Erratum

Erratum to: Characterization of Generalized Gradient Young Measures Generated by Sequences in $W^{1,1}$ and BV

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We thank Emil Wiedemann for pointing out that in the proof of the sufficiency part of Theorem 9 (p. 584) in [2] the reduction to the case with underlying deformation zero does not work. This is so, because in the definition of μ it might happen that μ_x^∞ is not a positive measure. The current proof works only if one restricts attention to the subclass of Young measures μ with underlying deformation $u \in \text{BV}(\Omega; \mathbb{R}^m)$ that admit a generating sequence of the special form $(u + \varphi_j)$, where $(\varphi_j) \subset W^{1,1}(\Omega; \mathbb{R}^m)$ with $\varphi_j \xrightarrow{*} 0$. Instead, to prove Theorem 9 in the stated generality, we extend Step 1 in the proof to the case of Young measures with arbitrary underlying deformation.

For $u \in \text{BV}(\Omega; \mathbb{R}^m)$ we let

$$\begin{aligned} \mathbf{GY}_{Du}(\Omega; \mathbb{R}^{m \times d}) := \{v \in \mathbf{GY}(\Omega; \mathbb{R}^{m \times d}) : \langle \text{id}, v_x \rangle \mathcal{L}^d \llcorner \Omega \\ + \langle \text{id}, v_x^\infty \rangle \lambda_v \llcorner \Omega = Du\}, \end{aligned}$$

that is, the set of gradient Young measures that can be generated by sequences converging weakly* to u in $\text{BV}(\Omega; \mathbb{R}^m)$.

First we need the following approximation lemma in $\text{BV}(\Omega; \mathbb{R}^m)$.

Lemma 1 *Let $u \in \text{BV}(\Omega; \mathbb{R}^m)$ and $\varepsilon > 0$. Then, there exists a countable family $\mathcal{Q}$ of (rotated) rectangles $Q \subset \Omega$ and a mapping $v \in \text{BV}(\Omega; \mathbb{R}^m)$ with the following properties:*

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¹ Here and in the following the numbers refer to those in [2], and we also use the notation introduced there.

- (i) The rectangles in $\mathcal{Q}$ are pairwise disjoint and $(\mathcal{L}^d + |Du|)(\Omega \setminus \bigcup \mathcal{Q}) = 0$.
- (ii) For each $Q \in \mathcal{Q}$ the restriction $v|_Q$ is affine.
- (iii) $\|u - v\|_{L^1(\Omega; \mathbb{R}^m)} + \big| |Dv|(\Omega) - |Du|(\Omega) \big| < \varepsilon$.
- (iv) $\sum_{Q \in \mathcal{Q}} \|u - v\|_{L^1(\partial Q; \mathcal{H}^{d-1}; \mathbb{R}^m)} < \varepsilon$.

Moreover, given any positive and locally finite measure γ on Ω , we can arrange that $\sum_{Q \in \mathcal{Q}} \gamma(\partial Q) = 0$.

Proof. We start by introducing some convenient notation. Let $C = (-1, 1)^d$ and put for $x_0 \in \mathbb{R}^d$ and $r > 0$, $Q(x_0, r) := x_0 + rC$. For a unit vector $e \in \mathbb{S}^{d-1}$ select a rotation $R \in \text{SO}(d)$ with $Re_1 = e$, where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^d$. Let $Q_e(x_0, r) := x_0 + rR(C)$, so that $Q_e(x_0, r)$ is an open cube with center x_0 , side length $2r$, and two faces orthogonal to e .

For $x_0 \in \Omega$ and $0 < r < \text{dist}(x_0, \partial\Omega)/\sqrt{d}$ put

$$u_r(x) := \frac{u(x_0 + rx) - \tilde{u}(x_0)}{r}, \quad x \in C,$$

where $\tilde{u}(x_0)$ is the value of the precise representative of u at x_0 . We recall that for $\mathcal{L}^d$ -almost all $x_0 \in \Omega$, u is approximately differentiable at x_0 and as $r \downarrow 0$,

$$u_r \rightarrow h \quad \text{strictly in } \text{BV}(C; \mathbb{R}^m) \quad \text{with} \quad h(x) := \nabla u(x_0)x, \quad x \in C. \quad (1)$$

Hence, invoking the Trace Theorem we also have that

$$\int_{\partial C} |u_r - h| \, d\mathcal{H}^{d-1} \rightarrow 0 \quad \text{as } r \downarrow 0; \quad (2)$$

see for instance [1, pp. 53–54] and the references given there. Let us denote the set of such points $x_0 \in \Omega$ by G_1 .

Next, for $|Du|$ -almost all $x_0 \in \Omega$ we have $\frac{dDu}{d|Du|}(x_0) = b \otimes e$ for $b \in \mathbb{R}^m$ and a unit vector $e \in \mathbb{S}^{d-1}$, and $\alpha_r := \frac{|Du|(Q_e(x_0, r))}{r^d} \rightarrow \infty$ as $r \downarrow 0$. Moreover, defining

$$\tilde{u}_r(x) := \frac{u(x_0 + rx) - u_{x_0, r}}{r\alpha_r}, \quad x \in Q_e(0, 1),$$

where $u_{x_0, r} = \int_{Q_e(x_0, r)} u \, dx$, we can find a sequence of r 's going to 0 (not specifically labelled) such that

$$\tilde{u}_r \rightarrow u_0 \quad \text{strictly in } \text{BV}(Q_e(0, 1); \mathbb{R}^m) \quad \text{and} \quad u_0(x) = b\psi(x \cdot e), \quad x \in Q_e(0, 1), \quad (3)$$

where $\psi: (-1, 1) \rightarrow \mathbb{R}$ is increasing and bounded. It is not hard to see that we additionally may assume that the sequence of r 's is chosen such that $(|Du| + \gamma)(\partial Q_e(x_0, r)) = 0$. We again refer to [1, pp. 54–55] for these assertions. Denote the set of such points $x_0 \in \Omega$ by G_2 , and observe that $(\mathcal{L}^d + |Du|)(\Omega \setminus (G_1 \cup G_2)) = 0$.

Fix $x_0 \in G_2$. For the function ψ appearing in (3) (extended by continuity to the endpoints ± 1) we select a partition $-1 = t_0 < t_1 < \dots < t_k = 1$ consisting of continuity points of ψ such that $\max_j (t_{j+1} - t_j) < \varepsilon/m$, where

m will be specified below. Define φ as the piecewise affine function satisfying $\varphi(t_j) = \psi(t_j)$ for each $0 \leq j \leq k$ (with inner limits at the endpoints), and note that with $D_j := R((t_j, t_{j+1}) \times (-1, 1)^{d-1})$ (the rotation R as above) we have

$$\begin{aligned} & \sum_{j=0}^{k-1} \int_{\partial D_j} |u_0(x) - b\varphi(x \cdot e)| \, d\mathcal{H}^{d-1}(x) \\ &= |b| \sum_{j=0}^{k-1} \int_{\partial D_j} |\psi(x \cdot e) - \varphi(x \cdot e)| \, d\mathcal{H}^{d-1}(x) \\ &\leq \frac{(d-1)2^{d-1}|b|\varepsilon}{m} \sum_{j=0}^{k-1} \psi(t_{j+1}) - \psi(t_j) < \varepsilon, \end{aligned}$$

provided we choose $m > (d-1)2^{d-1}|b|(\psi(1) - \psi(-1))$. Next, in view of (3) and our choice of partition points t_j we infer from the Trace Theorem that

$$\sum_{j=0}^{k-1} \int_{\partial D_j} |\tilde{u}_r - u_0| \, d\mathcal{H}^{d-1} \rightarrow 0 \text{ as } r \downarrow 0. \quad (4)$$

From (1), (2) it follows that for all $x_0 \in G_1$ there exists $r(x_0) > 0$ such that with $a(x) = \tilde{u}(x_0) + \nabla u(x_0)(x - x_0)$, $x \in Q(x_0, r)$, we have for $r < r(x_0)$,

$$\begin{aligned} & \int_{Q(x_0, r)} |u - a| \, dx + \left| |Du|(Q(x_0, r)) - \int_{Q(x_0, r)} |Da| \, dx \right| < \varepsilon \mathcal{L}^d(Q(x_0, r)), \\ & \int_{\partial Q(x_0, r)} |u - a| \, d\mathcal{H}^{d-1} < \varepsilon \mathcal{L}^d(Q(x_0, r)). \end{aligned} \quad (5)$$

For a point $x_0 \in G_2$, note that the mapping $x \mapsto b\varphi(e \cdot x)$ defined above is piecewise affine. Hence, changing variables in (3), (4) it follows that there exists $r(x_0) > 0$ such that for $r < r(x_0)$, and with the rotated cube $Q_e(x_0, r)$ split into k rectangles $D_j(x_0, r)$ with corresponding affine maps $a_j(x) := r\alpha_r b\varphi(r^{-1}(x - x_0) \cdot e) + u_{x_0, r}$, $x \in D_j(x_0, r) := x_0 + rD_j$,

$$\begin{aligned} & \sum_{j=0}^{k-1} \int_{D_j(x_0, r)} |u - a_j| \, dx < \varepsilon |Du|(Q_e(x_0, r)), \\ & \sum_{j=0}^{k-1} \left| |Du|(D_j(x_0, r)) - \int_{D_j(x_0, r)} |Da_j| \, dx \right| < \varepsilon |Du|(Q_e(x_0, r)), \\ & \sum_{j=0}^{k-1} \int_{\partial D_j(x_0, r)} |u - a_j| \, d\mathcal{H}^{d-1} < \varepsilon |Du|(Q_e(x_0, r)). \end{aligned} \quad (6)$$

We now invoke the Morse Covering Theorem with the above family of (rotated) cubes (some of which are then subsequently divided into the rectangles $D_j(x_0, r)$).

We hereby find the family $\mathcal{Q}$ satisfying (i) in the statement of Lemma 1. For the remaining properties (ii)–(v), we let

$$v := \sum_{Q \in \mathcal{Q}} a_Q \mathbb{1}_Q,$$

where a_Q denotes the affine map corresponding to the rectangle Q . Because the rectangles are disjoint, the map v is well-defined and integrable, and it is clear that (ii) is satisfied. For the remaining assertions we consider for $j \in \mathbb{N}$ the mapping

$$v_j = u \mathbb{1}_{H_j} + \sum_{\mathcal{L}^d(Q) > \frac{1}{j}} a_Q \mathbb{1}_Q, \quad \text{with } H_j = \Omega \setminus \left(\bigcup_{\mathcal{L}^d(Q) > \frac{1}{j}} \overline{Q} \right).$$

Since the above sums are finite, we infer that $v_j \in \text{BV}(\Omega; \mathbb{R}^m)$, and

$$\begin{aligned} Dv_j &= Du \llcorner H_j + u \otimes v_{H_j} \mathcal{H}^{d-1} \llcorner \partial H_j \\ &\quad + \sum_{\mathcal{L}^d(Q) > \frac{1}{j}} \left(\nabla a_Q \mathcal{L}^d \llcorner Q + a_Q \otimes v_Q \mathcal{H}^{d-1} \llcorner \partial Q \right) \\ &= Du \llcorner H_j + \sum_{\mathcal{L}^d(Q) > \frac{1}{j}} \left(\nabla a_Q \mathcal{L}^d \llcorner Q + (a_Q - u) \otimes v_Q \mathcal{H}^{d-1} \llcorner \partial Q \right). \end{aligned}$$

Because the rectangles of $\mathcal{Q}$ are disjoint, and employing (5), (6),

$$\begin{aligned} |Dv_j|(\Omega) &= |Du|(H_j) + \sum_{\mathcal{L}^d(Q) > \frac{1}{j}} \left(|\nabla a_Q| \mathcal{L}^d(Q) + \int_{\partial Q} |a_Q - u| \, d\mathcal{H}^{d-1} \right) \\ &= |Du|(H_j) + \sum_{\mathcal{L}^d(Q) > \frac{1}{j}} |\nabla a_Q| \mathcal{L}^d(Q) + O(\varepsilon)(\mathcal{L}^d + |Du|)(\Omega). \end{aligned}$$

Since $|Du|(H_j) \rightarrow 0$ as $j \rightarrow \infty$, we see that $v \in \text{BV}(\Omega; \mathbb{R}^m)$ and by similar calculations, (iii) and (iv) also hold (with a slightly different ε). The addition is clear.

With this approximation result at hand we can show:

Proposition 2 *Let $u \in \text{BV}(\Omega; \mathbb{R}^m)$. Then $\mathbf{GY}_{Du}(\Omega; \mathbb{R}^{m \times d})$ is a weakly* closed and convex subset of $\mathbf{E}(\Omega; \mathbb{R}^m)^*$.*

Proof. The weak* closedness follows just like in Proposition 9 by a diagonal argument.

For the convexity, let $\mu, \nu \in \mathbf{GY}_{Du}(\Omega; \mathbb{R}^{m \times d})$ and $\theta \in (0, 1)$. For a given $\varepsilon > 0$, apply the previous approximation lemma to u and with $\gamma := \lambda_\mu + \lambda_\nu$. Because $\sum_{Q \in \mathcal{Q}} \gamma(\partial Q) = 0$, we can adjust generating sequences $(u_j), (v_j) \subset$

$W^{1,1}(\Omega; \mathbb{R}^m)$ for μ, ν to agree with u on $\Gamma := \bigcup_{Q \in \mathcal{Q}} \partial Q$ (this follows in a similar fashion to Lemma 4). Then, we can further adjust the generating sequences to agree with v (from the approximation lemma) around Γ , yielding sequences $(\tilde{u}_j), (\tilde{v}_j) \subset \text{BV}(\Omega; \mathbb{R}^m)$ that generate gradient Young measures $\tilde{\mu}, \tilde{\nu}$. Of course, this introduces an error, but from the error estimates in the approximation lemma we have for all $f \in \mathbf{E}(\Omega; \mathbb{R}^m)$,

$$|\langle f, \tilde{\mu} \rangle - \langle f, \mu \rangle| \leq M\varepsilon, \quad |\langle f, \tilde{\nu} \rangle - \langle f, \nu \rangle| \leq M\varepsilon,$$

where $M > 0$ is the linear growth constant of f . Let us indicate more precisely how this is achieved. For each $Q \in \mathcal{Q}$ we select a smooth cut-off function $0 \leq \rho_Q \leq \mathbb{1}_Q$ which is a suitably good strict approximation of $\mathbb{1}_Q: \mathcal{L}^d(Q \setminus \{\rho_Q = 1\}) < \varepsilon \mathcal{L}^d(Q)$ and $||D\rho_Q|(\mathbb{R}^d) - \mathcal{H}^{d-1}(\partial Q)| < \varepsilon \mathcal{L}^d(Q)$. We then put

$$\tilde{u}_j = \sum_{Q \in \mathcal{Q}} (\rho_Q u_j + (\mathbb{1}_Q - \rho_Q)v).$$

Then it is not hard to see that the $\tilde{u}_j$ have the above mentioned properties. Indeed

$$\begin{aligned} D\tilde{u}_j - Du_j &= \sum_{Q \in \mathcal{Q}} ((\rho_Q - \mathbb{1}_Q)(Du_j - Dv) \\ &\quad + (u_j - v) \otimes D\rho_Q + v \otimes n_Q \mathcal{H}^{d-1} \llcorner \partial Q) - Du_j \llcorner \Gamma. \end{aligned}$$

Here, v is the inner trace in Q and n_Q is the unit inner normal to ∂Q . Thus, using the properties of v, λ_v and $u_j = u$ near Γ :

$$\begin{aligned} |D\tilde{u}_j - Du|(\Omega) &\leq \sum_{Q \in \mathcal{Q}} \int_Q (1 - \rho_Q)(|Du_j| + |Dv|) + |u_j - v| |D\rho_Q| \, dx \\ &\quad + \left| \sum_{Q \in \mathcal{Q}} v \otimes n_Q \mathcal{H}^{d-1} \llcorner \partial Q - (u_j^+ - u_j^-) \otimes n_\Gamma \mathcal{H}^{d-1} \llcorner \Gamma \right|(\Omega) \\ &\leq O(\varepsilon) + 2 \sum_{Q \in \mathcal{Q}} \int_{\partial Q} |v - u| \, d\mathcal{H}^{d-1} = O(\varepsilon). \end{aligned}$$

The above estimates follow from this. Similarly for $\tilde{v}_j$.

By a diagonal argument we can now assume that there exist generating sequences, denoted again by $(u_j), (v_j) \subset \text{BV}(\Omega; \mathbb{R}^m)$, of μ and ν , respectively, which additionally satisfy the property that u_j, v_j are affine on the boundary of rectangles in a covering $\mathcal{Q}_j$ of Ω , and $\sup_{Q \in \mathcal{Q}_j} \text{diam}(Q) \rightarrow 0$.

Because our generating sequences are affine on the boundary on the rectangles in $\mathcal{Q}_j$, by the Riemann–Lebesgue Lemma in Corollary 3 (also see Remark 5), we may average the elementary Young measure ε_{Du_j} inside each cube $Q \in \mathcal{Q}_j$, and recombine the averaged Young measures to get a gradient Young measure $\mu_j \in \mathbf{GY}(\Omega; \mathbb{R}^{m \times d})$ that satisfies

$$\langle f, \mu_j \rangle = \sum_{Q \in \mathcal{Q}_j} \int_Q \int_Q f(x, \nabla u_j(y)) \, dy \, dx$$

for all $f \in \mathbf{E}(\Omega; \mathbb{R}^{m \times d})$; likewise for v_j . Moreover, $\mu_j \xrightarrow{*} \mu$, because for all $\varphi \otimes h \in \mathbf{E}(\Omega; \mathbb{R}^{m \times d})$ (like in Proposition 8),

$$\begin{aligned} \langle \varphi \otimes h, \mu_j \rangle &= \sum_{Q \in \mathcal{Q}_j} \int_Q \varphi(x) \, dx \int_Q h(\nabla u_j(y)) \, dy \\ &= \sum_{Q \in \mathcal{Q}_j} \int_Q (\varphi(y) + o(1)) h(\nabla u_j(y)) \, dy \\ &= \langle \varphi \otimes h, \varepsilon_{\nabla u_j} \rangle + M(|\Omega| + \|\nabla u_j\|_{L^1(\Omega; \mathbb{R}^{m \times d})}) o(1), \end{aligned}$$

where $o(1)$ denotes a Q -independent term, which converges to zero as $j \rightarrow \infty$ (by the uniform continuity of φ and $\sup_{Q \in \mathcal{Q}_j} \text{diam}(Q) \rightarrow 0$). Analogously, $v_j \xrightarrow{*} v$.

As in Proposition 10 (on the covering $\mathcal{Q}_j$ and without letting $j \rightarrow \infty$) split up each $Q \in \mathcal{Q}_j$ into a part with area $\theta|Q|$ and one with area $(1 - \theta)|Q|$, then cover each part with rescaled copies of $u_j|_Q$ and $v_j|_Q$ (with smaller and smaller diameters), and finally average this sequence again over all of Q in order to get that also $\theta\mu_j + (1 - \theta)v_j \in \mathbf{GY}(\Omega; \mathbb{R}^{m \times d})$. Thus, for all $f \in \mathbf{E}(\Omega; \mathbb{R}^{m \times d})$,

$$\langle f, \theta\mu_j + (1 - \theta)v_j \rangle = \theta \langle f, \mu_j \rangle + (1 - \theta) \langle f, v_j \rangle \rightarrow \theta \langle f, \mu \rangle + (1 - \theta) \langle f, v \rangle,$$

so by the weak* closedness of $\mathbf{GY}_{Du}(\Omega; \mathbb{R}^{m \times d})$, also $\theta\mu + (1 - \theta)v \in \mathbf{GY}_{Du}(\Omega; \mathbb{R}^{m \times d})$, hence the latter set is convex.

In the proof of Theorem 9, we can now adapt Step 1 to the more general setting with arbitrary underlying deformations (Step 2 is not needed anymore). For this, we take an arbitrary weakly* closed half-space (all of them have this form)

$$H := \{e^* \in \mathbf{E}(\Omega; \mathbb{R}^{m \times d})^* : \langle f_H, e^* \rangle \geq \kappa\} \quad \text{where } f_H \in \mathbf{E}(\Omega; \mathbb{R}^{m \times d}), \quad \kappa \in \mathbb{R},$$

satisfying $\mathbf{GY}_{Du}(\Omega; \mathbb{R}^{m \times d}) \subset H$, in particular $\langle f_H, \mu \rangle \geq \kappa$ for all $\mu \in \mathbf{GY}_{Du}(\Omega; \mathbb{R}^{m \times d})$. We have to show that a Young measure ν satisfying (i)–(iii) from the statement of Theorem 9 fulfills $\langle f_H, \nu \rangle \geq \kappa$. Set $g_\varepsilon(x, A) := f_H(x, A) + \varepsilon|\cdot|$ and take a countable covering (C_k) of $\overline{\Omega}$ ($k \in \mathbb{N}$) consisting of closed Borel sets with Lipschitz boundaries and such that

- (A) $|Tg_\varepsilon(x, A) - Tg_\varepsilon(y, A)| \leq \delta$ whenever $x, y \in C_k$, $A \in \overline{\mathbb{B}^{m \times d}}$,
- (B) $\lambda_\nu(\partial C_k) = 0$ for all $k \in \mathbb{N}$,
- (C) $\sum_{k=1}^{\infty} \int_{C_k} Qf_H(z_k, 0) \, dx \leq \int_{\Omega} Qf_H(x, 0) \, dx + 1$.

The existence of the $z_k \in C_k$ in (C) follows by a standard argument using Lebesgue points. Estimating, we get

$$\begin{aligned}
 \|f_H, v\| &\geq \sum_{k=1}^{\infty} \left(\int_{C_k} \langle g_{\varepsilon}(z_k, \cdot), v_x \rangle dx + \int_{C_k} \langle g_{\varepsilon}^{\infty}(z_k, \cdot), v_x^{\infty} \rangle d\lambda_v(x) \right) \\
 &\quad - (\varepsilon + \delta) \|1 + |\cdot|, v\| \\
 &\geq \sum_{k=1}^{\infty} \underbrace{\left(\int_{C_k} Qg_{\varepsilon}(z_k, \nabla u(x)) dx + \int_{C_k} (Qg_{\varepsilon})^{\#} \left(z_k, \frac{D^s u}{|D^s u|}(x) \right) d|D^s u|(x) \right)}_{=: \mathcal{H}(u; C_k)} \\
 &\quad - (\varepsilon + \delta) \|1 + |\cdot|, v\|.
 \end{aligned}$$

Now, contrary to the strategy for zero underlying deformation, we use the main result of [1] (Theorem 1) to get recovery sequences $(w_j^{(k)}) \subset W_u^{1,1}(C_k; \mathbb{R}^m)$ with $w_j^{(k)} \xrightarrow{*} u$ in $BV(C_k; \mathbb{R}^m)$ such that

$$\int_{C_k} g_{\varepsilon}(z_k, \nabla w_j^{(k)}) dx \rightarrow \mathcal{H}(u; C_k)$$

as $j \rightarrow \infty$. This sequence then also satisfies for all k and $j \geq j(k)$ the bound

$$\|\nabla w_j^{(k)}\|_{L^1(C_k; \mathbb{R}^{m \times d})} \leq \frac{1}{\varepsilon} \left(\mathcal{H}(u; C_k) - \int_{C_k} Qf_H(z_k, 0) dx + (L|Du| + \mathcal{L}^d)(C_k) \right),$$

which can be seen as follows: By the Lipschitz continuity of $Qf_H(z_k, \cdot)$ with Lipschitz constant L , say, (also suppose $L \geq M$, the linear growth constant of Qf_H), we have for each fixed k from a certain step $j \geq j(k)$,

$$\begin{aligned}
 &\mathcal{H}(u; C_k) + \mathcal{L}^d(C_k) \\
 &\geq \int_{C_k} g_{\varepsilon}(z_k, \nabla w_j^{(k)}) dx = \int_{C_k} f_H(z_k, \nabla w_j^{(k)}) dx + \varepsilon \|\nabla w_j^{(k)}\|_{L^1(C_k; \mathbb{R}^{m \times d})} \\
 &\geq \int_{C_k} Qf_H(z_k, \nabla w_j^{(k)} - \nabla u) dx + \int_{C_k} (Qf_H)^{\#} \left(z_k, -\frac{D^s u}{|D^s u|} \right) d|D^s u| \\
 &\quad - L|Du|(C_k) + \varepsilon \|\nabla w_j^{(k)}\|_{L^1(C_k; \mathbb{R}^{m \times d})}.
 \end{aligned}$$

By quasiconvexity in BV (see for example Proposition 1 in [1]) in conjunction with $w_j^{(k)} - u \in BV_0(C_k; \mathbb{R}^m)$, this implies

$$\mathcal{H}(u; C_k) + \mathcal{L}^d(C_k) \geq \int_{C_k} Qf_H(z_k, 0) dx - L|Du|(C_k) + \varepsilon \|\nabla w_j^{(k)}\|_{L^1(C_k; \mathbb{R}^{m \times d})},$$

hence, as $Qf_H(z_k, 0) > -\infty$ (see the proof of Lemma 6), the above j -uniform bound on the $w_j^{(k)}$ follows.

Defining w_j by $w_{\max\{j, j(k)\}}^{(k)}$ on C_k ($k \in \mathbb{N}$) as before, we have from the estimates above, (C), and the Poincaré inequality in BV,

$$\limsup_{j \rightarrow \infty} \|w_j\|_{W^{1,1}(\Omega; \mathbb{R}^{m \times d})} \leq C(\varepsilon)$$

with $C(\varepsilon)$ independent of the choice of the covering $\{C_k\}$ (hence independent of δ). Thus, passing to a subsequence if necessary, $\nabla w_j \xrightarrow{\mathbf{Y}} \mu$ in $\mathbf{Y}(\Omega; \mathbb{R}^{m \times d})$ and clearly $\mu \in \mathbf{GY}_{Du}(\Omega; \mathbb{R}^{m \times d}) \subset H$. Hence, combining all the above assertions, we may derive, analogously to the original argument in [2],

$$\begin{aligned}
 \langle f_H, \nu \rangle &\geq \limsup_{j \rightarrow \infty} \sum_{k=1}^{\infty} \int_{C_k} g_{\varepsilon}(z_k, \nabla w_j(x)) \, dx - (\varepsilon + \delta) \langle 1 + |\cdot|, \nu \rangle, \\
 &\geq \limsup_{j \rightarrow \infty} \sum_{k=1}^{\infty} \int_{C_k} g_{\varepsilon}(x, \nabla w_j(x)) \, dx \\
 &\quad - \sup_{j \in \mathbb{N}} \delta (|\Omega| + \|\nabla w_j\|_{L^1(\Omega; \mathbb{R}^{m \times d})}) - (\varepsilon + \delta) \langle 1 + |\cdot|, \nu \rangle, \\
 &\geq \lim_{j \rightarrow \infty} \int_{\Omega} f_H(x, \nabla w_j(x)) \, dx - \delta (|\Omega| + C(\varepsilon)) - (\varepsilon + \delta) \langle 1 + |\cdot|, \nu \rangle \\
 &= \langle f_H, \mu \rangle - \delta (|\Omega| + C(\varepsilon)) - (\varepsilon + \delta) \langle 1 + |\cdot|, \nu \rangle \\
 &\geq \kappa - \delta (|\Omega| + C(\varepsilon)) - (\varepsilon + \delta) \langle 1 + |\cdot|, \nu \rangle.
 \end{aligned}$$

Letting first $\delta \downarrow 0$ and then $\varepsilon \downarrow 0$, we arrive at $\langle f_H, \nu \rangle \geq \kappa$, and hence, since H was arbitrary, we conclude by the Hahn–Banach Theorem that $\nu \in \mathbf{GY}_{Du}(\Omega; \mathbb{R}^{m \times d})$. $\square$

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