Erratum

Erratum to: Existence and Regularity of Very Weak Solutions of the Stationary Navier–Stokes Equations

HYUNSEOK KIM

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The purpose of this erratum is to correct some mistakes of Remark 1, Lemma 1 and their consequences in the paper [Arch Rational Mech Anal 193 (2009) 117–152].

The major contribution of the paper is to establish existence and regularity of very weak solutions in $L^q(\Omega)$ of the Dirichlet problem (NS) for the stationary Navier–Stokes equations with arbitrarily large boundary data g in $W^{-1/q,q}(\partial\Omega)$. In particular, it is shown in Theorem 1, the main existence result, that there exists at least one very weak solution $u \in L^{q_0}(\Omega)$ of (NS) in the sense of Definition 1 for the data f, k and g satisfying the very weak regularity (1). The proof of Theorem 1 is complete. Theorems 2 and 3, the main regularity results, are also correct. So are the corresponding results for the linear problems, i.e., Theorems 7, 9 and 10. However, due to the very weak regularity of f and k , the solution u neither satisfies (NS) in the usual distribution sense, nor does it have a well-defined trace, which contradicts the assertions in Remarks 2 and 3. In fact, Remarks 2 and 3 are wrong and should be corrected by imposing a stronger regularity assumption on f and k .

Let X and Y be Banach spaces. Assume that $X \hookrightarrow Y$, i.e., $X \subset Y$, and that there is a constant $C > 0$ such that $\|x\|_Y \leq C\|x\|_X$ for all $x \in X$. Then the restriction of each $f \in Y'$ to X , denoted $f|_X$, is a bounded linear functional on X' with the norm bounded by $C\|f\|_{Y'}$. Consider the space $Y'|_X = \{f|_X : f \in Y'\}$ equipped with the quotient norm

$$\|\hat{f}\|_{Y'|_X} = \inf \left\{ \|f\|_{Y'} : f \in Y', \hat{f} = f|_X \right\} \quad (\hat{f} \in Y'|_X).$$

It is easy to show that $Y'|_X \hookrightarrow X'$. Moreover, if X is dense in Y , then the restriction map $f \in Y' \mapsto f|_X$ is one-to-one and indeed an isometry from Y' onto $Y'|_X$, which

verifies $Y' \hookrightarrow X'$. However, this embedding result fails to hold in cases when X is not dense in Y . Therefore, Remark 1 is completely wrong and should be deleted; obviously, $W_0^{2,q'}(\Omega)$ ($W_0^{1,q'}(\Omega)$, respectively) is not dense in $W_0^{1,q'}(\Omega) \cap W^{2,q'}(\Omega)$ ($W^{1,q'}(\Omega)$, respectively).

Many of the results in the paper are based on the validity of Remark 1 and thus statement of these results should be corrected as follows:

- (i) In the vector equations in Remarks 2 and 9, f should be changed by $f|_{W_0^{2,q'}(\Omega)}$, the restriction of f to $W_0^{2,q'}(\Omega)$. To identify $f|_{W_0^{2,q'}(\Omega)}$ with f itself, we need to impose a stronger regularity assumption on f . In view of the Sobolev embedding theorem (see Remark 4), it suffices to assume that $1 < r \leq q < \infty$, $\frac{1}{n} + \frac{1}{q} \geq \frac{1}{r}$ and $f \in W^{-1,r}(\Omega)$. More generally, we may assume that $f \in Y'$, where Y is any Banach space such that $W_0^{1,q'}(\Omega) \cap W^{2,q'}(\Omega) \hookrightarrow Y$ and $W_0^{2,q'}(\Omega)$ is dense in Y .¹
- (ii) The hypothesis for the trace theorem, Theorem 5, (and similarly for Theorem 8) should be changed by $1 < r \leq q < \infty$, $\frac{1}{n} + \frac{1}{q} \geq \frac{1}{r}$, $v \in L^q(\Omega)$, $p \in W^{-1,q}(\Omega)$, $-\Delta v + \nabla p \in W^{-1,r}(\Omega)$ and $\operatorname{div} v \in L^r(\Omega)$.
- (iii) The hypotheses for Remark 3 and Theorem 6 should be changed by $1 < r \leq q < \infty$, $\frac{1}{n} + \frac{1}{q} \geq \frac{1}{r}$, $f \in W^{-1,r}(\Omega)$, $k \in L^r(\Omega)$ and $g \in W^{-1/q,q}(\Omega)$.
- (iv) Remark 12 should be corrected. Theorem 7 allows one to deduce the existence of a vector field $v \in L^q(\Omega)$ such that (9) holds for all $\varphi \in W^{1,q'}(\Omega)$ and $\|v\|_{L^q(\Omega)} \leq C\|(0, k, g)\|_{-2,q}$. If $1 < r \leq q < \infty$, $\frac{1}{n} + \frac{1}{q} \geq \frac{1}{r}$ and $k \in L^r(\Omega)$, then we can choose v so that $\operatorname{div} v = k$ in $L^r(\Omega)$ and $\gamma v = g$ in $W^{-1/q,q}(\partial\Omega)$ as well.

Lemma 1 and its consequences should be also corrected as follows. In the proof of Lemma 1, $u \in L^{q_0}(\Omega)$ is decomposed into $u = u_s + u_r$ with $C\|u_s\|_{L^{q_0}(\Omega)} \leq \varepsilon$ and $\|u_r\|_{L^\infty(\Omega)} \leq C_\varepsilon\|u\|_{L^{q_0}(\Omega)}$. In general, however, C_ε depends on u . Hence the constant in Lemma 1 (ii) should depend on u itself as well as its norm. Therefore, the constants in Lemmas 4–6 and Propositions 1–3 should depend on the given fields u_1 and u_2 .

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Department of Mathematics,
Sogang University,
Seoul 121-742, Korea.
e-mail: kimh@sogang.ac.kr

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¹ This is studied in more detail by C. Amrouche and M.A. Rodríguez-Bellido [Stokes, Oseen and Navier–Stokes equations with singular data, preprint].