{E} = 0$ in the Hilbert space $H^s(\mathbf{R}^3)$ (with $s \in \mathbf{R}$). Therefore, via the second equation in (VSE), we get

$$\begin{aligned} \mathbf{E}_1 \wedge \mathbf{B}(\mathbf{E}_1) - \mathbf{E}_2 \wedge \mathbf{B}(\mathbf{E}_2) &= (\mathbf{E}_1 - \mathbf{E}_2) \wedge \mathbf{B}(\mathbf{E}_1) \\ &+ \mathbf{E}_2 \wedge \mathcal{F}^{-1} \left[\frac{i\eta}{|\xi|^2 - \delta} (\xi \wedge (\xi \wedge \mathcal{F}((\mathbf{E}_1 - \mathbf{E}_2) \wedge \bar{\mathbf{E}}_1 + \mathbf{E}_2 \wedge (\bar{\mathbf{E}}_1 - \bar{\mathbf{E}}_2)))) \right]. \end{aligned}$$

In view of the fact that the operator $(\Delta + \delta)^{-1} \nabla \times \nabla \times$ is uniformly bounded on L^p ($\forall 1 < p < +\infty$) and the contraction mapping principle, we obtain that the above integral equation has the local wellposedness in $H^1(\mathbf{R}^3)$ by a standard argument (see, for example, [10, 12, 16–19, 24]). More precisely, we have

Proposition 2.1. *Let $\eta > 0$ and $\delta \leq 0$, $\mathbf{E}_0 \in H^1(\mathbf{R}^3)$. Then the Cauchy problem (VSE) has a unique solution $\mathbf{E} \in X_{4,\text{loc}}^1([0, T])$ for some positive time $T = T(\mathbf{E}_0)$, and for any $0 \leq T_1 < T_2 < T$, the mapping $\mathbf{E}_0 (\in H^1(\mathbf{R}^3)) \mapsto \mathbf{E}(t) (\in X_4^1([T_1, T_2]))$ is continuous. Moreover, there holds either $T = +\infty$, or $T < +\infty$ and $\lim_{t \rightarrow T} \|\mathbf{E}\|_{H^1(\mathbf{R}^3)} = +\infty$. Here, for any interval $I \subset \mathbf{R}$, $0 \leq \frac{2}{q} = 3(\frac{1}{2} - \frac{1}{r}) < 1$, $s \in \mathbf{R}$,*

$$\begin{aligned} X_r^s(I) &= (C \cap L^\infty)(I; H^s) \cap L^q(I; H_r^s) \quad \text{and} \\ X_{r,\text{loc}}^s(I) &= \{u; u \in X_r^s(J), \forall J \subset\subset I\}. \end{aligned}$$

As a consequence, following [5, 10, 12, 13, 19, 24] and by the properties of Fourier transform [16–18], we obtain that the Cauchy problem (1.1)–(1.3), for $\eta > 0$, $\delta \leq 0$ and $(E_{10}(x), E_{20}(x)) \in H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3)$, has a unique solution $(E_1, E_2) \in C([0, T]; H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3))$ for some $T \in (0, +\infty)$ with $T = +\infty$ or $T < +\infty$ and

$$\lim_{t \rightarrow T} (\|E_1\|_{H^1(\mathbf{R}^3)} + \|E_2\|_{H^1(\mathbf{R}^3)}) = +\infty.$$

2.2. Conservation Laws of Mass and Energy

Using the inner structure of Equations (1.1)–(1.2), we get the conservation laws of total mass and total energy.

Lemma 2.1. *If (E_1, E_2) is a smooth solution to the Cauchy problem (1.1)–(1.3), then total mass and total energy are conserved:*

$$\begin{aligned} \int_{\mathbf{R}^3} (|E_1|^2 + |E_2|^2) \, dx &= \int_{\mathbf{R}^3} (|E_{10}|^2 + |E_{20}|^2) \, dx, \\ \mathcal{H}(E_1, E_2) &= \int_{\mathbf{R}^3} (|\nabla E_1|^2 + |\nabla E_2|^2) \, dx - \frac{1}{2} \int_{\mathbf{R}^3} (|E_1|^4 + |E_2|^4) \, dx \\ &\quad - \int_{\mathbf{R}^3} |E_1|^2 |E_2|^2 \, dx \\ &\quad - \frac{1}{2} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \left(|\mathcal{F}(E_1 \bar{E}_2)|^2 + |\mathcal{F}(\bar{E}_1 E_2)|^2 \right) \, d\xi \end{aligned} \tag{2.1}$$

$$\begin{aligned}
& + \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \overline{\mathcal{F}(\overline{E_1} E_2)} \, d\xi \\
& = \mathcal{H}(E_{10}, E_{20}).
\end{aligned} \tag{2.2}$$

Proof. Multiplying (1.1) by $\overline{E_1}$ and (1.2) by $\overline{E_2}$, taking the imaginary part, then integrating with respect to the space variable x on $\mathbf{R}^3$, we get

$$\begin{aligned}
& \operatorname{Im} \int_{\mathbf{R}^3} i \partial_t E_1 \overline{E_1} \, dx + \operatorname{Im} \int_{\mathbf{R}^3} \Delta E_1 \overline{E_1} \, dx + \operatorname{Im} \int_{\mathbf{R}^3} (|E_1|^2 + |E_2|^2) E_1 \overline{E_1} \, dx \\
& + \operatorname{Im} \int_{\mathbf{R}^3} i \partial_t E_2 \overline{E_2} \, dx + \operatorname{Im} \int_{\mathbf{R}^3} \Delta E_2 \overline{E_2} \, dx + \operatorname{Im} \int_{\mathbf{R}^3} (|E_1|^2 + |E_2|^2) E_2 \overline{E_2} \, dx \\
& + \operatorname{Im} \int_{\mathbf{R}^3} E_2 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \right] \overline{E_1} \, dx \\
& - \operatorname{Im} \int_{\mathbf{R}^3} E_2 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\overline{E_1} E_2) \right] \overline{E_1} \, dx \\
& + \operatorname{Im} \int_{\mathbf{R}^3} E_1 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\overline{E_1} E_2) \right] \overline{E_2} \, dx \\
& - \operatorname{Im} \int_{\mathbf{R}^3} E_1 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \right] \overline{E_2} \, dx = 0.
\end{aligned} \tag{A-1}$$

Noting that

$$\begin{aligned}
& \operatorname{Im} \int_{\mathbf{R}^3} E_2 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \right] \overline{E_1} \, dx \\
& + \operatorname{Im} \int_{\mathbf{R}^3} E_1 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\overline{E_1} E_2) \right] \overline{E_2} \, dx \\
& = \operatorname{Im} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} (|\mathcal{F}(E_1 \overline{E_2})|^2 + |\mathcal{F}(\overline{E_1} E_2)|^2) \, d\xi = 0,
\end{aligned}$$

and

$$\begin{aligned}
& \operatorname{Im} \int_{\mathbf{R}^3} E_2 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\overline{E_1} E_2) \right] \overline{E_1} \, dx \\
& + \operatorname{Im} \int_{\mathbf{R}^3} E_1 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \right] \overline{E_2} \, dx \\
& = \operatorname{Im} \left[2 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \overline{\mathcal{F}(\overline{E_1} E_2)} \, d\xi \right] = 0,
\end{aligned}$$

we conclude (2.1).

Multiplying (1.1) by $2\partial_t \overline{E_1}$ and (1.2) by $2\partial_t \overline{E_2}$, taking the real parts, and then integrating with respect to space variable x on $\mathbf{R}^3$, we obtain

$$\begin{aligned}
& 2\operatorname{Re} \int_{\mathbf{R}^3} i \partial_t E_1 \cdot \partial_t \overline{E_1} \, dx + 2\operatorname{Re} \int_{\mathbf{R}^3} i \partial_t E_2 \cdot \partial_t \overline{E_2} \, dx \\
&= 2\operatorname{Re} \int_{\mathbf{R}^3} (-\Delta E_1) \partial_t \overline{E_1} \, dx + 2\operatorname{Re} \int_{\mathbf{R}^3} (-\Delta E_2) \partial_t \overline{E_2} \, dx \\
&\quad - 2\operatorname{Re} \int_{\mathbf{R}^3} (|E_1|^2 + |E_2|^2) (E_1 \partial_t \overline{E_1} + E_2 \partial_t \overline{E_2}) \, dx \\
&\quad - 2\operatorname{Re} \int_{\mathbf{R}^3} E_2 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \right] \partial_t \overline{E_1} \, dx \\
&\quad - 2\operatorname{Re} \int_{\mathbf{R}^3} E_1 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\overline{E_1} E_2) \right] \partial_t \overline{E_2} \, dx \\
&\quad + 2\operatorname{Re} \int_{\mathbf{R}^3} E_2 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\overline{E_1} E_2) \right] \partial_t \overline{E_1} \, dx \\
&\quad + 2\operatorname{Re} \int_{\mathbf{R}^3} E_1 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \right] \partial_t \overline{E_2} \, dx.
\end{aligned}$$

In view of

$$\begin{aligned}
& -2\operatorname{Re} \int_{\mathbf{R}^3} E_2 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \right] \partial_t \overline{E_1} \, dx \\
& -2\operatorname{Re} \int_{\mathbf{R}^3} E_1 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\overline{E_1} E_2) \right] \partial_t \overline{E_2} \, dx \\
&= -\frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \left(|\mathcal{F}(E_1 \overline{E_2})|^2 + |\mathcal{F}(\overline{E_1} E_2)|^2 \right) d\xi,
\end{aligned}$$

and

$$\begin{aligned}
& 2\operatorname{Re} \int_{\mathbf{R}^3} E_2 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\overline{E_1} E_2) \right] \partial_t \overline{E_1} \, dx \\
& + 2\operatorname{Re} \int_{\mathbf{R}^3} E_1 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \right] \partial_t \overline{E_2} \, dx \\
&= \operatorname{Re} \frac{d}{dt} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \overline{\mathcal{F}(\overline{E_1} E_2)} \, d\xi,
\end{aligned}$$

we conclude (2.2).

This completes the proof of Lemma 2.1. $\square$

2.3. Variational Structures

For $(u, v) \in H_r^1(\mathbf{R}^3) \times H_r^1(\mathbf{R}^3)$ (or $(u, v) \in H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3)$), a pair of complex-valued functions, we define the functionals $S(u, v)$ and $R(u, v)$ by

$$\begin{aligned}
S(u, v) = & \frac{1}{2} \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) \, dx + \frac{1}{2} \omega \int_{\mathbf{R}^3} (|u|^2 + |v|^2) \, dx \\
& - \frac{1}{4} \int_{\mathbf{R}^3} (|u|^4 + |v|^4) \, dx - \frac{1}{2} \int_{\mathbf{R}^3} |u|^2 |v|^2 \, dx \\
& - \frac{1}{4} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) \, d\xi \\
& + \frac{1}{2} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} \, d\xi, \tag{2.3}
\end{aligned}$$

$$\begin{aligned}
R(u, v) = & \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) \, dx - \frac{3}{4} \int_{\mathbf{R}^3} (|u|^4 + |v|^4) \, dx - \frac{3}{2} \int_{\mathbf{R}^3} |u|^2 |v|^2 \, dx \\
& - \frac{3}{4} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) \, d\xi \\
& + \frac{\delta}{2} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) \, d\xi \\
& + \frac{3}{2} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} \, d\xi \\
& - \delta \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} \, d\xi \tag{2.4}
\end{aligned}$$

Furthermore, we define two sets M and M_1 by

$$M = \{(u, v) \in H_r^1(\mathbf{R}^3) \times H_r^1(\mathbf{R}^3) \setminus \{(0, 0)\}, R(u, v) = 0\}, \tag{2.5}$$

$$M_1 = \{(u, v) \in H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3) \setminus \{(0, 0)\}, R(u, v) = 0\}. \tag{2.6}$$

In view of (2.3), (2.4), (2.5) and (2.6), we define two constrained variational problems

$$d := \inf_{(u, v) \in M} S(u, v), \tag{2.7}$$

$$d_1 := \inf_{(u, v) \in M_1} S(u, v). \tag{2.8}$$

From $(u, v) \in H_r^1(\mathbf{R}^3) \times H_r^1(\mathbf{R}^3)$ (or $(u, v) \in H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3)$), $\eta > 0$, $\delta \leq 0$, the Sobolev's embedding theorem and the properties of Fourier transforms, it follows that functionals $S(u, v)$ and $R(u, v)$ are both well defined.

Remark 2.1. We note that for all $\theta \geq 1$,

$$\int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} |\mathcal{F}(u\bar{v})|^2 \, d\xi = \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} |\mathcal{F}(\bar{u}v)|^2 \, d\xi.$$

We also note that if (u, v) is a critical point of (2.3), and hence a solution of (1.4) and (1.5) then $(E_1, E_2) = (e^{i\omega t} u, e^{i\omega t} v)$ is a standing wave solution of (1.1)–(1.2).

For the constrained variational problem (2.8), we can obtain the following result:

Proposition 2.2. *Let $\eta > 0$ and $\delta \leq 0$. There holds $d_1 > 0$.*

Proof. (2.3), (2.4) and (2.6) imply that for $(u, v) \in M_1$,

$$\begin{aligned} S(u, v) &= \frac{1}{6} \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) \, dx + \frac{\omega}{2} \int_{\mathbf{R}^3} (|u|^2 + |v|^2) \, dx \\ &\quad - \frac{\delta}{6} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) \, d\xi \\ &\quad + \frac{\delta}{3} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} \, d\xi. \end{aligned} \quad (2.9)$$

Noting that

$$\begin{aligned} &\left| \frac{\delta}{3} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} \, d\xi \right| \\ &\leq \frac{|\delta|}{6} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) \, d\xi, \end{aligned}$$

we get for $(u, v) \in M_1$,

$$S(u, v) \geq \frac{1}{6} \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) \, dx + \frac{\omega}{2} \int_{\mathbf{R}^3} (|u|^2 + |v|^2) \, dx. \quad (2.10)$$

On the other hand, in view of

$$\begin{aligned} \frac{3}{4} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} - \frac{\delta}{2} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} &\leq \frac{5\eta |\xi|^2}{4(|\xi|^2 - \delta)}, \\ \frac{3}{2} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} - \delta \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} &\leq \frac{5\eta |\xi|^2}{2(|\xi|^2 - \delta)}, \end{aligned}$$

from $R(u, v) = 0$ and the Gagliardo–Nirenberg inequality

$$\|v\|_{L^4(\mathbf{R}^3)}^4 \leq c \|\nabla v\|_{L^2(\mathbf{R}^3)}^3 \|v\|_{L^2(\mathbf{R}^3)}, \quad v \in H^1(\mathbf{R}^3),$$

it follows that

$$\begin{aligned} \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) \, dx &= \frac{3}{4} \int_{\mathbf{R}^3} (|u|^4 + |v|^4) \, dx + \frac{3}{2} \int_{\mathbf{R}^3} |u|^2 |v|^2 \, dx \\ &\quad + \frac{3}{4} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) \, d\xi \\ &\quad - \frac{\delta}{2} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) \, d\xi \\ &\quad - \frac{3}{2} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} \, d\xi \\ &\quad + \delta \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} \, d\xi \\ &\leq \frac{3}{4} \int_{\mathbf{R}^3} (|u|^4 + |v|^4) \, dx + \frac{3}{2} \int_{\mathbf{R}^3} |u|^2 |v|^2 \, dx \end{aligned}$$

$$\begin{aligned}
& + \frac{5}{2} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) d\xi \\
& \leq c \int_{\mathbf{R}^3} (|u|^4 + |v|^4) dx \\
& \leq c \left(\int_{\mathbf{R}^3} |\nabla u|^2 dx \right)^{\frac{3}{2}} \left(\int_{\mathbf{R}^3} |u|^2 dx \right)^{\frac{1}{2}} \\
& \quad + \left(\int_{\mathbf{R}^3} |\nabla v|^2 dx \right)^{\frac{3}{2}} \left(\int_{\mathbf{R}^3} |v|^2 dx \right)^{\frac{1}{2}} \\
& \leq c \left(\int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) dx \right)^{\frac{3}{2}} \left(\int_{\mathbf{R}^3} (|u|^2 + |v|^2) dx \right)^{\frac{1}{2}},
\end{aligned}$$

which implies that

$$\left(\int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) dx \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^3} (|u|^2 + |v|^2) dx \right)^{\frac{1}{2}} \geq c. \quad (2.11)$$

Thus, by Young's inequality, we have

$$\begin{aligned}
& 2 \left(\int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) dx \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^3} (|u|^2 + |v|^2) dx \right)^{\frac{1}{2}} \\
& \leq \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) dx + \int_{\mathbf{R}^3} (|u|^2 + |v|^2) dx.
\end{aligned} \quad (2.12)$$

Combining (2.9), (2.10), (2.11) and (2.12), we conclude that

$$S(u, v) \geq c > 0. \quad (2.13)$$

Therefore, we get $d_1 > 0$ from (2.8). $\square$

3. Finite Time Blow-ups

3.1. Main Blow-Up Results

Let

$$\Sigma := \{(E_1, E_2) : (|x|E_1, |x|E_2) \in L^2(\mathbf{R}^3) \times L^2(\mathbf{R}^3)\} \cap H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3). \quad (3.1)$$

The main results of this section can be stated as follows:

Theorem 3.1. *Let $\eta > 0$ and $\delta \leq 0$. Assume that*

1. $(E_{10}, E_{20}) \in H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3)$ and $(E_1, E_2) \in C([0, T]; H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3))$ is a solution to the Cauchy problem (1.1)–(1.3);
2. $(|x|E_{10}, |x|E_{20}) \in L^2(\mathbf{R}^3) \times L^2(\mathbf{R}^3)$ and one of the following three conditions holds:

- (i) $\mathcal{H}(E_{10}, E_{20}) < 0$;
- (ii) $\mathcal{H}(E_{10}, E_{20}) = 0$,

$$\operatorname{Im} \int_{\mathbf{R}^3} (x \nabla E_{10}) \overline{E_{10}} \, dx + \operatorname{Im} \int_{\mathbf{R}^3} (x \nabla E_{20}) \overline{E_{20}} \, dx < 0;$$

- (iii) $\mathcal{H}(E_{10}, E_{20}) > 0$,

$$\begin{aligned} & \operatorname{Im} \int_{\mathbf{R}^3} (x \nabla E_{10}) \overline{E_{10}} \, dx + \operatorname{Im} \int_{\mathbf{R}^3} (x \nabla E_{20}) \overline{E_{20}} \, dx \\ & \leq - \left[2\mathcal{H}(E_{10}, E_{20}) \int_{\mathbf{R}^3} |x|^2 (|E_{10}|^2 + |E_{20}|^2) \, dx \right]^{\frac{1}{2}}. \end{aligned}$$

Then there exists $0 < T < +\infty$ such that

$$\lim_{t \rightarrow T} (\|E_1\|_{H^1(\mathbf{R}^3)} + \|E_2\|_{H^1(\mathbf{R}^3)}) = +\infty.$$

Furthermore, if the solution $(E_1(t, x), E_2(t, x))$ to the Cauchy problem (1.1)–(1.3) is radially symmetric with respect to x and $(E_1(t, x), E_2(t, x)) \in H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3)$, then the following finite time blow-up result holds.

Theorem 3.2. *Let $\eta > 0$, $\delta \leq 0$ and (E_1, E_2) be a classical and radially symmetric solution to the Cauchy problem (1.1)–(1.3) with $(E_{10}, E_{20}) \in \Sigma$. Assume that*

$$(I) \quad \mathcal{H}(E_{10}, E_{20}) \leq 0$$

and

$$(II) \quad \operatorname{Im} \int_{\mathbf{R}^3} r \overline{E_{10}} \partial_r E_{10} \, dx + \operatorname{Im} \int_{\mathbf{R}^3} r \overline{E_{20}} \partial_r E_{20} \, dx < 0.$$

Then there exists a finite time T such that

$$\lim_{t \rightarrow T} (\|\nabla E_1\|_{L^2(\mathbf{R}^3)}^2 + \|\nabla E_2\|_{L^2(\mathbf{R}^3)}^2) = +\infty.$$

3.2. Key Ingredients

The following Lemma 3.1 and Proposition 3.1 will play key roles in the proof of the main results of this section, Theorem 3.1 and Theorem 3.2.

Lemma 3.1. [26] *Let f be a scalar-valued function. If $|x|f$ and ∇f belong to $L^2(\mathbf{R}^3)$, then f is in $L^2(\mathbf{R}^3)$ and satisfies*

$$\int_{\mathbf{R}^3} |f|^2 \, dx \leq \frac{2}{3} \left(\int_{\mathbf{R}^3} |\nabla f|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^3} |x|^2 |f|^2 \, dx \right)^{\frac{1}{2}}.$$

Proposition 3.1. *Let $(E_{10}, E_{20}) \in \Sigma$ and let $(E_1, E_2) \in C([0, T]; \Sigma)$ be a solution to the Cauchy problem (1.1)–(1.3) on $[0, T)$. Putting*

$$J(t) := \int_{\mathbf{R}^3} |x|^2 (|E_1|^2 + |E_2|^2) dx, \quad (3.2)$$

one gets

$$\begin{aligned} \frac{d}{dt} J(t) &= 4 \operatorname{Im} \int_{\mathbf{R}^3} [(x \nabla E_1) \overline{E_1} + (x \nabla E_2) \overline{E_2}] dx, \\ \frac{d^2}{dt^2} J(t) &= 8 \int_{\mathbf{R}^3} (|\nabla E_1|^2 + |\nabla E_2|^2) dx \\ &\quad - 6 \int_{\mathbf{R}^3} (|E_1|^4 + |E_2|^4) dx - 12 \int_{\mathbf{R}^3} |E_1|^2 |E_2|^2 dx \\ &\quad - 6 \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} (|\mathcal{F}(E_1 \overline{E_2})|^2 + |\mathcal{F}(\overline{E_1} E_2)|^2) d\xi \\ &\quad + 4\delta \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(E_1 \overline{E_2})|^2 + |\mathcal{F}(\overline{E_1} E_2)|^2) d\xi \\ &\quad + 12 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \overline{\mathcal{F}(\overline{E_1} E_2)} d\xi \\ &\quad - 8\delta \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(E_1 \overline{E_2}) \overline{\mathcal{F}(\overline{E_1} E_2)} d\xi \\ &= 8\mathcal{H}(E_{10}, E_{20}) \\ &\quad - 2 \int_{\mathbf{R}^3} (|E_1|^4 + |E_2|^4) dx - 4 \int_{\mathbf{R}^3} |E_1|^2 |E_2|^2 dx \\ &\quad - 2 \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} (|\mathcal{F}(E_1 \overline{E_2})|^2 + |\mathcal{F}(\overline{E_1} E_2)|^2) d\xi \\ &\quad + 4\delta \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(E_1 \overline{E_2})|^2 + |\mathcal{F}(\overline{E_1} E_2)|^2) d\xi \\ &\quad + 4 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \overline{\mathcal{F}(\overline{E_1} E_2)} d\xi \\ &\quad - 8\delta \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(E_1 \overline{E_2}) \overline{\mathcal{F}(\overline{E_1} E_2)} d\xi \\ &= -4 \int_{\mathbf{R}^3} (|\nabla E_1|^2 + |\nabla E_2|^2) dx + 12\mathcal{H}(E_{10}, E_{20}) \\ &\quad + 4\delta \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(E_1 \overline{E_2})|^2 + |\mathcal{F}(\overline{E_1} E_2)|^2) d\xi \\ &\quad - 8\delta \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(E_1 \overline{E_2}) \overline{\mathcal{F}(\overline{E_1} E_2)} d\xi. \end{aligned} \quad (3.4)$$

Proof of Proposition 3.1. Since $(E_{10}, E_{20}) \in \Sigma$, by GINIBRE-VELO [10], one has $(E_1, E_2) \in \Sigma$, where $(E_1, E_2) \in C([0, T]; \Sigma)$ is a solution to the Cauchy problem

(1.1)–(1.3). From (1.1)–(1.2) and (3.2) it follows that

$$\begin{aligned}
\frac{d}{dt} J(t) &= \int_{\mathbf{R}^3} |x|^2 [(\partial_t E_1 \overline{E_1} + E_1 \partial_t \overline{E_1}) + (\partial_t E_2 \overline{E_2} + E_2 \partial_t \overline{E_2})] dx \\
&= 4\text{Im} \int_{\mathbf{R}^3} [(x \nabla E_1) \overline{E_1} + (x \nabla E_2) \overline{E_2}] dx \\
&\quad - 2\text{Im} \int_{\mathbf{R}^3} |x|^2 \overline{E_1} E_2 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \right] dx \\
&\quad - 2\text{Im} \int_{\mathbf{R}^3} |x|^2 E_1 \overline{E_2} \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\overline{E_1} E_2) \right] dx \\
&\quad + 2\text{Im} \int_{\mathbf{R}^3} |x|^2 \overline{E_1} E_2 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\overline{E_1} E_2) \right] dx \\
&\quad + 2\text{Im} \int_{\mathbf{R}^3} |x|^2 E_1 \overline{E_2} \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \right] dx.
\end{aligned}$$

A direct calculation shows

$$\begin{aligned}
&-2\text{Im} \int_{\mathbf{R}^3} |x|^2 \overline{E_1} E_2 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \right] dx \\
&\quad - 2\text{Im} \int_{\mathbf{R}^3} |x|^2 E_1 \overline{E_2} \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\overline{E_1} E_2) \right] dx \\
&= -8\text{Im} \left[\text{Re} \int_{\mathbf{R}^3} \frac{\eta \xi}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \partial_{\xi} \overline{\mathcal{F}(E_1 \overline{E_2})} d\xi \right] \\
&\quad + 8\text{Im} \left[\text{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2 \xi}{(|\xi|^2 - \delta)^2} \mathcal{F}(E_1 \overline{E_2}) \partial_{\xi} \overline{\mathcal{F}(E_1 \overline{E_2})} d\xi \right] \\
&\quad - 4\text{Im} \left[\text{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \partial_{\xi} \mathcal{F}(E_1 \overline{E_2}) \partial_{\xi} \overline{\mathcal{F}(E_1 \overline{E_2})} d\xi \right] \\
&= 0.
\end{aligned}$$

In addition, one can check that

$$\begin{aligned}
&2\text{Im} \int_{\mathbf{R}^3} |x|^2 \overline{E_1} E_2 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\overline{E_1} E_2) \right] dx \\
&\quad + 2\text{Im} \int_{\mathbf{R}^3} |x|^2 E_1 \overline{E_2} \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \right] dx \\
&= 8\text{Im} \left[\text{Re} \int_{\mathbf{R}^3} \frac{\eta \xi}{|\xi|^2 - \delta} \mathcal{F}(\overline{E_1} E_2) \partial_{\xi} \overline{\mathcal{F}(E_1 \overline{E_2})} d\xi \right] \\
&\quad - 8\text{Im} \left[\text{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2 \xi}{(|\xi|^2 - \delta)^2} \mathcal{F}(\overline{E_1} E_2) \partial_{\xi} \overline{\mathcal{F}(E_1 \overline{E_2})} d\xi \right] \\
&\quad + 4\text{Im} \left[\text{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \partial_{\xi} \mathcal{F}(\overline{E_1} E_2) \partial_{\xi} \overline{\mathcal{F}(E_1 \overline{E_2})} d\xi \right] \\
&= 0.
\end{aligned}$$

Hence, we have

$$\frac{d}{dt} J(t) = 4\text{Im} \int_{\mathbf{R}^3} [(x \nabla E_1) \overline{E_1} + (x \nabla E_2) \overline{E_2}] \, dx. \quad (3.5)$$

Differentiating (3.5) again with respect to t , after a careful computation and proper groupings, we get

$$\begin{aligned} \frac{d^2}{dt^2} J(t) &= 4\text{Im} \int_{\mathbf{R}^3} \left\{ x [\nabla E_1 \partial_t \overline{E_1} + \nabla (\partial_t E_1) \overline{E_1}] + x [\nabla E_2 \partial_t \overline{E_2} + \nabla (\partial_t E_2) \overline{E_2}] \right\} \, dx \\ &= 4\text{Re} \int_{\mathbf{R}^3} \left\{ x [\nabla \overline{E_1} (i \partial_t E_1) - \nabla (i \partial_t E_1) \overline{E_1}] \right. \\ &\quad \left. + x [\nabla \overline{E_2} (i \partial_t E_2) - \nabla (i \partial_t E_2) \overline{E_2}] \right\} \, dx \\ &= 8\text{Re} \int_{\mathbf{R}^3} x \nabla \overline{E_1} \left\{ -\Delta E_1 - (|E_1|^2 + |E_2|^2) E_1 \right. \\ &\quad \left. - E_2 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2} - \overline{E_1} E_2) \right] \right\} \, dx \\ &\quad + 8\text{Re} \int_{\mathbf{R}^3} x \nabla \overline{E_2} \left\{ -\Delta E_2 - (|E_1|^2 + |E_2|^2) E_2 \right. \\ &\quad \left. - E_1 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\overline{E_1} E_2 - E_1 \overline{E_2}) \right] \right\} \, dx \\ &\quad + 12\text{Re} \int_{\mathbf{R}^3} \overline{E_1} \left\{ -\Delta E_1 - (|E_1|^2 + |E_2|^2) E_1 \right. \\ &\quad \left. - E_2 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2} - \overline{E_1} E_2) \right] \right\} \, dx \\ &\quad + 12\text{Re} \int_{\mathbf{R}^3} \overline{E_2} \left\{ -\Delta E_2 - (|E_1|^2 + |E_2|^2) E_2 \right. \\ &\quad \left. - E_1 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\overline{E_1} E_2 - E_1 \overline{E_2}) \right] \right\} \, dx. \end{aligned} \quad (\text{C-1})$$

These nonlocal expressions have to be handled with extra care. We verify them one by one, as follows:

Using integration by parts, the Parseval identity, and properties of Fourier transforms, we derive that

$$\begin{aligned} &\text{Re} \int_{\mathbf{R}^3} x \nabla \overline{E_1} \left\{ -E_2 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \right] \right\} \, dx \\ &\quad + \text{Re} \int_{\mathbf{R}^3} x \nabla \overline{E_2} \left\{ -E_1 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\overline{E_1} E_2) \right] \right\} \, dx \\ &= \frac{3}{4} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \left(|\mathcal{F}(E_1 \overline{E_2})|^2 + |\mathcal{F}(\overline{E_1} E_2)|^2 \right) \, d\xi \\ &\quad + \frac{\delta}{2} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \left(|\mathcal{F}(E_1 \overline{E_2})|^2 + |\mathcal{F}(\overline{E_1} E_2)|^2 \right) \, d\xi, \end{aligned} \quad (\text{C-2})$$

$$\begin{aligned}
& \operatorname{Re} \int_{\mathbf{R}^3} \overline{E_1} \left\{ -E_2 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \right] \right\} d\xi \\
& + \operatorname{Re} \int_{\mathbf{R}^3} \overline{E_2} \left\{ -E_1 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\overline{E_1} E_2) \right] \right\} d\xi \\
& = - \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \left(|\mathcal{F}(E_1 \overline{E_2})|^2 + |\mathcal{F}(\overline{E_1} E_2)|^2 \right) d\xi, \tag{C-3}
\end{aligned}$$

$$\begin{aligned}
& \operatorname{Re} \int_{\mathbf{R}^3} x \nabla \overline{E_1} \left\{ E_2 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\overline{E_1} E_2) \right] \right\} d\xi \\
& + \operatorname{Re} \int_{\mathbf{R}^3} x \nabla \overline{E_2} \left\{ E_1 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \right] \right\} d\xi \\
& = -\frac{3}{2} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \overline{\mathcal{F}(\overline{E_1} E_2)} d\xi \\
& - \delta \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(E_1 \overline{E_2}) \overline{\mathcal{F}(\overline{E_1} E_2)} d\xi, \tag{C-4}
\end{aligned}$$

$$\begin{aligned}
& \operatorname{Re} \int_{\mathbf{R}^3} \overline{E_1} \left\{ E_2 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\overline{E_1} E_2) \right] \right\} d\xi \\
& + \operatorname{Re} \int_{\mathbf{R}^3} \overline{E_2} \left\{ E_1 \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \right] \right\} d\xi \\
& = 2 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(E_1 \overline{E_2}) \overline{\mathcal{F}(\overline{E_1} E_2)} d\xi. \tag{C-5}
\end{aligned}$$

(C-1)–(C-5) imply the conclusion of Proposition 3.1. $\square$

3.3. Proofs of Theorem 3.1 and Theorem 3.2

With the preparations in the previous subsection, we can now prove Theorems 3.1 and 3.2.

Proof of Theorem 3.1. We show this theorem by contradiction. Assume that the maximal existence time T of the solution to the Cauchy problem (1.1)–(1.3) is infinity. For $\theta = 1, 2$, we have, in particular, that

$$\begin{aligned}
& \left| 2 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} \mathcal{F}(E_1 \overline{E_2}) \overline{\mathcal{F}(\overline{E_1} E_2)} d\xi \right| \\
& \leq 2 \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} |\mathcal{F}(E_1 \overline{E_2})| |\overline{\mathcal{F}(\overline{E_1} E_2)}| d\xi \\
& \leq \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} (|\mathcal{F}(E_1 \overline{E_2})|^2 + |\mathcal{F}(\overline{E_1} E_2)|^2) d\xi, \tag{3.7}
\end{aligned}$$

which together with $\eta > 0$, $\delta \leq 0$ and Proposition 3.1 implies

$$\frac{d^2}{dt^2} J(t) \leq 8\mathcal{H}(E_{10}, E_{20}), \quad (3.8)$$

where $J(t)$ is defined by (3.2). Integrating (3.8) twice with respect to t , we have

$$J(t) \leq 8\mathcal{H}(E_{10}, E_{20})t^2 + J'(0)t + J(0). \quad (3.9)$$

Under hypothesis (i), (ii) or (iii), (3.3) and (3.9), we conclude that if (E_1, E_2) remains in $H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3)$, then there is a $T^* < +\infty$ such that

$$\lim_{t \rightarrow T^*} \int_{\mathbf{R}^3} |x|^2 (|E_1|^2 + |E_2|^2) dx = 0. \quad (3.10)$$

On the other hand, by (2.1) and Lemma 3.1, one has

$$\begin{aligned} \int_{\mathbf{R}^3} (|E_{10}|^2 + |E_{20}|^2) dx &= \int_{\mathbf{R}^3} (|E_1|^2 + |E_2|^2) dx \\ &\leq \frac{4}{3} \left(\int_{\mathbf{R}^3} (|\nabla E_1|^2 + |\nabla E_2|^2) dx \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^3} |x|^2 (|E_1|^2 + |E_2|^2) dx \right)^{\frac{1}{2}}, \end{aligned}$$

which together with (3.10) yields

$$\lim_{t \rightarrow T^*} \int_{\mathbf{R}^3} (|\nabla E_1|^2 + |\nabla E_2|^2) dx = +\infty. \quad (3.11)$$

The latter implies that the maximal existence time $T_{\max} \leq T^*$, and hence is a contradiction. $\square$

Proof of Theorem 3.2. Let

$$F(t) = -\operatorname{Im} \int_{\mathbf{R}^3} r(\overline{E_1} \partial_r E_1) dx - \operatorname{Im} \int_{\mathbf{R}^3} r(\overline{E_2} \partial_r E_2) dx. \quad (3.12)$$

From assumption (II) in Theorem 3.2, it follows that $F(0) > 0$. By Proposition 3.1 we have

$$\begin{aligned} \frac{d}{dt} F(t) &= \int_{\mathbf{R}^3} (|\nabla E_1|^2 + |\nabla E_2|^2) dx - 3\mathcal{H}(E_{10}, E_{20}) \\ &\quad - \delta \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \left(|\mathcal{F}(E_1 \overline{E_2})|^2 + |\mathcal{F}(\overline{E_1} E_2)|^2 \right) d\xi \\ &\quad + 2\delta \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(E_1 \overline{E_2}) \overline{\mathcal{F}(\overline{E_1} E_2)} d\xi. \end{aligned} \quad (3.13)$$

By (3.7), $\eta > 0$, $\delta \leq 0$ and assumption (I) in Theorem 3.2, we obtain that

$$\frac{d}{dt} F(t) \geq \int_{\mathbf{R}^3} (|\nabla E_1|^2 + |\nabla E_2|^2) dx, \quad (3.14)$$

since $F(0) > 0$, (3.14) implies that $F(t) > 0$ whenever (E_1, E_2) exists. On the other hand, from (3.3) we have

$$\frac{d}{dt} \int_{\mathbf{R}^3} r^2 (|E_1|^2 + |E_2|^2) dx = -4F(t) < 0. \quad (3.15)$$

Therefore,

$$\int_{\mathbf{R}^3} r^2 (|E_1|^2 + |E_2|^2) dx \leq \int_{\mathbf{R}^3} r^2 (|E_{10}|^2 + |E_{20}|^2) dx = c_0^2 < +\infty. \quad (3.16)$$

The Schwartz inequality then implies

$$|F(t)| = F(t) \leq c_* \left(\|\nabla E_1\|_{L^2(\mathbf{R}^3)}^2 + \|\nabla E_2\|_{L^2(\mathbf{R}^3)}^2 \right)^{\frac{1}{2}}, \quad (3.17)$$

where $c_* \geq 2c_0$. Combining (3.14) and (3.17), we conclude

$$\frac{d}{dt} F(t) \geq \frac{F^2(t)}{c_*^2} \text{ with } F(0) > 0, \quad (3.18)$$

and consequently

$$F(t) \geq \frac{c_*^2 F(0)}{c_*^2 - F(0)t}, \quad t \in [0, c_*^2/F(0)].$$

In particular, one has

$$\lim_{t \rightarrow T} \left(\|\nabla E_1\|_{L^2(\mathbf{R}^3)}^2 + \|\nabla E_2\|_{L^2(\mathbf{R}^3)}^2 \right) = +\infty$$

for some $T \leq T_0 < +\infty$, where $T_0 = \frac{c_*^2}{F(0)}$. $\square$

4. Existence of Standing Waves

In this section, we discuss the existence of minimal energy standing waves of the system (1.1)–(1.2) for $\eta > 0$ and $\delta \leq 0$. The main result can be stated as:

Theorem 4.1. *For $\eta > 0$ and $\delta \leq 0$, there exists $(Q, P) \in M$ such that*

1. $S(Q, P) = d = \inf_{(u,v) \in M} S(u, v)$;
2. (Q, P) is a ground state solution to (1.4) and (1.5)
3. $(Q(x), P(x))$ are functions of $|x|$ alone and decay exponentially at the infinity.

Remark 4.1. For $\eta > 0$ and $\delta \leq 0$, since

$$\int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} |\mathcal{F}(Q\bar{P})|^2 d\xi = \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} |\mathcal{F}(\bar{Q}P)|^2 d\xi, \quad (4.1)$$

and

$$\begin{aligned}
& \left| \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(Q\bar{P}) \overline{\mathcal{F}(\bar{Q}P)} \, d\xi \right| \\
& \leq \frac{1}{2} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(Q\bar{P})|^2 + |\mathcal{F}(\bar{Q}P)|^2) \, d\xi \\
& = \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} |\mathcal{F}(Q\bar{P})|^2 \, d\xi,
\end{aligned} \tag{4.2}$$

it follows from Theorem 4.1 and virial identity that

$$\begin{aligned}
S(Q, P) &= \frac{1}{6} \int_{\mathbf{R}^3} (|\nabla Q|^2 + |\nabla P|^2) \, dx + \frac{\omega}{2} \int_{\mathbf{R}^3} (|Q|^2 + |P|^2) \, dx \\
&\quad - \frac{\delta}{3} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \left[|\mathcal{F}(Q\bar{P})|^2 \right. \\
&\quad \left. - \operatorname{Re} \left(\mathcal{F}(Q\bar{P}) \overline{\mathcal{F}(\bar{Q}P)} \right) \right] \, d\xi > 0.
\end{aligned} \tag{4.3}$$

In order to prove Theorem 4.1, we need a series of basic facts. First of all, one has the following well-known result:

Lemma 4.1. (STRAUSS [25]) *For $1 < \sigma < 6$, the embedding*

$$H_r^1(\mathbf{R}^3) \mapsto L_r^\sigma(\mathbf{R}^3)$$

is compact, where

$$H_r^1(\mathbf{R}^3) = \{f(x) \in H^1(\mathbf{R}^3) : f(x) = f(|x|) \text{ is a function of } |x| \text{ alone}\}.$$

Next, we have the following proposition.

Proposition 4.1. *Let $\eta > 0$, $\delta \leq 0$. Then the non-trivial solution of (1.4) and (1.5) belongs to M .*

Proof. Let (u, v) be a non-trivial solution of (1.4) and (1.5). Multiplying (1.4) by $\bar{u}$ and (1.5) by $\bar{v}$, then integrating with respect to x on $\mathbf{R}^3$, we get

$$\begin{aligned}
& -\omega \int_{\mathbf{R}^3} |u|^2 \, dx - \omega \int_{\mathbf{R}^3} |v|^2 \, dx - \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) \, dx \\
& + \int_{\mathbf{R}^3} (|u|^4 + |v|^4) \, dx + 2 \int_{\mathbf{R}^3} |u|^2 |v|^2 \, dx \\
& + \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) \, d\xi \\
& - 2 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} \, d\xi = 0.
\end{aligned} \tag{4.4}$$

On the other hand, multiplying (1.4) by $x \nabla \bar{u}$ and (1.5) by $x \nabla \bar{v}$, then integrating with respect to x on $\mathbf{R}^3$ and taking the real parts for the resulting equations, we obtain

$$\begin{aligned}
& \operatorname{Re} \int_{\mathbf{R}^3} (-\omega u x \nabla \bar{u} - \omega v x \nabla \bar{v}) \, dx + \operatorname{Re} \int_{\mathbf{R}^3} (\Delta u x \nabla \bar{u} + \Delta v x \nabla \bar{v}) \, dx \\
& + \operatorname{Re} \int_{\mathbf{R}^3} (|u|^2 + |v|^2)(u x \nabla \bar{u} + v x \nabla \bar{v}) \, dx \\
& + \operatorname{Re} \int_{\mathbf{R}^3} v x \nabla \bar{u} \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(u \bar{v}) \right] \, dx \\
& + \operatorname{Re} \int_{\mathbf{R}^3} u x \nabla \bar{v} \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\bar{u} v) \right] \, dx \\
& - \operatorname{Re} \int_{\mathbf{R}^3} v x \nabla \bar{u} \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\bar{u} v) \right] \, dx \\
& - \operatorname{Re} \int_{\mathbf{R}^3} u x \nabla \bar{v} \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(u \bar{v}) \right] \, dx = 0.
\end{aligned} \tag{4.5}$$

By a direct calculation, we find

$$\operatorname{Re} \int_{\mathbf{R}^3} (-\omega u x \nabla \bar{u} - \omega v x \nabla \bar{v}) \, dx = \frac{3\omega}{2} \int_{\mathbf{R}^3} (|u|^2 + |v|^2) \, dx, \tag{S-1}$$

$$\begin{aligned}
& \operatorname{Re} \int_{\mathbf{R}^3} \Delta u x \nabla \bar{u} \, dx + \operatorname{Re} \int_{\mathbf{R}^3} \Delta v x \nabla \bar{v} \, dx \\
& = \frac{1}{2} \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) \, dx,
\end{aligned} \tag{S-2}$$

$$\begin{aligned}
& \operatorname{Re} \int_{\mathbf{R}^3} (|u|^2 + |v|^2) u x \nabla \bar{u} \, dx + \operatorname{Re} \int_{\mathbf{R}^3} (|u|^2 + |v|^2) v x \nabla \bar{v} \, dx \\
& = -\frac{3}{4} \int_{\mathbf{R}^N} (|u|^4 + |v|^4) \, dx - \frac{3}{2} \int_{\mathbf{R}^N} |u|^2 |v|^2 \, dx,
\end{aligned} \tag{S-3}$$

$$\begin{aligned}
& \operatorname{Re} \int_{\mathbf{R}^3} v x \nabla \bar{u} \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(u \bar{v}) \right] \, dx \\
& + \operatorname{Re} \int_{\mathbf{R}^3} u x \nabla \bar{v} \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\bar{u} v) \right] \, dx \\
& = -\frac{3}{4} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} (|\mathcal{F}(u \bar{v})|^2 + |\mathcal{F}(\bar{u} v)|^2) \, d\xi \\
& - \frac{\delta}{2} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(u \bar{v})|^2 + |\mathcal{F}(\bar{u} v)|^2) \, d\xi,
\end{aligned} \tag{S-4}$$

$$\begin{aligned}
& \operatorname{Re} \int_{\mathbf{R}^3} vx \nabla \bar{u} \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\bar{u}v) \right] dx \\
& + \operatorname{Re} \int_{\mathbf{R}^3} ux \nabla \bar{v} \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \right] dx \\
& = -\frac{3}{2} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi \\
& - \delta \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi.
\end{aligned} \tag{S-5}$$

Putting (S-1)–(S-5) into (4.5), we obtain

$$\begin{aligned}
& \frac{3\omega}{2} \int_{\mathbf{R}^3} (|u|^2 + |v|^2) dx + \frac{1}{2} \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) dx \\
& - \frac{3}{4} \int_{\mathbf{R}^3} (|u|^4 + |v|^4) dx - \frac{3}{2} \int_{\mathbf{R}^3} |u|^2 |v|^2 dx \\
& - \frac{3}{4} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) d\xi \\
& - \frac{\delta}{2} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) d\xi \\
& + \frac{3}{2} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi \\
& + \delta \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi = 0.
\end{aligned} \tag{4.6}$$

From (2.4), (4.4) and (4.6) it follows that $R(u, v) = 0$, and hence $(u, v) \in M$. $\square$

Proposition 4.2. *The functional S is bounded from below on M for $\eta > 0$ and $\delta \leq 0$.*

Proof. From (2.3), (2.4) and (2.5), for $(u, v) \in M$, one has

$$\begin{aligned}
S(u, v) &= \frac{1}{6} \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) dx + \frac{\omega}{2} \int_{\mathbf{R}^3} (|u|^2 + |v|^2) dx \\
& - \frac{\delta}{6} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) d\xi \\
& + \frac{\delta}{3} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi.
\end{aligned} \tag{4.7}$$

Since for $\theta = 1, 2$,

$$\begin{aligned}
& \left| \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi \right| \\
& \leq \frac{1}{2} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) d\xi,
\end{aligned} \tag{4.8}$$

where $\eta > 0$ and $\delta \leq 0$, one obtains

$$\begin{aligned}
& -\frac{\delta}{6} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) d\xi \\
& + \frac{\delta}{3} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi \\
& \geq -\frac{\delta}{6} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) d\xi \\
& + \frac{\delta}{6} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) d\xi \\
& \geq 0.
\end{aligned}$$

The latter, together with (4.7) and $\omega > 0$, implies on M that

$$S(u, v) \geq \frac{1}{6} \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) dx + \frac{\omega}{2} \int_{\mathbf{R}^3} (|u|^2 + |v|^2) dx > 0. \quad (4.9)$$

This completes the proof of Proposition 4.2. $\square$

Proposition 4.3. *Assume that $\eta > 0$, $\delta \leq 0$ and let $(u, v) \in H_r^1(\mathbf{R}^3) \times H_r^1(\mathbf{R}^3) \setminus \{(0, 0)\}$. For $\lambda > 0$, let $u_\lambda(x) = \lambda^{\frac{3}{2}} u(\lambda x)$, $v_\lambda(x) = \lambda^{\frac{3}{2}} v(\lambda x)$, then there exists a unique $\mu > 0$ (depending on (u, v)) such that $R(u_\mu, v_\mu) = 0$. Moreover,*

$$\begin{aligned}
R(u_\lambda, v_\lambda) &> 0 \quad \text{for } \lambda \in (0, \mu); \\
R(u_\lambda, v_\lambda) &< 0 \quad \text{for } \lambda \in (\mu, \infty); \\
S(u_\mu, v_\mu) &\geq S(u_\lambda, v_\lambda), \quad \forall \lambda > 0.
\end{aligned}$$

Proof. Via (2.3), (2.4), $\eta > 0$, $\delta \leq 0$, the quantities $S(u_\lambda, v_\lambda)$ and $R(u_\lambda, v_\lambda)$ can be expressed as follows:

$$\begin{aligned}
S(u_\lambda, v_\lambda) &= \frac{1}{2} \lambda^2 \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) dx + \frac{\omega}{2} \int_{\mathbf{R}^3} (|u|^2 + |v|^2) dx \\
&- \frac{1}{4} \lambda^3 \int_{\mathbf{R}^3} (|u|^4 + |v|^4) dx - \frac{1}{2} \lambda^3 \int_{\mathbf{R}^3} |u|^2 |v|^2 dx \\
&- \frac{1}{4} \lambda^3 \int_{\mathbf{R}^3} \frac{\eta \lambda^2 |\xi|^2}{\lambda^2 |\xi|^2 - \delta} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) d\xi \\
&+ \frac{1}{2} \lambda^3 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta \lambda^2 |\xi|^2}{\lambda^2 |\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi, \quad (4.10) \\
R(u_\lambda, v_\lambda) &= \lambda^2 \left\{ \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) dx \right. \\
&- \frac{3}{4} \lambda \int_{\mathbf{R}^3} (|u|^4 + |v|^4) dx - \frac{3}{2} \lambda \int_{\mathbf{R}^3} |u|^2 |v|^2 dx \\
&- \frac{3}{4} \lambda \int_{\mathbf{R}^3} \frac{\eta \lambda^2 |\xi|^2}{\lambda^2 |\xi|^2 - \delta} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) d\xi
\end{aligned}$$

$$\begin{aligned}
& + \frac{\delta}{2} \lambda \int_{\mathbf{R}^3} \frac{\eta \lambda^2 |\xi|^2}{(\lambda^2 |\xi|^2 - \delta)^2} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) d\xi \\
& + \frac{3}{2} \lambda \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta \lambda^2 |\xi|^2}{\lambda^2 |\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi \\
& - \delta \lambda \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta \lambda^2 |\xi|^2}{(\lambda^2 |\xi|^2 - \delta)^2} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi \Big\} \\
& = \lambda^2 R^*(u_\lambda, v_\lambda).
\end{aligned} \tag{4.11}$$

It is clear that for $\theta = 1, 2$,

$$\begin{aligned}
& \left| \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta \lambda^2 |\xi|^2}{(\lambda^2 |\xi|^2 - \delta)^\theta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi \right| \\
& \leq \frac{1}{2} \int_{\mathbf{R}^3} \frac{\eta \lambda^2 |\xi|^2}{(\lambda^2 |\xi|^2 - \delta)^\theta} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) d\xi,
\end{aligned} \tag{4.12}$$

we thus obtain

$$\begin{aligned}
& - \frac{3}{4} \lambda^3 \int_{\mathbf{R}^3} \frac{\eta \lambda^2 |\xi|^2}{\lambda^2 |\xi|^2 - \delta} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) d\xi \\
& + \frac{\delta}{2} \lambda^3 \int_{\mathbf{R}^3} \frac{\eta \lambda^2 |\xi|^2}{(\lambda^2 |\xi|^2 - \delta)^2} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) d\xi \\
& + \frac{3}{2} \lambda^3 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta \lambda^2 |\xi|^2}{\lambda^2 |\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi \\
& - \delta \lambda^3 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta \lambda^2 |\xi|^2}{(\lambda^2 |\xi|^2 - \delta)^2} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi \leq 0.
\end{aligned} \tag{4.13}$$

To prove that there exists $\mu > 0$ such that $R(u_\mu, v_\mu) = 0$, we divide the discussion into two cases.

Case 1 $R(u, v) > 0$; **Case 2** $R(u, v) < 0$.

♠ If **Case 1** is true, and if there exists λ such that $R(u_\lambda, v_\lambda) = 0$, then $\lambda \in (1, \infty)$ according to (4.11) and (4.13). Let $G(\lambda^2) = \frac{3}{4} \frac{\eta \lambda^2 |\xi|^2}{\lambda^2 |\xi|^2 - \delta} - \frac{\delta}{2} \frac{\eta \lambda^2 |\xi|^2}{(\lambda^2 |\xi|^2 - \delta)^2}$. Then $G'(\lambda^2) = \frac{-\delta \eta \lambda^2 |\xi|^4 + 5\delta^2 \eta |\xi|^2}{4(\lambda^2 |\xi|^2 - \delta)^3}$, which together with $\eta > 0$ and $\delta \leq 0$ leads to $G'(\lambda^2) \geq 0$. Thus, $G(\lambda^2)$ is an increasing function of λ^2 ($\lambda^2 \in (1, \infty)$). Thus there exists $\mu \in (1, \infty)$ such that $R(u_\mu, v_\mu) = 0$.

♠ If **Case 2** is true, and if there exists λ such that $R(u_\lambda, v_\lambda) = 0$, then $\lambda \in (0, 1)$. Indeed, we consider functional $R^*(u_\lambda, v_\lambda)$ defined by (4.11). Since $R^*(u_\lambda, v_\lambda) \rightarrow R(u, v) < 0$ as $\lambda \rightarrow 1$, and $R^*(u_\lambda, v_\lambda) = \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) dx > 0$ as $\lambda \rightarrow 0$, one concludes that there exists $\mu \in (0, 1)$ such that $R^*(u_\mu, v_\mu) = 0$. The latter implies that $\mu^2 R^*(u_\mu, v_\mu) = R(u_\mu, v_\mu) = 0$.

In both cases, there always exists $\mu > 0$ such that $R(u_\mu, v_\mu) = 0$.

On the other hand, we can easily check

$$R(u_\lambda, v_\lambda) > 0 \quad \text{for } \lambda \in (0, \mu), \quad R(u_\lambda, v_\lambda) < 0 \quad \text{for } \lambda \in (\mu, \infty). \tag{4.14}$$

Since

$$\frac{d\left(\frac{\eta\lambda^5|\xi|^2}{\lambda^2|\xi|^2-\delta}\right)}{d\lambda} = \frac{3\eta\lambda^4|\xi|^2}{\lambda^2|\xi|^2-\delta} - \frac{2\eta\delta\lambda^4|\xi|^2}{(\lambda^2|\xi|^2-\delta)^2},$$

we get

$$\begin{aligned} \frac{d}{d\lambda} S(u_\lambda, v_\lambda) &= \lambda \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) dx \\ &\quad - \frac{3}{4}\lambda^2 \int_{\mathbf{R}^3} (|u|^4 + |v|^4) dx - \frac{3}{2}\lambda^2 \int_{\mathbf{R}^3} |u|^2 |v|^2 dx \\ &\quad - \frac{3}{4}\lambda^4 \int_{\mathbf{R}^3} \frac{\eta|\xi|^2}{\lambda^2|\xi|^2-\delta} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) d\xi \\ &\quad + \frac{\delta}{2}\lambda^4 \int_{\mathbf{R}^3} \frac{\eta|\xi|^2}{(\lambda^2|\xi|^2-\delta)^2} (|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2) d\xi \\ &\quad + \frac{3}{2}\lambda^4 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta\lambda^2|\xi|^2}{\lambda^2|\xi|^2-\delta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi \\ &\quad - \delta\lambda^4 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta\lambda^2|\xi|^2}{(\lambda^2|\xi|^2-\delta)^2} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi \\ &= \lambda^{-1} R(u_\lambda, v_\lambda). \end{aligned} \tag{4.15}$$

By $R(u_\mu, v_\mu) = 0$, (4.14) and (4.15) imply that

$$S(u_\mu, v_\mu) \geq S(u_\lambda, v_\lambda), \quad \forall \lambda > 0.$$

This completes the proof of Proposition 4.3. $\square$

Now, we begin to prove Theorem 4.1.

Step 1: Proof of Theorem 4.1, part (1).

Let $\{(Q_n, P_n), n \in \mathbf{N}\} \subset M$ be a minimizing sequence for (2.7), that is, $R(Q_n, P_n) = 0$ and

$$S(Q_n, P_n) \rightarrow \inf_{(u,v) \in M} S(u, v) \quad \text{as } n \rightarrow \infty. \tag{4.16}$$

In view of $\eta > 0$ and $\delta \leq 0$, we get

$$\begin{aligned} & -\frac{\delta}{6} \int_{\mathbf{R}^3} \frac{\eta|\xi|^2}{(|\xi|^2-\delta)^2} (|\mathcal{F}(u_n\bar{v}_n)|^2 + |\mathcal{F}(\bar{u}_n v_n)|^2) d\xi \\ & + \frac{\delta}{3} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta|\xi|^2}{(|\xi|^2-\delta)^2} \mathcal{F}(u_n\bar{v}_n) \overline{\mathcal{F}(\bar{u}_n v_n)} d\xi \geq 0, \end{aligned} \tag{4.17}$$

which together with (4.7) and (4.16) implies that $\|Q_n\|_{H^1_r(\mathbf{R}^3)}$ and $\|P_n\|_{H^1_r(\mathbf{R}^3)}$ are both bounded for all $n \in \mathbf{N}$. Then there exists a subsequence

$$\{Q_{nk}, k \in \mathbf{N}\} \subset \{Q_n, n \in \mathbf{N}\} \tag{4.18}$$

such that, as $k \rightarrow \infty$,

$$Q_{nk} \rightharpoonup Q_\infty \text{ weakly in } H_r^1(\mathbf{R}^3), \quad Q_{nk} \rightarrow Q_\infty \text{ almost everywhere in } \mathbf{R}^3, \quad (4.19)$$

and

$$P_{nk} \rightharpoonup P_\infty \text{ weakly in } H_r^1(\mathbf{R}^3), \quad P_{nk} \rightarrow P_\infty \text{ almost everywhere in } \mathbf{R}^3. \quad (4.20)$$

For simplicity, we still denote $\{(Q_{nk}, P_{nk}), k \in \mathbf{N}\}$ by $\{(Q_n, P_n), n \in \mathbf{N}\}$. From Lemma 4.1, (4.19) and (4.20), it follows that

$$Q_n \rightarrow Q_\infty \text{ strongly in } L_r^4(\mathbf{R}^3), \quad P_n \rightarrow P_\infty \text{ strongly in } L_r^4(\mathbf{R}^3). \quad (4.21)$$

Since

$$\|Q_n\|_{L_r^2(\mathbf{R}^3)} \leq c, \quad \|P_n\|_{L_r^2(\mathbf{R}^3)} \leq c,$$

the boundedness of $\{(Q_n, P_n), n \in \mathbf{N}\}$ in $H_r^1(\mathbf{R}^3) \times H_r^1(\mathbf{R}^3)$ and the Gagliardo–Nirenberg inequality

$$\|v\|_{L_r^{p+1}(\mathbf{R}^N)}^{p+1} \leq c \|\nabla v\|_{L_r^2(\mathbf{R}^N)}^{\frac{N}{2}(p-1)} \|v\|_{L_r^2(\mathbf{R}^N)}^{p+1-\frac{N}{2}(p-1)}, \quad v \in H_r^1(\mathbf{R}^N), \quad 1 \leq p < \frac{N+2}{N-2},$$

imply, in particular, that

$$\int_{\mathbf{R}^3} |Q_n|^4 dx \leq c \left(\int_{\mathbf{R}^3} |\nabla Q_n|^2 dx \right)^{\frac{3}{2}}, \quad \int_{\mathbf{R}^3} |P_n|^4 dx \leq c \left(\int_{\mathbf{R}^3} |\nabla P_n|^2 dx \right)^{\frac{3}{2}}. \quad (4.22)$$

Here and henceforth, $c > 0$ denotes various positive constants. Via (4.22), $\eta > 0, \delta \leq 0$,

$$R(Q_n, P_n) = 0, \quad \int_{\mathbf{R}^3} |Q_n|^2 |P_n|^2 dx \leq \frac{1}{2} \int_{\mathbf{R}^3} (|Q_n|^4 + |P_n|^4) dx, \quad (4.23)$$

and

$$\begin{aligned} 0 &\leq \frac{3}{4} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} (|\mathcal{F}(Q_n \overline{P_n})|^2 + |\mathcal{F}(\overline{Q_n} P_n)|^2) d\xi \\ &\quad - \frac{\delta}{2} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(Q_n \overline{P_n})|^2 + |\mathcal{F}(\overline{Q_n} P_n)|^2) d\xi \\ &\quad - \frac{3}{2} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(Q_n \overline{P_n}) \overline{\mathcal{F}(\overline{Q_n} P_n)} d\xi \\ &\quad + \delta \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(Q_n \overline{P_n}) \overline{\mathcal{F}(\overline{Q_n} P_n)} d\xi \\ &\leq \int_{\mathbf{R}^3} \frac{3\eta |\xi|^4 - 5\eta \delta |\xi|^2}{2(|\xi|^2 - \delta)^2} (|\mathcal{F}(Q_n \overline{P_n})|^2 + |\mathcal{F}(\overline{Q_n} P_n)|^2) d\xi \\ &\leq \int_{\mathbf{R}^3} \frac{\eta |\xi|^2 (3|\xi|^2 - 5\delta)}{2(|\xi|^2 - \delta)^2} (|\mathcal{F}(Q_n \overline{P_n})|^2 + |\mathcal{F}(\overline{Q_n} P_n)|^2) d\xi \\ &\leq 5\eta \int_{\mathbf{R}^3} |Q_n|^2 |P_n|^2 dx \leq \frac{5}{2} \eta \int_{\mathbf{R}^3} (|Q_n|^4 + |P_n|^4) dx, \end{aligned} \quad (4.24)$$

we obtain

$$\begin{aligned} \int_{\mathbf{R}^3} (|\nabla Q_n|^2 + |\nabla P_n|^2) dx &\leq c \left(\int_{\mathbf{R}^3} |\nabla Q_n|^2 dx \right)^{\frac{3}{2}} + c \left(\int_{\mathbf{R}^3} |\nabla P_n|^2 dx \right)^{\frac{3}{2}} \\ &\leq c \left(\int_{\mathbf{R}^3} (|\nabla Q_n|^2 + |\nabla P_n|^2) dx \right)^{\frac{3}{2}}. \end{aligned} \quad (4.25)$$

Equation (4.25) implies that $\|\nabla Q_n\|_{L_r^2(\mathbf{R}^3)}^2 + \|\nabla P_n\|_{L_r^2(\mathbf{R}^3)}^2$ is bounded away from 0. Furthermore, we claim that $(Q_\infty, P_\infty) \neq (0, 0)$. Assume to the contrary that $(Q_\infty, P_\infty) \equiv (0, 0)$, from (4.21) it follows that

$$Q_n \rightarrow 0 \text{ strongly in } L_r^4(\mathbf{R}^3), \quad P_n \rightarrow 0 \text{ strongly in } L_r^4(\mathbf{R}^3). \quad (4.26)$$

Thus, (4.23) and (4.24) imply that as $n \rightarrow \infty$,

$$\begin{aligned} \int_{\mathbf{R}^3} |Q_n|^2 |P_n|^2 dx &\rightarrow 0, \quad \int_{\mathbf{R}^3} (|Q_n|^4 + |P_n|^4) dx \rightarrow 0, \\ \int_{\mathbf{R}^3} \left[\frac{3}{4} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} - \frac{\delta}{2} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \right] (|\mathcal{F}(Q_n \overline{P_n})|^2 + |\mathcal{F}(\overline{Q_n} P_n)|^2) d\xi \\ - \operatorname{Re} \int_{\mathbf{R}^3} \left[\frac{3}{2} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} - \frac{\eta \delta |\xi|^2}{(|\xi|^2 - \delta)^2} \right] \mathcal{F}(Q_n \overline{P_n}) \overline{\mathcal{F}(\overline{Q_n} P_n)} d\xi &\rightarrow 0. \end{aligned}$$

In view of $R(Q_n, P_n) = 0$, one would then conclude that as $n \rightarrow \infty$,

$$\int_{\mathbf{R}^3} (|\nabla Q_n|^2 + |\nabla P_n|^2) dx \rightarrow 0,$$

which contradicts (4.25). Thus $(Q_\infty, P_\infty) \neq (0, 0)$.

Let $Q = (Q_\infty)_\mu$, $P = (P_\infty)_\mu$ with $\mu > 0$ uniquely determined by the condition $R(Q, P) = R[(Q_\infty)_\mu, (P_\infty)_\mu] = 0$, where $(Q_\infty)_\mu$ and $(P_\infty)_\mu$ are defined by Proposition 4.3. Then Lemma 4.1 concludes

$$\begin{cases} (Q_n)_\mu \rightarrow Q, & (P_n)_\mu \rightarrow P \text{ strongly in } L_r^4(\mathbf{R}^3), \\ (Q_n)_\mu \rightharpoonup Q, & (P_n)_\mu \rightharpoonup P \text{ weakly in } H_r^1(\mathbf{R}^3), \end{cases} \quad (4.27)$$

whereas $R(Q_n, P_n) = 0$ and Proposition 4.3 imply

$$S[(Q_n)_\mu, (P_n)_\mu] \leq S(Q_n, P_n). \quad (4.28)$$

Hence, from (4.27) and (4.28) one concludes

$$S(Q, P) \leq \lim_{n \rightarrow \infty} S[(Q_n)_\mu, (P_n)_\mu] \leq \lim_{n \rightarrow \infty} S(Q_n, P_n) = \inf_{(u,v) \in M} S(u, v), \quad (4.29)$$

which together with $R(Q, P) = 0$ yields $(Q, P) \in M$. Therefore, (Q, P) solves the minimization problem

$$S(Q, P) = \min_{(u,v) \in M} S(u, v). \quad (4.30)$$

This completes the proof of Theorem 4.1, part (1).

Step 2: Proofs of Theorem 4.1, parts (2) and (3)

We first prove (2). Since (Q, P) is a solution of the minimization problem (4.30), there exists a Lagrange multiplier Λ such that

$$\delta'_Q[S(Q, P) + \Lambda R(Q, P)] = 0, \quad \delta'_{\overline{P}}[S(Q, P) + \Lambda R(Q, P)] = 0. \quad (4.31)$$

Here, $\delta'_u T(u, v)$ denotes the variation of $T(u, v)$ with respect to u . According to the following four identities for $\theta = 1, 2$:

$$\begin{aligned} & \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} \mathcal{F}(Q\overline{P}) \overline{\mathcal{F}(\overline{Q}P)} \, d\xi \\ &= \frac{1}{2} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} \mathcal{F}(Q\overline{P}) \overline{\mathcal{F}(\overline{Q}P)} \, d\xi \\ & \quad + \frac{1}{2} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} \overline{\mathcal{F}(Q\overline{P})} \mathcal{F}(\overline{Q}P) \, d\xi \\ &= \frac{1}{2} \int_{\mathbf{R}^3} \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} \mathcal{F}(Q\overline{P}) \right] Q\overline{P} \, dx \\ & \quad + \frac{1}{2} \int_{\mathbf{R}^3} \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} \mathcal{F}(\overline{Q}P) \right] \overline{Q}P \, dx, \end{aligned} \quad (\text{I-1})$$

$$\int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} |\mathcal{F}(\overline{Q}P)|^2 \, d\xi = \int_{\mathbf{R}^3} \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} \mathcal{F}(\overline{Q}P) \right] Q\overline{P} \, dx, \quad (\text{I-2})$$

$$\int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} |\mathcal{F}(Q\overline{P})|^2 \, d\xi = \int_{\mathbf{R}^3} \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} \mathcal{F}(Q\overline{P}) \right] \overline{Q}P \, dx, \quad (\text{I-3})$$

and

$$\int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} |\mathcal{F}(Q\overline{P})|^2 \, d\xi = \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} |\mathcal{F}(\overline{Q}P)|^2 \, d\xi, \quad (\text{I-4})$$

by the formula $\delta'_u T(u, v) = \frac{\partial}{\partial \zeta} T(u + \zeta \delta' u, v)|_{\zeta=0}$, and by taking $\delta' \overline{Q} = \overline{Q}$, $\delta' \overline{P} = \overline{P}$, we obtain by (4.31)

$$B_1(Q, P) = 0, \quad B_2(Q, P) = 0, \quad (4.32)$$

where

$$\begin{aligned} B_1(Q, P) &= -(1 + 2\Lambda)\Delta Q + \omega Q - (1 + 3\Lambda)|Q|^2 Q - (1 + 3\Lambda)Q|P|^2 \\ & \quad - (1 + 3\Lambda)P\mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(Q\overline{P}) \right] + (1 + 3\Lambda)P\mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\overline{Q}P) \right] \\ & \quad + 2\Lambda\delta P\mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(Q\overline{P}) \right] - 2\Lambda\delta P\mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(\overline{Q}P) \right], \end{aligned} \quad (4.33)$$

$$\begin{aligned}
B_2(Q, P) = & -(1 + 2\Lambda)\Delta P + \omega P - (1 + 3\Lambda)|P|^2 P - (1 + 3\Lambda)P|Q|^2 \\
& - (1 + 3\Lambda)Q\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2 - \delta}\mathcal{F}(\overline{Q}P)\right] + (1 + 3\Lambda)Q\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2 - \delta}\mathcal{F}(Q\overline{P})\right] \\
& + 2\Lambda\delta Q\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{(|\xi|^2 - \delta)^2}\mathcal{F}(\overline{Q}P)\right] - 2\Lambda\delta Q\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{(|\xi|^2 - \delta)^2}\mathcal{F}(Q\overline{P})\right].
\end{aligned} \tag{4.34}$$

Let

$$B_1(Q, P) = S_1(Q, P) + \Lambda P_1(Q, P) = 0, \tag{4.35}$$

$$B_2(Q, P) = S_2(Q, P) + \Lambda P_2(Q, P) = 0, \tag{4.36}$$

where

$$\begin{aligned}
S_1(Q, P) &= -\Delta Q + \omega Q - |Q|^2 Q - Q|P|^2 \\
&\quad - P\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2 - \delta}\mathcal{F}(Q\overline{P})\right] + P\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2 - \delta}\mathcal{F}(\overline{Q}P)\right],
\end{aligned} \tag{4.37}$$

$$\begin{aligned}
P_1(Q, P) &= -2\Delta Q - 3|Q|^2 Q - 3Q|P|^2 \\
&\quad - 3P\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2 - \delta}\mathcal{F}(Q\overline{P})\right] + 3P\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2 - \delta}\mathcal{F}(\overline{Q}P)\right] \\
&\quad + 2\delta P\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{(|\xi|^2 - \delta)^2}\mathcal{F}(Q\overline{P})\right] - 2\delta P\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{(|\xi|^2 - \delta)^2}\mathcal{F}(\overline{Q}P)\right],
\end{aligned} \tag{4.38}$$

$$\begin{aligned}
S_2(Q, P) &= -\Delta P + \omega P - |P|^2 P - P|Q|^2 \\
&\quad - Q\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2 - \delta}\mathcal{F}(\overline{Q}P)\right] + Q\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2 - \delta}\mathcal{F}(Q\overline{P})\right],
\end{aligned} \tag{4.39}$$

$$\begin{aligned}
P_2(Q, P) &= -2\Delta P - 3|P|^2 P - 3P|Q|^2 \\
&\quad - 3Q\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2 - \delta}\mathcal{F}(\overline{Q}P)\right] + 3Q\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2 - \delta}\mathcal{F}(Q\overline{P})\right] \\
&\quad + 2\delta Q\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{(|\xi|^2 - \delta)^2}\mathcal{F}(\overline{Q}P)\right] - 2\delta Q\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{(|\xi|^2 - \delta)^2}\mathcal{F}(Q\overline{P})\right].
\end{aligned} \tag{4.40}$$

Multiplying (4.35) by $\overline{Q}$ and (4.36) by $\overline{P}$, then integrating the resulting equations with respect to x on $\mathbf{R}^3$, we get

$$T_1(Q, P) + \Lambda T_2(Q, P) = 0, \tag{4.41}$$

where

$$\begin{aligned}
 T_1(Q, P) = & \int_{\mathbf{R}^3} (|\nabla Q|^2 + |\nabla P|^2) \, dx + \omega \int_{\mathbf{R}^3} (|Q|^2 + |P|^2) \, dx \\
 & - \int_{\mathbf{R}^3} (|Q|^4 + |P|^4) \, dx - 2 \int_{\mathbf{R}^3} |Q|^2 |P|^2 \, dx \\
 & - \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} (|\mathcal{F}(Q\bar{P})|^2 + |\mathcal{F}(\bar{Q}P)|^2) \, d\xi \\
 & + 2\text{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(Q\bar{P}) \overline{\mathcal{F}(\bar{Q}P)} \, d\xi, \tag{4.42}
 \end{aligned}$$

$$\begin{aligned}
 T_2(Q, P) = & 2 \int_{\mathbf{R}^3} (|\nabla Q|^2 + |\nabla P|^2) \, dx - 3 \int_{\mathbf{R}^3} (|Q|^4 \\
 & + |P|^4) \, dx - 6 \int_{\mathbf{R}^3} |Q|^2 |P|^2 \, dx \\
 & - 3 \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} (|\mathcal{F}(Q\bar{P})|^2 + |\mathcal{F}(\bar{Q}P)|^2) \, d\xi \\
 & + 6\text{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(Q\bar{P}) \overline{\mathcal{F}(\bar{Q}P)} \, d\xi \\
 & + 2\delta \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(Q\bar{P})|^2 + |\mathcal{F}(\bar{Q}P)|^2) \, d\xi \\
 & - 4\delta\text{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(Q\bar{P}) \overline{\mathcal{F}(\bar{Q}P)} \, d\xi. \tag{4.43}
 \end{aligned}$$

On the other hand, multiplying (4.35) by $x \nabla \bar{Q}$ and (4.36) by $x \nabla \bar{P}$, then integrating the resulting terms with respect to x on $\mathbf{R}^3$ and taking the real part, one obtains

$$T_3(Q, P) + \Lambda T_4(Q, P) = 0, \tag{4.44}$$

where

$$\begin{aligned}
 T_3(Q, P) = & -\frac{1}{2} \int_{\mathbf{R}^3} (|\nabla Q|^2 + |\nabla P|^2) \, dx - \frac{3\omega}{2} \int_{\mathbf{R}^3} (|Q|^2 + |P|^2) \, dx \\
 & + \frac{3}{4} \int_{\mathbf{R}^3} (|Q|^4 + |P|^4) \, dx + \frac{3}{2} \int_{\mathbf{R}^3} |Q|^2 |P|^2 \, dx \\
 & + \frac{3}{4} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} (|\mathcal{F}(Q\bar{P})|^2 + |\mathcal{F}(\bar{Q}P)|^2) \, d\xi \\
 & - \frac{3}{2} \text{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(Q\bar{P}) \overline{\mathcal{F}(\bar{Q}P)} \, d\xi \\
 & + \frac{\delta}{2} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(Q\bar{P})|^2 + |\mathcal{F}(\bar{Q}P)|^2) \, d\xi \\
 & - \delta \text{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(Q\bar{P}) \overline{\mathcal{F}(\bar{Q}P)} \, d\xi, \tag{4.45}
 \end{aligned}$$

$$\begin{aligned}
T_4(Q, P) = & - \int_{\mathbf{R}^3} (|\nabla Q|^2 + |\nabla P|^2) \, dx + \frac{9}{4} \int_{\mathbf{R}^3} (|Q|^4 \\
& + |P|^4) \, dx + \frac{9}{2} \int_{\mathbf{R}^3} |Q|^2 |P|^2 \, dx \\
& + \frac{9}{4} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} (|\mathcal{F}(Q\bar{P})|^2 + |\mathcal{F}(\bar{Q}P)|^2) \, d\xi \\
& + \frac{3}{2} \delta \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(Q\bar{P})|^2 + |\mathcal{F}(\bar{Q}P)|^2) \, d\xi \\
& - \frac{9}{2} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(Q\bar{P}) \overline{\mathcal{F}(\bar{Q}P)} \, d\xi \\
& - \frac{3}{2} \delta \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(Q\bar{P}) \overline{\mathcal{F}(\bar{Q}P)} \, d\xi \\
& - 3\delta \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(Q\bar{P})|^2 + |\mathcal{F}(\bar{Q}P)|^2) \, d\xi \\
& + \frac{1}{2} \delta \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(Q\bar{P})|^2 + |\mathcal{F}(\bar{Q}P)|^2) \, d\xi \\
& - 2\delta^2 \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^3} (|\mathcal{F}(Q\bar{P})|^2 + |\mathcal{F}(\bar{Q}P)|^2) \, d\xi \\
& + 5\delta \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(Q\bar{P}) \overline{\mathcal{F}(\bar{Q}P)} \, d\xi \\
& + 4\delta^2 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^3} \mathcal{F}(Q\bar{P}) \overline{\mathcal{F}(\bar{Q}P)} \, d\xi. \tag{4.46}
\end{aligned}$$

Thus, we have by (4.41)

$$\frac{3}{2} T_1(Q, P) + \frac{3}{2} \Lambda T_2(Q, P) = 0. \tag{4.47}$$

Noting that

$$\begin{aligned}
\frac{3}{2} \times T_1(Q, P) + T_3(Q, P) = & \int_{\mathbf{R}^3} (|\nabla Q|^2 + |\nabla P|^2) \, dx \\
& - \frac{3}{4} \int_{\mathbf{R}^3} (|Q|^4 + |P|^4) \, dx - \frac{3}{2} \int_{\mathbf{R}^3} |Q|^2 |P|^2 \, dx \\
& - \frac{3}{4} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} (|\mathcal{F}(Q\bar{P})|^2 + |\mathcal{F}(\bar{Q}P)|^2) \, d\xi \\
& + \frac{\delta}{2} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(Q\bar{P})|^2 + |\mathcal{F}(\bar{Q}P)|^2) \, d\xi \\
& + \frac{3}{2} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(Q\bar{P}) \overline{\mathcal{F}(\bar{Q}P)} \, d\xi \\
& - \delta \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(Q\bar{P}) \overline{\mathcal{F}(\bar{Q}P)} \, d\xi \\
= & R(Q, P) = 0, \tag{4.48}
\end{aligned}$$

by (4.44), (4.47) and (4.48), it is evident that

$$\frac{3}{2}\Lambda T_2(Q, P) + \Lambda T_4(Q, P) = 0, \quad (4.49)$$

that is,

$$\Lambda \left[\frac{3}{2}T_2(Q, P) + T_4(Q, P) \right] = \Lambda T_5(Q, P) = 0, \quad (4.50)$$

where

$$\begin{aligned} T_5(Q, P) = & 2 \int_{\mathbf{R}^3} (|\nabla Q|^2 + |\nabla P|^2) \, dx - \frac{9}{4} \int_{\mathbf{R}^3} (|Q|^4 \\ & + |P|^4) \, dx - \frac{9}{2} \int_{\mathbf{R}^3} |Q|^2 |P|^2 \, dx \\ & - \frac{9}{4} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} (|\mathcal{F}(Q\bar{P})|^2 + |\mathcal{F}(\bar{Q}P)|^2) \, d\xi \\ & + \frac{9}{2} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(Q\bar{P}) \overline{\mathcal{F}(\bar{Q}P)} \, d\xi \\ & + 2\delta \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(Q\bar{P})|^2 + |\mathcal{F}(\bar{Q}P)|^2) \, d\xi \\ & - \frac{5}{2} \delta \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(Q\bar{P}) \overline{\mathcal{F}(\bar{Q}P)} \, d\xi \\ & - 2\delta^2 \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^3} (|\mathcal{F}(Q\bar{P})|^2 + |\mathcal{F}(\bar{Q}P)|^2) \, d\xi \\ & + 4\delta^2 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^3} \mathcal{F}(Q\bar{P}) \overline{\mathcal{F}(\bar{Q}P)} \, d\xi. \end{aligned} \quad (4.51)$$

By a direct calculation, we obtain

$$\begin{aligned} T_5(Q, P) = & 3R(Q, P) - \int_{\mathbf{R}^3} (|\nabla Q|^2 + |\nabla P|^2) \, dx \\ & + \frac{1}{2} \delta \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(Q\bar{P})|^2 + |\mathcal{F}(\bar{Q}P)|^2) \, d\xi \\ & - \delta \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(Q\bar{P}) \overline{\mathcal{F}(\bar{Q}P)} \, d\xi \\ & - 2\delta^2 \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^3} (|\mathcal{F}(Q\bar{P})|^2 + |\mathcal{F}(\bar{Q}P)|^2) \, d\xi \\ & + 4\delta^2 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^3} \mathcal{F}(Q\bar{P}) \overline{\mathcal{F}(\bar{Q}P)} \, d\xi. \end{aligned} \quad (4.52)$$

In view of (4.48), $(Q, P) \neq (0, 0)$, $R(Q, P) = 0$ and

$$\begin{aligned} & \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} \mathcal{F}(Q\bar{P}) \overline{\mathcal{F}(\bar{Q}P)} \, d\xi \\ & \leq \frac{1}{2} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^\theta} (|\mathcal{F}(Q\bar{P})|^2 + |\mathcal{F}(\bar{Q}P)|^2) \, d\xi, \quad (\theta = 2, 3), \end{aligned}$$

we get by $\eta > 0$ and $\delta \leq 0$,

$$T_5(Q, P) \leq - \int_{\mathbf{R}^3} (|\nabla Q|^2 + |\nabla P|^2) dx < 0,$$

which implies that $T_5(Q, P) \neq 0$. Thus, we derive that $\Lambda = 0$ from (4.50). Therefore, from (4.32), (4.33) and (4.34), it follows that (Q, P) satisfies the following equations:

$$\begin{aligned} & -\omega Q + \Delta Q + (|Q|^2 + |P|^2)Q \\ & + P \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(Q\bar{P}) \right] - P \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\bar{Q}P) \right] = 0, \quad (4.53) \\ & -\omega P + \Delta P + (|Q|^2 + |P|^2)P \\ & + Q \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(\bar{Q}P) \right] - Q \mathcal{F}^{-1} \left[\frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(Q\bar{P}) \right] = 0. \end{aligned}$$

That is, (Q, P) is a solution of (1.4) and (1.5). As (1.4) and (1.5) comprise the Euler-Lagrange equations of the functional $S(Q, P)$, applying Proposition 4.1, we conclude (Q, P) is a ground state solution of (1.4) and (1.5). Furthermore, it is evident that (Q, P) are functions of $|x|$ alone. By [3, 4], (Q, P) have exponential decays at infinity.

We thus complete the proof of Theorem 4.1.

5. Sharp Threshold of Global Existence in $\mathbf{R}^3$

In this section, we discuss the sharp threshold of global existence to the Cauchy problem (1.1)–(1.3). As in [1, 2], our arguments are based on the local well-posedness (Proposition 2.1) established in Section 2. The main result of this section can be described by the following.

Theorem 5.1. *Let $\delta \leq 0, \eta > 0, (E_{10}(x), E_{20}(x)) \in H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3)$ and satisfy*

$$S(E_{10}, E_{20}) < d_1, \quad (5.1)$$

where d_1 is defined by (2.8). Then we have

(I) *If $(E_{10}(x), E_{20}(x))$ satisfies*

$$R(E_{10}, E_{20}) < 0, \quad (5.2)$$

and $(|x|E_{10}, |x|E_{20}) \in L^2(\mathbf{R}^3) \times L^2(\mathbf{R}^3)$, then the solution $(E_1(t, x), E_2(t, x))$ of the Cauchy problem (1.1)–(1.3) blows up in a finite time.

(II) *If $(E_{10}(x), E_{20}(x))$ satisfies*

$$R(E_{10}, E_{20}) > 0, \quad (5.3)$$

then the solution $(E_1(t, x), E_2(t, x))$ of the Cauchy problem (1.1)–(1.3) exists globally on $t \in [0, +\infty)$. Moreover, $(E_1(t, x), E_2(t, x))$ satisfies, for any $t \in [0, +\infty)$,

$$\begin{aligned} & \omega \left(\|E_1(t)\|_{L^2(\mathbf{R}^3)}^2 + \|E_2(t)\|_{L^2(\mathbf{R}^3)}^2 \right) \\ & + \frac{1}{3} \left(\|\nabla E_1(t)\|_{L^2(\mathbf{R}^3)}^2 + \|\nabla E_2(t)\|_{L^2(\mathbf{R}^3)}^2 \right) \leq 2d_1. \end{aligned} \quad (5.4)$$

We start with two propositions which are key to the proof of Theorem 5.1.

Proposition 5.1. *Let $\delta \leq 0, \eta > 0$ and*

$$K_1 = \{(u, v) \in H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3), R(u, v) > 0, S(u, v) < d_1\},$$

$$K_2 = \{(u, v) \in H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3), R(u, v) < 0, S(u, v) < d_1\}.$$

Then K_1 and K_2 are invariant under the flow generated by the Cauchy problem (1.1)–(1.3).

Proof. Let $(E_{10}(x), E_{20}(x)) \in K_1$ and $(E_1(t), E_2(t))$ be the solution to the Cauchy problem (1.1)–(1.3) with initial data $(E_{10}(x), E_{20}(x))$. By (2.1) and (2.2) one has

$$S(E_1(t), E_2(t)) = S(E_{10}, E_{20}), \quad t \in [0, T]. \quad (5.5)$$

From $S(E_{10}, E_{20}) < d_1$, it follows that

$$S(E_1(t), E_2(t)) < d_1, \quad t \in [0, T]. \quad (5.6)$$

To show

$$R(E_1(t), E_2(t)) > 0, \quad t \in [0, T], \quad (5.7)$$

we argue by contradiction. If (5.7) is not true, by continuity, $\eta > 0, \delta \leq 0$ and $R(E_{10}, E_{20}) > 0$, there would exist a $t^* \in (0, T)$ such that

$$R(E_1(t^*), E_2(t^*)) = 0, \quad (E_1(t^*), E_2(t^*)) \neq (0, 0). \quad (5.8)$$

This implies $(E_1(t^*), E_2(t^*)) \in M_1$. In view of (5.6), we obtain

$$S(E_1(t^*), E_2(t^*)) < d_1.$$

This is impossible by (2.8) and Proposition 2.2. Therefore, K_1 is invariant under the flow generated by the Cauchy problem (1.1)–(1.3).

Similarly, we can prove that K_2 is also invariant under the flow generated by the Cauchy problem (1.1)–(1.3). $\square$

Proposition 5.2. *Let $\delta \leq 0, \eta > 0$ and $u_\lambda(x) = \lambda^{\frac{3}{2}}u(\lambda x), v_\lambda(x) = \lambda^{\frac{3}{2}}v(\lambda x)$ for $\lambda > 0$. Suppose $(u, v) \in H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3) \setminus \{(0, 0)\}$ and $(u, v) \in K_2$ (K_2 is defined by proposition 5.1). Then there exists $0 < \mu < 1$ such that $R(u_\mu, v_\mu) = 0$ and*

$$S(u, v) - S(u_\mu, v_\mu) \geq \frac{1}{2}R(u, v). \quad (5.9)$$

Here, $S(u, v)$ and $R(u, v)$ are defined by (2.3) and (2.4), respectively.

Proof. By (2.3) and (2.4), one can express $S(u_\lambda, v_\lambda)$ and $R(u_\lambda, v_\lambda)$ as follows:

$$\begin{aligned}
 S(u_\lambda, v_\lambda) &= \frac{1}{2}\lambda^2 \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) \, dx + \frac{\omega}{2} \int_{\mathbf{R}^3} (|u|^2 + |v|^2) \, dx \\
 &\quad - \frac{1}{4}\lambda^3 \int_{\mathbf{R}^3} (|u|^4 + |v|^4) \, dx - \frac{1}{2}\lambda^3 \int_{\mathbf{R}^3} |u|^2 |v|^2 \, dx \\
 &\quad - \frac{1}{4}\lambda^3 \int_{\mathbf{R}^3} \frac{\eta\lambda^2|\xi|^2}{\lambda^2|\xi|^2 - \delta} \left(|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2 \right) \, d\xi \\
 &\quad + \frac{1}{2}\lambda^3 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta\lambda^2|\xi|^2}{\lambda^2|\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} \, d\xi, \tag{5.10}
 \end{aligned}$$

$$\begin{aligned}
 R(u_\lambda, v_\lambda) &= \lambda^2 \left\{ \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) \, dx \right. \\
 &\quad - \frac{3}{4}\lambda \int_{\mathbf{R}^3} (|u|^4 + |v|^4) \, dx - \frac{3}{2}\lambda \int_{\mathbf{R}^3} |u|^2 |v|^2 \, dx \\
 &\quad - \frac{3}{4}\lambda \int_{\mathbf{R}^3} \frac{\eta\lambda^2|\xi|^2}{\lambda^2|\xi|^2 - \delta} \left(|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2 \right) \, d\xi \\
 &\quad + \frac{\delta}{2}\lambda \int_{\mathbf{R}^3} \frac{\eta\lambda^2|\xi|^2}{(\lambda^2|\xi|^2 - \delta)^2} \left(|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2 \right) \, d\xi \\
 &\quad + \frac{3}{2}\lambda \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta\lambda^2|\xi|^2}{\lambda^2|\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} \, d\xi \\
 &\quad \left. - \delta\lambda \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta\lambda^2|\xi|^2}{(\lambda^2|\xi|^2 - \delta)^2} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} \, d\xi \right\} \\
 &= \lambda^2 Q_\lambda(u, v). \tag{5.11}
 \end{aligned}$$

Since $(u, v) \in H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3) \setminus \{(0, 0)\}$ and $(u, v) \in K_2$, one can check easily that $Q_\lambda(u, v) > 0$ as $\lambda \rightarrow 0$ and $Q_\lambda(u, v) < 0$ as $\lambda \rightarrow 1$. Thus, by continuity, there exists $0 < \mu < 1$ such that $Q_\mu(u, v) = 0$, which together with (5.11) yields

$$R(u_\mu, v_\mu) = 0 \quad \text{for } 0 < \mu < 1. \tag{5.12}$$

On the other hand, (5.11) and (5.12) imply that

$$\begin{aligned}
 &S(u, v) - S(u_\mu, v_\mu) \\
 &= \frac{1}{2} \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) \, dx - \frac{1}{4} \int_{\mathbf{R}^3} (|u|^4 + |v|^4) \, dx - \frac{1}{2} \int_{\mathbf{R}^3} |u|^2 |v|^2 \, dx \\
 &\quad - \frac{1}{4} \int_{\mathbf{R}^3} \frac{\eta|\xi|^2}{|\xi|^2 - \delta} \left(|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2 \right) \, d\xi \\
 &\quad + \frac{1}{2} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta|\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} \, d\xi \\
 &\quad - \frac{1}{2}\mu^2 \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) \, dx + \frac{1}{4}\mu^3 \int_{\mathbf{R}^3} (|u|^4 \\
 &\quad + |v|^4) \, dx + \frac{1}{2}\mu^3 \int_{\mathbf{R}^3} |u|^2 |v|^2 \, dx
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \mu^3 \int_{\mathbf{R}^3} \frac{\eta \mu^2 |\xi|^2}{\mu^2 |\xi|^2 - \delta} \left(|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2 \right) d\xi \\
& - \frac{1}{2} \mu^3 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta \mu^2 |\xi|^2}{\mu^2 |\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi \\
& = \frac{1}{2} \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) dx - \frac{1}{4} \int_{\mathbf{R}^3} (|u|^4 + |v|^4) dx - \frac{1}{2} \int_{\mathbf{R}^3} |u|^2 |v|^2 dx \\
& - \frac{1}{4} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \left(|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2 \right) d\xi \\
& + \frac{1}{2} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi \\
& - \left(\frac{3}{8} - \frac{1}{4} \right) \mu^3 \int_{\mathbf{R}^3} (|u|^4 + |v|^4) dx - \left(\frac{3}{4} - \frac{1}{2} \right) \mu^3 \int_{\mathbf{R}^3} |u|^2 |v|^2 dx \\
& - \left(\frac{3}{8} - \frac{1}{4} \right) \mu^3 \int_{\mathbf{R}^3} \frac{\eta \mu^2 |\xi|^2}{\mu^2 |\xi|^2 - \delta} \left(|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2 \right) d\xi \\
& + \frac{\delta}{4} \mu^3 \int_{\mathbf{R}^3} \frac{\eta \mu^2 |\xi|^2}{(\mu^2 |\xi|^2 - \delta)^2} \left(|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2 \right) d\xi \\
& + \left(\frac{3}{4} - \frac{1}{2} \right) \mu^3 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta \mu^2 |\xi|^2}{\mu^2 |\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi \\
& - \frac{\delta}{2} \mu^3 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta \mu^2 |\xi|^2}{(\mu^2 |\xi|^2 - \delta)^2} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi \\
& \geq \frac{1}{2} \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) dx - \frac{3}{8} \int_{\mathbf{R}^3} (|u|^4 + |v|^4) dx - \frac{3}{4} \int_{\mathbf{R}^3} |u|^2 |v|^2 dx \\
& - \frac{1}{4} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \left(|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2 \right) d\xi \\
& + \frac{1}{2} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi \\
& - \frac{1}{8} \mu^3 \int_{\mathbf{R}^3} \frac{\eta \mu^2 |\xi|^2}{\mu^2 |\xi|^2 - \delta} \left(|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2 \right) d\xi \\
& + \frac{\delta}{4} \mu^3 \int_{\mathbf{R}^3} \frac{\eta \mu^2 |\xi|^2}{(\mu^2 |\xi|^2 - \delta)^2} \left(|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2 \right) d\xi \\
& + \frac{1}{4} \mu^3 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta \mu^2 |\xi|^2}{\mu^2 |\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi \\
& - \frac{\delta}{2} \mu^3 \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta \mu^2 |\xi|^2}{(\mu^2 |\xi|^2 - \delta)^2} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} d\xi. \tag{5.13}
\end{aligned}$$

Let $G_1(\mu) = \frac{\eta \mu^5 |\xi|^2}{\mu^2 |\xi|^2 - \delta}$, $G_2(\mu) = \frac{\eta \mu^5 |\xi|^2}{(\mu^2 |\xi|^2 - \delta)^2}$. Then $G'_1(\mu) = \frac{\eta \mu^4 |\xi|^2 (3\mu^2 - 5\delta)}{(\mu^2 |\xi|^2 - \delta)^2}$, which together with $\delta \leq 0$, $\mu > 0$ and $\eta > 0$ implies that $G'_1(\mu) \geq 0$ for $\mu \geq 0$. Now, we show $G_2(\mu)$ is also increasing with respect to $\mu \geq 0$. Indeed, let $\mu_1 > \mu_2 > 0$, then

$$\begin{aligned}
& G_2(\mu_1) - G_2(\mu_2) \\
&= \frac{\eta\mu_1^4\mu_2^4|\xi|^6(\mu_1 - \mu_2) + \eta\delta^2|\xi|^2(\mu_1^5 - \mu_2^5) - 2\eta\delta\mu_1^2\mu_2^2|\xi|^4(\mu_1^5 - \mu_2^5)}{(\mu_1^2|\xi|^2 - \delta)^2(\mu_2^2|\xi|^2 - \delta)^2}.
\end{aligned}$$

Since $\delta \leq 0$, $\eta > 0$, $\mu_1 > \mu_2 > 0$, we see that $G_2(\mu_1) - G_2(\mu_2) > 0$. Therefore both $G_1(\mu)$ and $G_2(\mu)$ are increasing with respect to $\mu \geq 0$. In addition, for $\theta = 1, 2$, by Young's inequality we have

$$\begin{aligned}
& \left| \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta\mu^2|\xi|^2}{(\mu^2|\xi|^2 - \delta)^\theta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} \, d\xi \right| \\
& \leq \frac{1}{2} \int_{\mathbf{R}^3} \frac{\eta\mu^2|\xi|^2}{(\mu^2|\xi|^2 - \delta)^\theta} \left(|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2 \right) \, d\xi.
\end{aligned}$$

Hence, one deduces from (5.13) that for $\mu \in (0, 1)$,

$$\begin{aligned}
& S(u, v) - S(u_\mu, v_\mu) \\
& \geq \frac{1}{2} \int_{\mathbf{R}^3} (|\nabla u|^2 + |\nabla v|^2) \, dx - \frac{3}{8} \int_{\mathbf{R}^3} (|u|^4 + |v|^4) \, dx - \frac{3}{4} \int_{\mathbf{R}^3} |u|^2 |v|^2 \, dx \\
& \quad - \frac{3}{8} \int_{\mathbf{R}^3} \frac{\eta|\xi|^2}{|\xi|^2 - \delta} \left(|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2 \right) \, d\xi \\
& \quad + \frac{3}{4} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta|\xi|^2}{|\xi|^2 - \delta} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} \, d\xi \\
& \quad + \frac{\delta}{4} \int_{\mathbf{R}^3} \frac{\eta|\xi|^2}{(|\xi|^2 - \delta)^2} \left(|\mathcal{F}(u\bar{v})|^2 + |\mathcal{F}(\bar{u}v)|^2 \right) \, d\xi \\
& \quad - \frac{\delta}{2} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta|\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(u\bar{v}) \overline{\mathcal{F}(\bar{u}v)} \, d\xi \\
& = \frac{1}{2} R(u, v),
\end{aligned}$$

which completes the proof of Proposition 5.2. $\square$

Now we turn to the proof of Theorem 5.1.

We first show (I) by contradiction. Suppose that the maximal existence time T of the solution $(E_1(t), E_2(t))$ of the Cauchy problem (1.1)–(1.3) is infinity. Then Proposition 5.1, (5.1) and (5.2) imply

$$R(E_1(t), E_2(t)) < 0, \quad S(E_1(t), E_2(t)) < d_1. \quad (5.14)$$

Via (5.5), we have

$$S(E_1(t), E_2(t)) = S(E_{10}, E_{20}), \quad t \in [0, T). \quad (5.15)$$

Whereas, by a direct calculation, it follows from (5.14) that

$$\frac{d^2}{dt^2} \int_{\mathbf{R}^3} |x|^2 (|E_1|^2 + |E_2|^2) \, dx = 8R(E_1, E_2). \quad (5.16)$$

Let $\mu > 0$ be such that $R(E_{1\mu}, E_{2\mu}) = 0$ ($(E_{1\mu}, E_{2\mu})$ is defined by Proposition 5.2). Since $R(E_1, E_2) < 0$, we get $0 < \mu < 1$ from Proposition 5.2. Moreover, from (2.1), (2.2), (2.4), (2.8), (5.5), Proposition 2.2 and $R(E_{1\mu}, E_{2\mu}) = 0$, it follows that

$$S(E_1, E_2) = S(E_{10}, E_{20}), S(E_{1\mu}, E_{2\mu}) \geq d_1, \quad (5.17)$$

which together with Proposition 5.2, (5.14) and (5.15) leads to

$$R(E_1, E_2) \leq 2[S(E_{10}, E_{20}) - d_1] < 0. \quad (5.18)$$

Let

$$\beta = 2[S(E_{10}, E_{20}) - d_1] < 0, \quad (5.19)$$

and

$$J(t) = \int_{\mathbf{R}^3} |x|^2 (|E_1|^2 + |E_2|^2) dx. \quad (5.20)$$

Equations (5.16) and (5.18) give

$$\frac{d^2}{dt^2} J(t) = 8R(E_1, E_2) \leq 8\beta < 0. \quad (5.21)$$

Integrating (5.21) twice with respect to t , we obtain

$$J(t) \leq 8\beta t^2 + J'(0)t + J(0), \quad (5.22)$$

which implies that if (E_1, E_2) would remain in $H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3)$, then there is a $T^* < +\infty$ such that

$$\lim_{t \rightarrow T^*} \int_{\mathbf{R}^3} |x|^2 (|E_1|^2 + |E_2|^2) dx = 0. \quad (5.23)$$

On the other hand, (2.1) and Lemma 3.1 yield

$$\begin{aligned} \int_{\mathbf{R}^3} (|E_{10}|^2 + |E_{20}|^2) dx &= \int_{\mathbf{R}^3} (|E_1|^2 + |E_2|^2) dx \\ &\leq \frac{4}{3} \left(\int_{\mathbf{R}^3} (|\nabla E_1|^2 + |\nabla E_2|^2) dx \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^3} |x|^2 (|E_1|^2 + |E_2|^2) dx \right)^{\frac{1}{2}}. \end{aligned} \quad (5.24)$$

One concludes from (5.23) and (5.24) that

$$\lim_{t \rightarrow T^*} \int_{\mathbf{R}^3} (|\nabla E_1|^2 + |\nabla E_2|^2) dx = +\infty,$$

which completes the proof of (I) of Theorem 5.1.

To prove (II) of Theorem 5.1, we note that from (5.1) and (5.3), we have $(E_{10}, E_{20}) \in K_1$. Thus, Proposition 5.1 implies that the solution $(E_1(t), E_2(t))$ of the Cauchy problem (1.1)–(1.3) satisfies

$$(E_1(t), E_2(t)) \in K_1, \quad \text{for all } t \in [0, T]. \quad (5.25)$$

That is,

$$R(E_1(t), E_2(t)) > 0, \quad S(E_1(t), E_2(t)) < d_1. \quad (5.26)$$

Consequently, by (2.3) and (2.4), one has

$$\begin{aligned} & \omega \int_{\mathbf{R}^3} (|E_1|^2 + |E_2|^2) dx + \frac{1}{3} \int_{\mathbf{R}^3} (|\nabla E_1|^2 + |\nabla E_2|^2) dx \\ & - \frac{\delta}{3} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \left(|\mathcal{F}(E_1 \overline{E_2})|^2 + |\mathcal{F}(\overline{E_1} E_2)|^2 \right) d\xi \\ & + \frac{2\delta}{3} \operatorname{Re} \int_{\mathbf{R}^3} \frac{\eta |\xi|^2}{(|\xi|^2 - \delta)^2} \mathcal{F}(E_1 \overline{E_2}) \overline{\mathcal{F}(\overline{E_1} E_2)} d\xi < 2d_1. \end{aligned} \quad (5.27)$$

Furthermore, as $\eta > 0$ and $\delta \leq 0$, it gives

$$\omega \int_{\mathbf{R}^3} (|E_1|^2 + |E_2|^2) dx + \frac{1}{3} \int_{\mathbf{R}^3} (|\nabla E_1|^2 + |\nabla E_2|^2) dx < 2d_1. \quad (5.28)$$

Therefore, we have established the uniform boundedness of $(E_1(t), E_2(t))$ in $H^1(\mathbf{R}^3) \times H^1(\mathbf{R}^3)$ for $t \in [0, T)$. In particular, $T = +\infty$ by a continuity argument. In other words, the solution $(E_1(t), E_2(t))$ to the Cauchy problem (1.1)–(1.3) exists globally on $t \in [0, +\infty)$. Furthermore, (5.28) implies the estimate (5.4).

This completes the proof of (II) of Theorem 5.1.

6. Orbital Instability of Standing Waves in $\mathbf{R}^3$

In this section, we discuss the instability of standing waves of (1.1)–(1.2) in $\mathbf{R}^3$ obtained in Theorem 4.1. We have the following orbital instability result for the standing waves of the equations (1.1)–(1.2) with minimal action $S(Q, P)$.

Theorem 6.1. *For $\delta \leq 0$ and $\eta > 0$, let $(Q, P) \in M$ be given by Theorem 4.1. For arbitrary $\varepsilon > 0$, there exists $(E_{10}, E_{20}) \in H_r^1(\mathbf{R}^3) \times H_r^1(\mathbf{R}^3)$ with*

$$\|E_{10} - Q\|_{H_r^1(\mathbf{R}^3)} < \varepsilon, \quad \|E_{20} - P\|_{H_r^1(\mathbf{R}^3)} < \varepsilon \quad (6.1)$$

such that the solution (E_1, E_2) of the equations (1.1)–(1.2) with (1.3) has the following property: For some finite time $T < \infty$, (E_1, E_2) exists on $[0, T)$, $(E_1, E_2) \in C([0, T); H_r^1(\mathbf{R}^3) \times H_r^1(\mathbf{R}^3))$ and

$$\lim_{t \rightarrow T} \left(\|E_1\|_{H_r^1(\mathbf{R}^3)} + \|E_2\|_{H_r^1(\mathbf{R}^3)} \right) = +\infty. \quad (6.2)$$

Proof of Theorem 6.1. Let $(E_1, E_2) \in H_r^1(\mathbf{R}^3) \times H_r^1(\mathbf{R}^3)$ be a solution of the equations (1.1)–(1.2) with (1.3) on $[0, T)$. By (2.1) and (2.2), we get

$$S(E_1(t), E_2(t)) = S(E_{10}, E_{20}), \quad t \in [0, T), \quad (6.3)$$

where S is defined by (2.3). Let

$$J(t) = \int_{\mathbf{R}^3} |x|^2 (|E_1|^2 + |E_2|^2) dx. \quad (6.4)$$

By a direct calculation, it follows from (2.4) that

$$\frac{d^2}{dt^2} J(t) = 8R(E_1, E_2). \quad (6.5)$$

We want to show, for some initial data, that the right-hand side of (6.5) is strictly negative. One first notices that

$$S(Q, P) = \min_{(u,v) \in M} S(u, v) > 0. \quad (6.6)$$

Let

$$K = \{(u, v) \in H_r^1(\mathbf{R}^3) \times H_r^1(\mathbf{R}^3), R(u, v) < 0, S(u, v) < S(Q, P)\},$$

and let $(E_{10}, E_{20}) \in K$ such that

$$(|x|E_{10}, |x|E_{20}) \in L_r^2(\mathbf{R}^3) \times L_r^2(\mathbf{R}^3). \quad (6.7)$$

We shall see later that such (E_{10}, E_{20}) exists. We claim there is a finite T such that

$$\lim_{t \rightarrow T} (\|E_1\|_{H_r^1(\mathbf{R}^3)} + \|E_2\|_{H_r^1(\mathbf{R}^3)}) = +\infty. \quad (6.8)$$

Indeed, for such $(E_{10}, E_{20}) \in K$, one has

$$S(E_1, E_2) = S(E_{10}, E_{20}) < S(Q, P), \quad 0 \leq t < T, \quad (6.9)$$

and

$$R(E_1, E_2) < 0, \quad 0 \leq t < T. \quad (6.10)$$

The latter is true, for otherwise, by continuity, there would exist a $\bar{t} > 0$ such that $0 < \bar{t} < T$ and

$$R(E_1(\bar{t}), E_2(\bar{t})) = 0, \quad (6.11)$$

which implies $(E_1(\bar{t}), E_2(\bar{t})) \in M$. This contradicts Theorem 4.1 and Eqn. (6.9).

Next, for a fixed $t \in [0, T)$, $(E_1, E_2) = (E_1(t), E_2(t))$, and let $0 < \mu < 1$ be such that $R(E_{1\mu}, E_{2\mu}) = 0$, $(E_{1\mu}(x), E_{2\mu}(x)) = (\mu^{\frac{3}{2}} E_1(\mu x), \mu^{\frac{3}{2}} E_2(\mu x))$ (see Proposition 4.3). Since

$$S(E_{1\mu}, E_{2\mu}) \geq S(Q, P), \quad S(E_1, E_2) = S(E_{10}, E_{20}), \quad (6.12)$$

making a proof similar to that of Proposition 5.2, we have

$$R(E_1, E_2) \leq 2[S(E_{10}, E_{20}) - S(Q, P)] = \varphi < 0. \quad (6.13)$$

In view of (6.5) and (6.13), we conclude

$$\frac{d^2}{dt^2} J(t) \leq 8\varphi < 0, \quad (6.14)$$

which implies that T must be finite and that

$$\lim_{t \rightarrow T} (\|E_1\|_{H_r^1(\mathbf{R}^3)} + \|E_2\|_{H_r^1(\mathbf{R}^3)}) = +\infty.$$

In order to complete the proof of Theorem 6.1, we need to show $(E_{10}, E_{20}) \in K$ with (6.7).

Let

$$E_{10}(x) = \lambda^{\frac{3}{2}} Q(\lambda x), \quad E_{20}(x) = \lambda^{\frac{3}{2}} P(\lambda x), \quad \lambda > 1. \quad (6.15)$$

By Proposition 4.3, the functions $E_{10}(x)$ and $E_{20}(x)$ verify

$$R(E_{10}, E_{20}) < 0, \quad S(E_{10}, E_{20}) < S(Q, P), \quad \lambda > 1. \quad (6.16)$$

Furthermore, by Theorem 4.1, one sees $(Q(x), P(x))$ has an exponential decay at infinity, and hence $(|x|E_{10}, |x|E_{20}) \in L_r^2(\mathbf{R}^3) \times L_r^2(\mathbf{R}^3)$. As $\lambda \rightarrow 1$, $\|E_{10} - Q\|_{H_r^1(\mathbf{R}^3)}, \|E_{20} - P\|_{H_r^1(\mathbf{R}^3)}$ can be made arbitrarily small. We thus complete the proof of Theorem 6.1. $\square$

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