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On learning equilibria

Received: 27 April 2005 / Accepted: 24 October 2005 / Published online: 1 December 2005
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Abstract Bullard (1994) and Schönhofer (1999) show that endogenous business cycles may emerge in an inflationary overlapping generations model where households predict future inflation rates by running a least squares regression on prices. We show that given the same beliefs but under an alternative, more natural, estimation procedure based upon inflation rates the monetary steady state will be globally stable for a large set of savings functions. We also study an evolutionary competition between the two estimation procedures. Although the dynamics are stabilized for a large set of parameter values, endogenous business cycles may still emerge in this heterogeneous beliefs framework.

Keywords Overlapping generations models · Learning · Endogenous business cycles · Heterogeneous beliefs · Evolutionary dynamics

JEL Classification Numbers D83 · D84 · D90 · E31 · E32

1 Introduction

There is a growing literature pointing out the limitations of the rational expectations hypothesis. In particular, it has been perceived as unsatisfactory that this hypothesis endows economic agents with precise information about the structure of the economy and the beliefs of other agents as well as unbounded reasoning abilities

This paper benefitted much from discussions with Jim Bullard, Maurice Bun, Cees Diks, Cars Hommes, Jan Kiviet and Claus Weddepohl. The constructive remarks of an anonymous referee led to significant improvements. This research has been supported by the Netherlands Organisation for Scientific Research (NWO) under a MaGW-Pionier grant.

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to deal with this information. A number of authors have suggested that the rational expectations hypothesis still is valid as a description of long run behavior, since economic agents learn over time and eventually arrive at a rational expectations steady state. The rational expectations hypothesis can therefore be supported by a learning story (see Lucas 1986; Marcet and Sargent 1989; Evans and Honkapohja 2001). In such a learning model *boundedly rational* agents are generally assumed to have no structural information about their economic environment other than time series observations on certain economic variables. They use these observations to make inferences about the economic environment. In his book on bounded rationality Sargent (1993, p. 22) writes: "We can interpret the idea of bounded rationality broadly as a research program to build models populated by agents who behave like working economists or econometricians." Since the perceptions of agents influence their behavior, the learning feeds back into the actual realizations of economic variables. Hence, the learning procedure itself is one of the determinants of the evolution of the economic variables. With respect to this learning procedure Bullard (1994, p. 468) states:

"A common research question, asked increasingly often in the recent literature, is how this learning takes place, and more importantly, if it makes any difference for inferences from dynamic general equilibrium models whether the learning is explicitly modeled."

In this paper Bullard (1994) shows, in an overlapping generations framework, that modelling agents explicitly as econometricians might create equilibrium paths that do not converge to the rational expectations steady state. Some of these *learning equilibria* can be characterized by endogenous fluctuations in inflation rates and agents beliefs. Moreover, Schönhofer (1999) shows that *chaotic* learning equilibria exist.

The objective of the present paper is twofold. First, we show that the non-convergence results of Bullard (1994) and Schönhofer (1999) depend heavily on the estimation procedure their agents use. The net effect of their procedure is that agents weight their past observations with an exponentially (over time) decreasing factor. In other words, their agents forget quickly. But then it is not surprising that the rational expectations steady state may not be learned. In fact, we will show that for an estimation procedure that seems to be more natural from a statistical point of view, the steady state of the learning model is globally stable for a large set of savings functions (including those studied by Bullard 1994; Schönhofer 1999). An important observation is that the perceived law of motion is the same for both procedures, the only difference is in the way this perceived law is estimated. The main point is that it might be more appropriate from an econometric point of view to run a regression on a stationary time series rather than on a nonstationary time series. The fact that in the presented model agents want to predict the inflation rates, together with the observation that the time series of price levels is nonstationary, suggests that the estimation procedure should be in terms of inflation rates instead of price levels. Indeed, we prove that for an estimation procedure based upon inflation rates the monetary steady state is globally stable for a large set of savings functions. The learning equilibria are therefore driven by the estimation procedure agents use and not by the fact that they are learning nor by the structure of their beliefs.

In this paper we identify a different source of complicated dynamics in this inflationary overlapping generations framework. We are interested whether all households will persistently use estimation procedures that do not lead to consistent and converging estimates. Therefore, the second objective of our paper is to study an environment where both types of estimation procedures are available to the households, and where they choose one of them on the basis of their past performance (with past performance being measured in terms of forecasting accuracy). The main research question then is whether the estimation procedure suggested by Bullard (1994) and Schönhofer (1999) will still be viable when a more natural procedure is available, or whether the latter will drive out the former. We investigate this issue by employing the Brock and Hommes (1997) model of evolutionary competition between heterogeneous beliefs. Contrary to our initial expectations, we find that both rules survive this evolutionary competition and that endogenous business cycles are still possible in this heterogeneous beliefs framework, albeit for a significantly smaller set of overlapping generation economies. The intuition behind this result is that, far away from the steady state, the “stable” estimation procedure will perform better (in terms of forecast errors) and eventually most households of future generations will use this rule, which will stabilize the inflation rates. However, for inflation rates close to the steady state, evolutionary pressure against the “unstable” estimation procedure will diminish and more households of future generations will start using it, which may destabilize the inflation dynamics. After this the whole story repeats.

The paper is organized as follows. Section 2 describes the overlapping generations model and the learning equilibria studied in Bullard (1994). In Section 3 the main stability results for the alternative estimation procedure are given. Section 4 introduces evolutionary competition between the two different estimation procedures and Section 5 concludes.

2 Learning equilibria

We consider a standard two period overlapping generations model with only one commodity, where in each period a generation is born that lives for two periods. The generation born in period t solves

$$\max_{c_0, c_1} U(c_0, c_1) \quad \text{subject to} \quad p_t c_0 + p_{t+1}^e c_1 \leq p_t w_0 + p_{t+1}^e w_1, \quad (1)$$

where $U : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is a strictly monotone, strictly quasi concave utility function, $c_0(w_0)$ and $c_1(w_1)$ are consumption (endowment) of the commodity in the first and second period of the agent's life, p_t is the price of the commodity in period t and p_{t+1}^e the price expected for period $t + 1$. Optimal saving of the young generation is given by the savings function $S\left(\frac{p_{t+1}^e}{p_t}\right) = w_0 - c_0\left(\frac{p_{t+1}^e}{p_t}\right)$, where $c_0 = c_0\left(\frac{p_{t+1}^e}{p_t}\right)$ solves (1). From now on we will assume that the savings function is twice differentiable and positive, i.e. $S(\xi) > 0$ for all ξ .¹ The demand for real balances in

¹ Any continuous function corresponds to a savings function (which is an aggregate excess demand function) consistent with utility maximization, if we extend the number of different agents per generation to at least two (see Sonnenschein, 1973).

period t is given by

$$\frac{M_t}{p_t} = S \left(\frac{p_{t+1}^e}{p_t} \right). \quad (2)$$

The only means of saving is money. The money stock M_t is controlled by the government and grows over time to finance government expenditures. The monetary policy rule is

$$M_t = \vartheta M_{t-1}. \quad (3)$$

Combining the demand for real balances (2) with the monetary policy rule (3), and defining the gross inflation rate $\pi_t \equiv \frac{p_{t+1}}{p_t}$ we arrive at the following market clearing condition

$$\pi_{t-1} S(\pi_t^e) = \vartheta S(\pi_{t-1}^e). \quad (4)$$

At the monetary steady state, $\pi^* = \vartheta$, the inflation rate is equal to the money growth rate.

The model is closed by specifying the way in which agents form expectations about future inflation rates. Under rational expectations or perfect foresight we have $\pi_{t+1}^e = \pi_{t+1}$. It is well-known that for a downward sloping savings function the monetary steady state ϑ is unstable under perfect foresight. In this case the steady state is also said to be *locally determinate*, i.e. it is the locally unique perfect foresight solution. For non-monotonic savings functions more complicated perfect foresight dynamics, such as cycles and chaotic fluctuations, may occur (see e.g. Grandmont 1985).

The assumption of perfect foresight requires that agents exactly know the market equilibrium equations as well as other agents' beliefs and are able to use this information to compute the market clearing prices for the future. An alternative approach is to assume that economic agents make inferences about their environment by means of a learning procedure. Such a procedure uses time series observations to make forecasts about the future development of variables. Consider for example the case where agents believe that the inflation rate is constant, that is, in terms of prices, they have the following *perceived law of motion*

$$p_t = \beta p_{t-1}. \quad (5)$$

The agents have no a priori knowledge about β , however. Bullard (1994) assumes that agents run a least squares regression on prices in order to estimate β and that they use this estimate to form predictions on the inflation rate. The least squares regression estimate for agents born in period t , using data available up to time $t-1$, is

$$\beta_t = \frac{\sum_{s=1}^{t-1} p_{s-1} p_s}{\sum_{s=1}^{t-1} p_{s-1}^2}, \quad (6)$$

and hence their forecast of the inflation rate is $\pi_t^e = \beta_t$. Given this forecast, the *implied actual law of motion* for the price dynamics of the model becomes

$$p_t = \vartheta \frac{S(\beta_{t-1})}{S(\beta_t)} p_{t-1}. \quad (7)$$

Equations (6) and (7) together form an *expectations feedback system*. Realized prices influence how agents perceive their economic environment and these perceptions feed back into the actual dynamics and determine which prices will be realized. By introducing the variables $g_t = p_{t-1}^2 [\sum_{s=1}^t p_{s-1}^2]^{-1}$ and $\gamma_t \equiv \beta_{t-1}$ equations (6) and (7) can be written as

$$\begin{aligned}\beta_{t+1} &= \beta_t + g_t \left[\vartheta \frac{S(\gamma_t)}{S(\beta_t)} - \beta_t \right], \\ \gamma_{t+1} &= \beta_t, \\ g_{t+1} &= \left[g_t^{-1} \left(\vartheta \frac{S(\gamma_t)}{S(\beta_t)} \right)^{-2} + 1 \right]^{-1}.\end{aligned}\tag{8}$$

Define the *inflation elasticity of savings* as $a(\pi) = -\pi \frac{S'(\pi)}{S(\pi)}$. We then have

Proposition 1 (Bullard 1994) *Assume $\vartheta > 1$ and $S(\cdot)$ is twice differentiable and downward sloping. Then (8) generically undergoes a Hopf bifurcation at the monetary steady state at that value ϑ^* of ϑ , for which*

$$(1 - \vartheta^{-2}) a(\vartheta) = 1.\tag{9}$$

Moreover, if $(1 - \vartheta^{-2}) a(\vartheta) < (>) 1$ the monetary steady state is locally stable (unstable).

It will be useful for us to investigate the details of this result a little further. The Jacobian matrix of (8) evaluated at the monetary steady state $(\beta^*, \gamma^*, g^*) = (\vartheta, \vartheta, 1 - \vartheta^{-2})$ is

$$\mathbf{J} = \begin{pmatrix} \vartheta^{-2} + (1 - \vartheta^{-2}) a(\vartheta) - (1 - \vartheta^{-2}) a(\vartheta) & 0 & 0 \\ \frac{\partial g}{\partial \beta} & 0 & 0 \\ \frac{\partial g}{\partial \gamma} & \vartheta^{-2} & \end{pmatrix}.\tag{10}$$

One of the eigenvalues is equal to ϑ^{-2} and hence lies inside the unit circle for $\vartheta > 1$. The other two eigenvalues are complex and lie on the unit circle when $(1 - \vartheta^{-2}) a(\vartheta) = 1$. Moreover, the eigenvalues cross the unit circle with positive speed as ϑ changes. The Hopf bifurcation described in the proposition leads to an invariant closed curve around the steady state of the learning dynamics. This closed curve can be attracting or repelling, and motion on the closed curve can be periodic or quasi-periodic.^{2,3} Bullard (1994) calls these cycles “learning equilibria” since they correspond to equilibria of the learning dynamics, which are not equilibria under rational expectations. Their existence can therefore be attributed

² Quasi-periodic motion on a circle is obtained if, measuring the position on the circle by an angle ζ , the dynamics are given as $\zeta_{t+1} = \zeta_t + 2\pi\kappa$, where κ is not a rational number. Then the iterates ζ_t are all distinct from each other, and it can be shown that they fill the circle densely. Invariant quasi-periodic circles are surprisingly stable under perturbations. For details, see Arnol'd (1983) or Devaney (1989).

³ If the savings function is nonmonotonic similar phenomena occur. In that case, the monetary steady state may also lose stability through a *period-doubling* bifurcation. This happens at that value ϑ^* of ϑ for which $\frac{\vartheta^2-1}{\vartheta^2+1} a(\vartheta) = -\frac{1}{2}$.

to the learning process. If ϑ is increased further, the time series of the inflation rates can become even more complicated. Schönhofer (1999) gives, for some specific examples, numerical evidence for the existence of homoclinic orbits and chaos in the learning dynamics.

Let us now try to understand why the recursive least squares estimates do not converge to the monetary steady state. Ordinary least squares algorithms are so-called *decreasing gains* algorithms. Different observations receive the same weights in the regression which implies that, as time goes on and the number of observations increases, the impact or gain of individual new observations becomes smaller. In (8) this gain is represented by the variable $g_t = p_{t-1}^2 / \sum_{s=1}^t p_{s-1}^2$. If price levels are bounded g_t will converge to 0 which, if it does not result in convergence to the monetary steady state, at least leads to ever smaller changes in the estimate of β . In the present model, however, price levels are unbounded and in fact grow at a constant rate $\vartheta > 1$ at the steady state, leading to a strictly positive equilibrium value of g_t , $g^* = 1 - \vartheta^{-2} > 0$. Hence, even after many observations, one new observation on the price level may lead to a significant change in the beliefs of the agents, which therefore may keep on fluctuating, implying endogenous and persisting fluctuations in the inflation rates. All observations still get the same weight in the regression, but the fact that these observations themselves become larger prevent the gain from converging to 0.

In fact, least squares learning on *price levels* is closely related to the *adaptive expectations rule on inflation rates*. Adaptive expectations (Nerlove 1958) correspond to updating the expectation in the direction of the last observation, i.e., $\pi_{t+1}^e = \pi_t^e + \alpha (\pi_{t-1} - \pi_t^e)$, with $0 < \alpha \leq 1$. Notice that the weight α is constant, and adaptive expectations therefore correspond to a *constant gains* algorithm. Introducing adaptive expectations into (4) yields the following second order difference equation

$$\beta_{t+1} = \beta_t + \alpha \left(\vartheta \frac{S(\beta_{t-1})}{S(\beta_t)} - \beta_t \right). \quad (11)$$

Notice that the only difference between (8) and (11) is that for the latter the weight α is constant whereas for the former it depends upon the realization of the prices. However, if the weight in (11) equals $\alpha = g^* = 1 - \vartheta^{-2}$, (11) has the same local stability properties as (8).⁴ Hence, the learning scheme proposed by Bullard (1994) turns out to be closely related to adaptive expectations. Although the weight or gain g_t in (8) is not constant, it is certainly not (monotonically) decreasing over time.

The residuals or forecast errors from the regression (6) turn out to be

$$e_t = p_t - \beta_{t-1} p_{t-1} = \left(\vartheta \frac{S(\beta_{t-1})}{S(\beta_t)} - \beta_{t-1} \right) p_{t-1}. \quad (12)$$

Recall the concept of ω -limit set $\omega(x_0)$ of a point x_0 in a dynamical system Φ . This is the ‘asymptotic tail’ of the orbit through x_0 , that is, the set of accumulation points of the orbit of Φ through x_0 ; in symbols $\omega(x_0) = \bigcap_{t>0} \overline{\{y | y = \Phi^s(x_0), s > t\}}$, where

the upper bar indicates closure of the set. We now have

⁴ This follows from the fact that the upper 2×2 matrix of (10) (which is the relevant part) is equal to the Jacobian of (11) evaluated at the monetary steady state.

Lemma 2 *Consider the dynamical system (8) for $\vartheta > 1$. If the economy evolves along an ω -limit evolution other than the monetary steady state ϑ and if the β_t remain bounded for all t , then the forecast errors (12) grow without bound.*

Proof Note that the implied actual law (7) yields

$$p_{t+n} = \vartheta^n \frac{S(\beta_t)}{S(\beta_{t+n})} p_t. \quad (13)$$

By assumption, there is a constant $M > 0$ such that $0 < \beta_t < M$ for all t . Let moreover $m_1 = \min_{0 \leq \beta \leq M} S(\beta)$ and $m_2 = \max_{0 \leq \beta \leq M} S(\beta)$. Then it follows from (13) that

$$\frac{m_1}{m_2} p_0 \leq p_t \vartheta^{-t} \leq \frac{m_2}{m_1} p_0.$$

We see that the prices p_t grow exponentially, asymptotically with rate ϑ . Therefore, if forecast errors are bounded, then from (12) we have $\lim_{t \rightarrow \infty} \vartheta \frac{S(\beta_{t-1})}{S(\beta_t)} - \beta_{t-1} = 0$. For an evolution on an ω -limit set, the limit is attained already, and $\vartheta \frac{S(\beta_{t-1})}{S(\beta_t)} - \beta_{t-1} = 0$, for all t . But then the learning model (8) implies that $\beta_{t+1} = \beta_t$ for all t , and consequently, that $\beta_t = \vartheta$, contrary to the assumption that the dynamics is not at the monetary steady state. $\square$

This lemma provides us with another explanation for the nonconvergence of the recursive system (8): because the forecast errors grow indefinitely (in absolute value), the estimates keep changing significantly, despite the fact that the weight attached to each individual observation in the least squares regression (6) decreases as time goes by. Each new observation can upset the current estimate and lead to a radical change in the estimated perceived law of motion which, of course, is an unsatisfactory property of an estimation procedure. Given the exploding forecast errors, and the fluctuations in the beliefs, agents will be inclined to change their estimation procedure.

In sum, it is the fact that agents use a nonstationary time series for their regression that drives the possible instability of the learning process. Whether it is, from an econometrician's viewpoint, sensible to use this nonstationary time series depends upon how stochastic shocks enter the perceived law of motion. That is, if agents believe that the evolution of prices is governed by $p_t = \beta p_{t-1} + \varepsilon_t$ where ε_t is white noise, a least squares regression might in fact be appropriate. On the other hand however, the assumption that ε_t is white noise is inconsistent with the implication of Lemma 2 that the forecast errors grow without bound and it seems reasonable to assume that this will lead agents to use a different estimation procedure.

3 An alternative learning procedure

In the previous section it was argued that the nonstationary nature of the price time series may lead to endogenous and persisting fluctuations in inflation rates. According to the perceived law of motion (5) agents believe that the systematic

part of the inflation rate is constant. We can rewrite (5) into the following perceived law of motion

$$\pi_t = \beta. \quad (14)$$

Notice that the economic agents' perceptions underlying both (5) and (14) is that the inflation rate is constant.⁵

Now suppose agents try to learn the correct value of β in (14) by running a least squares regression of inflation rates on a constant, which corresponds to averaging over past inflation rates, that is $\beta_{t+1} = \frac{1}{t} \sum_{s=0}^{t-1} \pi_s = \left(1 - \frac{1}{t}\right) \beta_t + \frac{1}{t} \pi_{t-1}$. The evolution of inflation rates and dynamics is therefore described by

$$\beta_{t+1} = \beta_t + \frac{1}{t} \left(\vartheta \frac{S(\beta_{t-1})}{S(\beta_t)} - \beta_t \right). \quad (15)$$

This updating rule is closely related to (8) and (11), the main difference lying in the fact that the weight factor $1/t$ approaches 0 as t goes to infinity. Hence, the contribution of new observations will decrease over time and the stability properties of (15) turn out to be dramatically different from the stability properties of (8). In an earlier version of the current paper (Tuinstra and Wagener 2000) we showed that the monetary steady state ϑ is globally stable under (15) for a large set of savings functions (including those studied in Bullard 1994; Schönhofer 1999).⁶

In the present version of this paper we consider a slightly different algorithm. Note that the total relative price increase between period 0 and period t can be written as $\frac{p_t}{p_0} = \frac{p_t}{p_{t-1}} \cdot \frac{p_{t-1}}{p_{t-2}} \cdots \frac{p_1}{p_0} = \prod_{s=0}^{t-1} \pi_s$. The average inflation rate over this

period should therefore be computed as its *geometric* mean $\left(\prod_{s=0}^{t-1} \pi_s\right)^{\frac{1}{t}}$ and this is what our agents use as a predictor for the future inflation rate β_{t+1} . Notice that this can be written recursively as

$$\ln \beta_{t+1} = \frac{1}{t} \sum_{s=0}^{t-1} \ln \pi_s = \ln \beta_t + \frac{1}{t} (\ln \pi_{t-1} - \ln \beta_t), \quad (16)$$

which corresponds to the *arithmetic* mean of the logarithm of the inflation rate.

Now define $x_t = \ln \beta_t - \ln \vartheta$, and $\sigma(x_t) = \ln S(\vartheta \exp[x_t])$. Using (7), (16) can be written as the following nonautonomous second order difference equation

$$x_{t+1} = x_t + \frac{1}{t} (\sigma(x_{t-1}) - \sigma(x_t) - x_t). \quad (17)$$

⁵ From an econometric point of view there is a distinction between the perceived laws of motions (5) and (14) however, depending upon how shocks are perceived to enter the system; as $p_t = \beta p_{t-1} + \varepsilon_t$, or as $\pi_t = \beta + \varepsilon_t$ (or as $\ln \pi_t = \ln \beta + \varepsilon_t$ for the system analyzed later in this section). From an econometric perspective it is, a priori, unclear which of these perceived laws of motion is the most appropriate. Note that all of these perceived laws of motions are misspecified, since the underlying actual law of motion (7) is a more complicated function of inflation rates and beliefs.

⁶ Also Evans and Honkapohja (2001, p. 358), when discussing the results of Bullard (1994), suggest that it is more natural to assume that β is estimated by the mean of past inflation rates. Moreover, they also claim that the steady state is always stable under this learning rule.

The remainder of this section is devoted to analyzing the stability properties of (17).

Recall that a function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is *locally Lipschitz continuous*, if it satisfies a Lipschitz condition on any bounded interval. In this section we show that the steady state $x = 0$ of the dynamic system (17) is locally stable for any locally Lipschitz continuous function σ , and that $x = 0$ is globally stable for any function satisfying a Lipschitz condition on $\mathbb{R}$.

The first property of our system is that, if a sufficiently long initial segment of the sequence $\{x_t\}$ is bounded then the whole sequence is bounded.

Proposition 3 *Let $K > 2$ be an arbitrary positive constant, and assume that $\sigma : [-K, K] \rightarrow \mathbb{R}$ is Lipschitz continuous with Lipschitz constant L . Let the sequence $\{x_t\}_{t=1}^\infty$ satisfy the recurrence relation (17). If for $1 \leq t \leq 10L + 1$ the condition*

$$|x_t| < \frac{3}{10}K \quad (18)$$

holds, then for all $t \geq 1$ we have $|x_t| < K$.

Proof Introduce the increments $\delta_{t+1} = x_{t+1} - x_t$, and note that rewriting the recurrence equation (17) in terms of the increments gives

$$\delta_{t+1} = \frac{1}{t} (\sigma(x_t - \delta_t) - \sigma(x_t) - x_t). \quad (19)$$

The idea of the proof: if all x_t were, for instance, larger than zero, then using Lipschitz continuity, the increment δ_{t+1} can be bounded from above by

$$\delta_{t+1} \leq \frac{L}{t} |\delta_t|. \quad (20)$$

For large t , the factor L/t will be smaller than unity. Inequality (20) then implies that the sequence x_t is bounded from above. However, care is needed to treat the case that the x_t change sign, and to deal with the fact that (20) only gives an upper bound.

Set $T = 10L + 1$. Note that $|\delta_{t+1}| \leq |x_{t+1}| + |x_t| \leq \frac{3}{5}K$ for all $1 \leq t \leq T - 1$ by assumption. Moreover, as a consequence of (20) and the choice of T , the δ_{t+1} have the property that whenever $x_t > 0$ for $t \geq T$, then $\delta_{t+1} \leq \frac{1}{10}|\delta_t|$. We claim that this estimate implies $x_t < x_{T-1} + \frac{2}{3}K < K$ for all t . Note generally that $x_t - x_{T-1} = \sum_{j=0}^{t-T} \delta_{T+j}$.

1. First consider the case that $x_t > 0$ for all $t \geq T$. Let $T \leq s \leq t - 1$ be such that $\delta_s \leq 0$ and that $\delta_{s+j} > 0$, for $j = 1, \dots, n$. Since $\sum_{j=1}^n \delta_{s+j} \leq \sum_{j=1}^n 10^{-j} |\delta_s| < -\delta_s$, it follows that $\sum_{j=0}^n \delta_{s+j} < 0$. Hence, the sum $\sum_{j=0}^{t-T} \delta_{T+j}$ is bounded from above by the contribution of the first l terms, where l is such that $\delta_{T+j} \geq 0$ for $0 \leq j \leq l$ and $\delta_{T+l+1} < 0$. Since $|\delta_T| < \frac{3}{5}K$, this contribution is smaller than $\sum_{j=0}^{t-T} \delta_{T+j} < \frac{3}{5}K / (1 - \frac{1}{10}) = \frac{2}{3}K$ and therefore $x_t = x_{T-1} + \sum_{j=0}^{t-T} \delta_{T+j} < (\frac{3}{10} + \frac{2}{3})K < K$.

2. In the case of a sign change of x_t , we argue by induction. It is assumed that $|x_j| < K$ for $1 \leq j \leq t$, and that $x_t < 0$ and $x_{t+j} \geq 0$ for some t and $j = 1, \dots, n$. Then, by (19), $x_t < |\delta_{t+1}| \leq \frac{L}{t} |x_t - x_{t-1}| + \frac{|x_t|}{t} \leq \frac{L}{10L+1} 2K + \frac{K}{10L+1} < \frac{3}{10} K$. Hence $x_{t+1} < \frac{3}{10} K$, and by the same argument as in 1. it follows that $\sum \delta_{t+j} < \frac{2}{3} K$ and consequently that $x_{t+j} < K$ for all $j = 1, \dots, n$.

The proof that $x_t > -K$ for all $t \geq T$ is completely analogous. $\square$

As a consequence we have

Corollary 4 *Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be (globally) Lipschitz continuous with Lipschitz constant L . If the sequence $\{x_t\}$ satisfies (17), then it is bounded.*

Proof In view of Proposition 3, all we have to know is that $\max_{1 \leq t \leq 10L+1} |x_t| < \infty$. But this follows from the Lipschitz condition. $\square$

In order to get some idea what types of savings functions are covered by Proposition 3 let $\xi = \vartheta \exp[x]$ and $\phi = \vartheta \exp[y]$. The condition that $\sigma(x)$ satisfies a global Lipschitz condition $|\sigma(x) - \sigma(y)| < L|x - y|$ is equivalent to $\left(\frac{\xi}{\phi}\right)^{-L} < \frac{S(\xi)}{S(\phi)} < \left(\frac{\xi}{\phi}\right)^L$. In particular, for $\phi = 1$ this condition reads as $S(1)\xi^{-L} < S(\xi) < S(1)\xi^L$. The first inequality says that S should not decrease faster than polynomially in ξ^{-1} as $\xi \rightarrow \infty$.

That the condition is quite sharp can be seen by considering $\sigma(x) = -x - x^2$. The corresponding savings function reads as $S(\xi) = (\xi/\vartheta)^{-(\ln \xi + c)}$, where $c = 1 - \ln \vartheta$. Note that here the exponent of ξ^{-1} grows beyond all bounds. For this choice of σ , the unbounded series $x_t = t$ is one of the solutions of (17). However, also for this savings function the trajectories under (17) remain bounded for a large set of initial conditions.

Bounded sequences have the following attractive property.

Proposition 5 *Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Any bounded orbit $\{x_t\}$ of (17) converges to 0.*

Proof Recall that 0 corresponds to the monetary steady state in the original variables. Introduce, as before, the increment $\delta_{t+1} = x_{t+1} - x_t$, and let $K > 0$ be such that $|x_t| < K$ for all $t \geq 1$. Note that it follows from (19) that $|\delta_t| < \frac{M}{t}$ for some $M > 0$.

We shall show that for arbitrary $\varepsilon > 0$, any point will move to the interval $[-2\varepsilon, 2\varepsilon]$, and that the points in this interval cannot escape too far if t is sufficiently large.

Fix $\varepsilon > 0$. Note that σ is uniformly continuous on $[-K, K]$, and hence that there exists an $\delta > 0$ such that if $|\delta_t| < \delta$, then $|\sigma(x_t - \delta_t) - \sigma(x_t)| < \varepsilon$. Choose $T > 0$ larger than M/ε and M/δ . Then for all $t > T$, we have that $|\delta_t| < \varepsilon$ and, from (19), that

$$\delta_{t+1} \leq \frac{\varepsilon - x_t}{t}. \quad (21)$$

Let $t_0 > T$. We ask whether it is possible that $x_t \notin [-2\varepsilon, 2\varepsilon]$ for all $t \geq t_0$. Since $|\delta_{t+1}| < \varepsilon$, points cannot move from positive to negative without going through

$[-2\varepsilon, 2\varepsilon]$, and we may restrict our attention to the case that the x_t are positive (the other case would be treated similarly).

From (21) and $x_t > 2\varepsilon$, the following estimate is obtained: $x_t = x_{t_0} + \sum_{s=t_0}^{t-1} \delta_{s+1} \leq x_{t_0} - \varepsilon \sum_{s=t_0}^{t-1} \frac{1}{s}$. Since the harmonic series $\sum s^{-1}$ diverges, the right hand side cannot be larger than 2ε for all t , and hence there must be a first moment in time t_1 such that $x_{t_1} \in [-2\varepsilon, 2\varepsilon]$. But note that then for $t > t_1$, the state x_t cannot escape the interval $[-3\varepsilon, 3\varepsilon]$ any more: if $|x_{t-1}| < 2\varepsilon < |x_t|$, then $|x_t| < |x_{t-1}| + |\delta_t| < 2\varepsilon + \varepsilon$, and the next iterates move the state back to the interval $[-2\varepsilon, 2\varepsilon]$, as can be seen by noting that from (21) we have $\delta_{t+1} < 0$ for $x_t > \varepsilon$. Similarly we have, from (19), $\delta_{t+1} > 0$ for $x_t < -\varepsilon$.

Finally, we remark that if σ is of the form $\ln S(\vartheta \exp[x])$, with S a positive savings function, then σ has the special property that $\lim_{x \rightarrow -\infty} \sigma(x) = \ln S(0)$. From this we obtain that an orbit $\{x_t\}$ of (17) will be bounded from below. To see this, fix $\varepsilon > 0$, and find $K > 0$ such that $|\sigma(x) - \ln S(0)| < \varepsilon$ if $x < -K$. Then for $x_t < -K$ we have that $x_{t+1} \geq x_t - \frac{x_t + 2\varepsilon}{t} \geq x_t + \frac{K - 2\varepsilon}{t} \geq x_t$, while for $|x_t| \leq K$ we have that $x_{t+1} \geq x_t - \frac{2C + K}{t} \geq -2C - 2K$, where $C = \max_{|x| \leq K} \sigma(x)$. Hence $x_t \geq \min(x_1, -2C - 2K)$, and these orbits cannot run off to infinity. $\square$

From these results we may conclude that our dynamic system converges to the monetary steady state ϑ for a large set of savings functions (including those from Bullard 1994; Schönhofner 1999). If the inflation dynamics do not converge to this equilibrium inflation rates diverge to infinity (which corresponds to autarky). This only happens if savings decline very fast as the (expected) inflation rate increases. Duffy (1994) studies an overlapping generations model where this autarkic steady state is also stable under learning.

4 Competition between learning procedures

As shown above, the procedure that agents use to estimate a perceived law of motion is pivotal for the stability properties of the full economic system. Moreover, the endogenous business cycles studied by Bullard (1994) and Schönhofner (1999) do not result from the perceptions of the agents per se, but from the way in which these perceptions are updated as new information becomes available. From Propositions 3 and 5 we know that these endogenous business cycles disappear when agents use inflation rates, instead of price levels, to update their perceptions.

However, the fact that an estimation procedure is destabilizing does not necessarily imply that economic agents will not use it. It is therefore important to investigate what happens when both procedures are available to the agents, a problem which we take up in this section. We assume that each newborn agent chooses an estimation procedure on the basis of its past performance. In fact, the lower the average quadratic forecast error of the predictions generated by a certain procedure, the higher the fraction of the newborn generation that will choose this procedure. Our objective is to study whether, in this *evolutionary competition* between estimation procedures, the “unstable” procedure will be driven out. A priori, this does not have to be the case, since the presence of this procedure may disrupt the inflation

dynamics to such an extent that the other estimation procedure will perform even worse.⁷

We follow the framework developed in Brock and Hommes (1997) for analyzing evolutionary competition between estimation procedures. There are two types of agents, those using prices to estimate future inflation rates (index 1), as discussed in Section 2, and those using past inflation rates (index 2), as discussed in Section 3. The fraction of agents of type 1 of the generation born in period t is denoted by n_t . Demand for real money balances follows as [cf. equation (2)] $\frac{M_t}{p_t} = n_t S(p_{1,t+1}^e/p_t) + (1 - n_t) S(p_{2,t+1}^e/p_t)$. Using the money growth rule (3), the market clearing condition becomes

$$\pi_{t-1} = \vartheta \frac{n_{t-1} S(\beta_{1,t-1}) + (1 - n_{t-1}) S(\beta_{2,t-1})}{n_t S(\beta_{1t}) + (1 - n_t) S(\beta_{2t})}, \quad (22)$$

where $\beta_{1t} = p_{1,t+1}^e/p_t$ and $\beta_{2t} = p_{2,t+1}^e/p_t$, are the expected inflation rates of type 1 and type 2 agents, respectively, which evolve according to [cf. equations (6) and (16)]

$$\begin{aligned} \beta_{1,t+1} &= \beta_{1t} + g_t [\pi_{t-1} - \beta_{1t}], \\ g_{t+1} &= [g_t^{-1} (\pi_{t-1})^{-2} + 1]^{-1} \text{ and} \\ \beta_{2,t+1} &= \beta_{2t}^{1-\frac{1}{i}} \pi_{t-1}^{\frac{1}{i}}. \end{aligned} \quad (23)$$

First consider the case where the fraction n_t is fixed and exogenously given. The proof of the following result, together with that of Proposition 7, is based upon a generalization of a result from Evans and Honkapohja (2000), which can be found in the appendix of this paper.

Lemma 6 *Consider the model given by (22) and (23), where the distribution of agents over learning procedures is fixed, i.e. $n_t = \bar{n}$ for all t . Assume furthermore that the savings function is positive but monotonically decreasing in the inflation rate and that $a'(\vartheta) > 0$. The monetary steady state ϑ then is locally stable if*

$$\bar{n} (1 - \vartheta^{-2}) a(\vartheta) < 1. \quad (24)$$

Proof First we linearize the system [i.e. equations (22) and (23)] around the steady state and then apply Proposition 9 from the appendix, where $\xi_t = \tilde{\beta}_{2t}$ and $x_t = (\tilde{\beta}_{1t}, \tilde{\beta}_{1,t-1}, \tilde{g}_t)$ are the linearized variables. Since, in the terminology of Proposition 9, $A = 0$ and $C = 0$ [see equations (37), (38)] local stability is determined by the eigenvalues of the matrix D , which in fact is the Jacobian matrix of the time invariant dynamical system

$$\begin{aligned} \beta_{t+1} &= \beta_t + g_t \left[\vartheta \frac{\bar{n} S(\gamma_t) + (1 - \bar{n}) S(\vartheta)}{\bar{n} S(\beta_t) + (1 - \bar{n}) S(\vartheta)} - \beta_t \right], \\ \gamma_{t+1} &= \beta_t \\ g_{t+1} &= \left[g_t^{-1} \left(\vartheta \frac{\bar{n} S(\gamma_t) + (1 - \bar{n}) S(\vartheta)}{\bar{n} S(\beta_t) + (1 - \bar{n}) S(\vartheta)} \right)^{-2} + 1 \right]^{-1}, \end{aligned}$$

⁷ This would be similar to the theoretical finding that on financial markets so-called “noise traders” cannot always be driven out by rational investors (see De Long, Shleifer, Summers and Waldmann 1990).

evaluated at the steady state. This Jacobian matrix is given by [compare with (10)]

$$D = \begin{pmatrix} \vartheta^{-2} + \bar{n} (1 - \vartheta^{-2}) a(\vartheta) & -\bar{n} (1 - \vartheta^{-2}) a(\vartheta) & 0 \\ 1 & 0 & 0 \\ 2\bar{n}\vartheta^{-3} (1 - \vartheta^{-2}) a(\vartheta) & -2\bar{n}\vartheta^{-3} (1 - \vartheta^{-2}) a(\vartheta) & \vartheta^{-2} \end{pmatrix} = \begin{pmatrix} D_1 & 0 \\ D_2 & \vartheta^{-2} \end{pmatrix}, \quad (25)$$

where D_1 is the 2×2 upper left submatrix of D . The eigenvalues of D are $\mu_3 = \vartheta^{-2}$ and those of D_1 . The determinant of D_1 is $\bar{n} (1 - \vartheta^{-2}) a(\vartheta)$ which, under condition (24), is smaller than 1. Moreover, condition (24) together with $\vartheta > 1$ implies that the trace of D_1 , which equals $\vartheta^{-2} + \bar{n} (1 - \vartheta^{-2}) a(\vartheta)$, lies between 0 and 2. The eigenvalues of D_1 are given by $\mu_{1,2} = \frac{1}{2} \text{Trace} \pm \frac{1}{2} \sqrt{(\text{Trace})^2 - 4 \text{Det}}$. If the eigenvalues are complex then their absolute value is equal to the determinant which is smaller than one. Therefore the eigenvalues lie inside the unit circle for that case. If the eigenvalues are real they have to be positive since the trace (which is equal to $\mu_1 + \mu_2$) and the determinant (which is equal to $\mu_1 \times \mu_2$) are both positive. As can be easily verified the largest eigenvalue is larger than one if and only if $\text{Trace} > 1 + \text{Det}$. However, since $\text{Trace} = \vartheta^{-2} + \text{Det}$ and $\vartheta > 1$ this is impossible. Therefore, under condition (24) all eigenvalues of D lie in the unit circle and it follows from Proposition 9 that the steady state of the model (22) and (23), with $n_t = \bar{n}$, is locally stable. $\square$

Clearly, a decrease in the fraction $\bar{n}$ of type 1 agents generates local stability of the monetary steady state for a larger set of savings functions and money growth rates ϑ .

Now we will endogenize n_t by assuming that the households from a newborn generation base their choice for the estimation procedure upon the past performance of these procedures. The appropriate performance measure is the *average quadratic forecast error*, since this is what agents are actually trying to minimize with their least squares algorithms.⁸ The (negative of the) average quadratic forecast error for learning procedure i is given as

$$w_{i,t+1} = -\frac{1}{t-1} (\beta_{i,t-1} - \pi_{t-1})^2 + \frac{t-2}{t-1} w_{it}, \quad i = 1, 2. \quad (26)$$

The relationship between $w_{1,t+1}$, $w_{2,t+1}$ and the fraction n_{t+1} is modelled as follows. We assume that each household i of the newborn generation can be identified by a parameter ε_i , which is distributed according to a distribution F , with mean 0. This parameter ε_i measures household i 's preference for (or bias towards) estimation procedure 2. This household then chooses rule 1 if and only if the performance of rule 1 is significantly higher than that of rule 2, that is, if $w_{1,t+1} - w_{2,t+1} \geq \frac{1}{\eta} \varepsilon_i$, where η is a measure of the dispersion of the bias, in the sense that as η increases the bias toward rule 2 is distributed more tightly around 0. Moreover, as η goes to $+\infty$, all households always choose the estimation procedure with the lowest average quadratic forecast error. Assuming that there are many households in each

⁸ Alternatively, we could let the choice of estimator depend upon the realized utility from the savings decision. This leads to qualitatively the same results.

generation and ε_i is drawn from the logistic distribution we obtain

$$n_{t+1} = \frac{1}{1 + \exp[\eta(w_{2,t+1} - w_{1,t+1})]}, \quad (27)$$

but other choices give qualitatively the same results. The parameter η is sometimes called the *intensity of choice* and can be interpreted as a measure of how sensitive agents' choice of estimation procedure is with respect to differences in forecasting accuracy. In fact, as η goes to infinity all households of the current generation will choose the estimation procedure with the best forecasting accuracy in the past. The parameter η therefore measures the level of "rationality" in the choice of estimation procedure. Furthermore, if estimation procedures perform equally well, agents will be distributed evenly over the estimation procedures, i.e. $n_t = \frac{1}{2}$. Note the timing of the process. When a new generation is born in period $t + 1$ the last price it observes is p_t and the last inflation rate it observes is π_{t-1} . The fraction n_{t+1} is determined by quadratic forecast errors, and the last observed forecast error therefore corresponds to $\beta_{i,t-1} - \pi_{t-1}$.

Combining (26) and (27) gives

$$n_{t+1} = \frac{1}{1 + \exp[\eta W_{t+1}]}, \quad (28)$$

where

$$W_{t+1} = \alpha_{t-1} ((\beta_{1,t-1}^2 - \beta_{2,t-1}^2) + 2\pi_{t-1} (\beta_{2,t-1} - \beta_{1,t-1})) + (1 - \alpha_{t-1}) \frac{1}{\eta} \ln \frac{1 - n_t}{n_t},$$

with $\alpha_t = \frac{1}{t}$. The full evolutionary model is now given by equations (22), (23) and (28). The steady state of the full model is $(\beta_1^*, g^*, \beta_2^*, n^*) = (\vartheta, 1 - \vartheta^{-2}, \vartheta, \frac{1}{2})$. Applying Proposition 9 from the appendix again we find the following.

Proposition 7 *Consider the evolutionary model given by equations (22), (23) and (28). Let the savings function S be positive and monotonically decreasing in the inflation rate. Then the monetary steady state is locally stable if*

$$(1 - \vartheta^{-2}) a(\vartheta) < 2. \quad (29)$$

Proof We apply Proposition 9 from the appendix again, where in this case $\xi_t = (\tilde{\beta}_{2t}, \tilde{n}_t)$ and $x_t = (\tilde{\beta}_{1t}, \tilde{\beta}_{1,t-1}, \tilde{g}_t)$. Since the matrices A and C both are equal to 0 local stability of the system given by (22), (23) and (28) is determined again by the eigenvalues of the matrix D from (25) with $\bar{n} = \frac{1}{2}$. Applying the reasoning of the proof of Lemma 6 we find that all eigenvalues lie inside the unit circle and then, by Proposition 9 the steady state of (22), (23) and (28) is stable. $\square$

Note the relation with condition (9) from Proposition 1 which characterises local stability in the case that only the type 1 estimation procedure is available. The interpretation underlying Proposition 7 is the following: if inflation rates converge to the monetary steady state, both types of forecasts will be correct and the fraction n_t will converge to $\frac{1}{2}$. However, if the monetary steady state is unstable when n is fixed at $\frac{1}{2}$ (apply Lemma 6 for $\bar{n} = \frac{1}{2}$) the autonomous dynamic system will be locally unstable at the monetary steady state ϑ . Also note that, although instability

Fig. 1 Attractor for the evolutionary model with $\vartheta = \frac{5}{2}$, $\eta = 3$ and CES utility function with $\rho = \frac{3}{4}$

of the steady state is still possible under evolutionary competition between the estimation procedures, this instability occurs only under a subset of savings functions and money growth rates for which instability arises in the original model. The introduction of type 2 agents therefore indeed tends to stabilize the inflation and learning dynamics. This is similar to the result by Branch (2002) who finds that adding a third predictor, corresponding to adaptive expectations, to the Brock and Hommes (1997) model of evolutionary competition between rational and naive predictors in a cobweb model, may stabilize the system.

Let us consider a typical numerical simulation to illustrate the global dynamics of the model when condition (29) does not hold. We use an example from Bullard (1994), where the savings function is derived from the well-known CES utility function $U(c_0, c_1) = (c_0^\rho + c_1^\rho)^{1/\rho}$ with endowments $w_0 = 1$ and $w_1 = 0$ and $\frac{1}{2} < \rho < 1$. Taking $\rho = \frac{3}{4}$, the savings function and inflation elasticity follow as $S(\pi) = (1 + \pi^3)^{-1}$ and $a(\pi) = 3(1 + \pi^{-3})^{-1}$. From Proposition 1 we find that, if all agents are of type 1 (i.e. $n_t = 1$ for all t), the monetary steady state is locally stable for $\vartheta < \frac{1}{2} + \frac{1}{2}\sqrt{3} \approx 1.366$. On the other hand, if n_t is determined by past performance, as in (28), the steady state is locally stable for $\vartheta < 2$, as can be seen from applying Proposition 7. Simulation results for the case with $\vartheta = \frac{5}{2}$ and $\eta = 3$ are shown in Figures 1 and 2.

Fig. 2 Time series for the evolutionary model with $\vartheta = \frac{5}{2}$, $\eta = 3$ and with CES utility function with $\rho = \frac{3}{4}$

The attractor in Figure 1 shows the long run behavior of the inflation rates for $\vartheta = \frac{5}{2}$ and $\eta = 3$. Inflation rates move quasi-periodically over an invariant closed curve. Figure 2 shows the time series for the forecasts β_{1t} and β_{2t} and the fraction n_t . These time series provide a nice illustration of the mechanism underlying the global dynamics. When inflation rates are far away from the monetary steady state, as in the first ten periods, predictor β_2 is much more accurate than β_1 . This decreases n_t as can be seen in the lower panel of Figure 2. The decreased fraction of type 1 agents stabilizes the inflationary dynamics. However, as inflation rates converge to the monetary steady state, there is no evolutionary pressure against β_1 and the fraction of type 1 agents will increase again, until n_t has taken a value such that the inflation rates become unstable (it follows from Lemma 6 that for $\vartheta = \frac{5}{2}$ the critical value of n is given by $n^c = \frac{19}{45} \approx 0.4222$) and the story repeats. Eventually the fraction n_t converges to $\hat{n} \approx 0.4248$. At this value of n the inflation rates and the forecasts of learning procedure 1 move over an invariant closed curve. Note that the fraction n_t converges because it is determined by the average forecast errors which converge as long as they are bounded.

Qualitatively similar results are obtained for simulations with different savings functions, parameter settings and/or initial conditions, although the transient behavior might be a little different.⁹ In all these examples n_t converges to a value below $\frac{1}{2}$.

⁹ Depending on the parameter values and the initial conditions, the transient behavior of the system might be quite long (this happens for example for high values of η).

5 Concluding remarks

Departing from the theory of rational expectations introduces infinitely many degrees of freedom in modelling agents' beliefs. This "wilderness of bounded rationality" can be restricted by considering agents that, if not unboundedly rational, at least are trying to be "sensible" in predicting the future development of economic variables. That is, they should have a perceived law of motion that is reasonable, in some sense, and they should use the appropriate econometric techniques to estimate this perceived law of motion. The learning equilibria obtained by Bullard (1994) and Schönhofer (1999), are due to applying a least squares regression to a nonstationary price time series. In this paper we have shown that a different and perhaps more reasonable estimation technique (estimating the perceived law of motion on the basis of the stationary time series of inflation rates) induces convergence to the monetary steady state. Recall that the perceived laws of motion and therefore agents' beliefs are the same for both models. Therefore, these learning equilibria are not the result of modelling agents as econometricians per se, but they are the result of the specific way in which these econometricians run their regressions.

Another approach to discipline the "wilderness of bounded rationality" is to consider an evolutionary competition between the different types of beliefs, and then let this competition decide which beliefs will eventually survive. Applying such an evolutionary competition to the two estimation procedures discussed in this paper shows that none of the estimation procedures will be driven out. This result can be explained by observing that at the steady state both learning procedures generate the same forecasts, and their performance will therefore be the same. At this steady state, therefore, there is no evolutionary pressure against the rule using a nonstationary time series, but its presence may nevertheless destabilize the inflation rate dynamics and cause endogenous business cycles to emerge.

In this paper we have therefore identified a new route to endogenous fluctuations in overlapping generations models. First we argued that in a homogeneous world, where all agents use the same estimation procedure, it does not seem to be reasonable to assume that they all use a procedure which uses nonstationary data, leads to exploding forecast errors and generates inconsistent and nonconverging estimators. Second, in a heterogeneous world, where some agents might use such a procedure whereas others do not, an evolutionary competition between the different rules might, in a very natural way, lead to endogenous fluctuations.

The goal of this paper has not been to provide a counter-example to possible instability of learning models (Wenzelburger 2002, for example, discusses for the same overlapping generations framework another learning process with nice stability properties). In fact, we saw in section 4 that learning processes may lead to endogenous business cycles. However, the underlying mechanism driving these business cycles is different from that in the original model of Bullard (1994) and certainly more robust. Other examples of learning models that might lead to endogenous fluctuations are, for example, provided by Hommes and Sorger (1998) and Tuinstra (2003). In these models the perceived law of motion of the agents con-

verges to some limit belief and given this limit belief prices keep fluctuating over some nontrivial attractor.

Appendix A: Local asymptotic stability of the evolutionary model

To show the local asymptotic stability of the evolutionary model defined by equations (22), (23) and (27), we have to generalize a result from Evans and Honkapohja (2000). They consider an iterative scheme of the form

$$x_t = x_{t-1} + \alpha_t (F(x_{t-1}, \alpha_t) - x_{t-1}), \quad (30)$$

with $x_t \in \mathbb{R}^m$, $\alpha_t \in \mathbb{R}$ and the function F continuously differentiable in a neighborhood of $(x, \alpha) = (0, 0)$, $F(0, \alpha) = 0$ and F continuous in a neighborhood of $(0, \alpha)$ for every $\alpha \in [0, 1]$. Moreover, they assume that $0 < \alpha_t < 1$, $\lim_{t \rightarrow \infty} \alpha_t = 0$ and $\sum_{t=1}^{\infty} \alpha_t = \infty$.

These assumptions imply that $x = 0$ is a fixed point of the scheme.

Theorem 8 (Evans and Honkapohja 2000, Proposition 1, p. 719) *If all the eigenvalues of the matrix $D_x F(0, 0)$ have real part smaller than one, the fixed point $x = 0$ of (30) is locally asymptotically stable; that is, there is a neighborhood U of $x = 0$, such that for all $x_0 \in U$, the sequence $\{x_t\}$ generated by (30) converges to 0.*

However, the form of the evolutionary model given by (22), (23) and (28) is more complicated than the Evans–Honkapohja iterative scheme (30). This can be seen most readily by introducing linearizing variables

$$\beta_{j,s} = \vartheta + \tilde{\beta}_{j,s}, \quad g_t = (1 - \vartheta^{-2}) + \tilde{g}_t, \quad n_t = \frac{1}{2} + \tilde{n}_t.$$

Expanding the evolutionary model around the steady state then yields

$$\tilde{\beta}_{1,t+1} = \tilde{\beta}_{1,t} + \left(1 - \frac{1}{\vartheta^2}\right) \left[-\frac{a(\vartheta)}{2} (\tilde{\beta}_{1,t-1} + \tilde{\beta}_{2,t-1} - \tilde{\beta}_{1,t} - \tilde{\beta}_{2,t}) - \tilde{\beta}_{1,t} \right] + \dots, \quad (31)$$

$$\tilde{\beta}_{2,t+1} = \tilde{\beta}_{2,t} + \frac{1}{t} \left[-\frac{a(\vartheta)}{2} (\tilde{\beta}_{1,t-1} + \tilde{\beta}_{2,t-1} - \tilde{\beta}_{1,t} - \tilde{\beta}_{2,t}) - \tilde{\beta}_{2,t} \right] + \dots, \quad (32)$$

$$\tilde{g}_{t+1} = -\vartheta^{-3} (1 - \vartheta^{-2}) a(\vartheta) (\tilde{\beta}_{1,t-1} + \tilde{\beta}_{2,t-1} - \tilde{\beta}_{1,t} - \tilde{\beta}_{2,t}) + \vartheta^{-2} \tilde{g}_t + \dots, \quad (33)$$

$$\tilde{n}_{t+1} = \left(1 - \frac{1}{t-1}\right) \tilde{n}_t + \dots, \quad (34)$$

where the dots indicate terms of higher than first order. This is a non-autonomous dynamical system in the state variables $(\tilde{\beta}_{1,t}, \tilde{\beta}_{1,t-1}, \tilde{\beta}_{2,t}, \tilde{\beta}_{2,t-1}, \tilde{g}_t, \tilde{n}_t, \tilde{n}_{t-1})$. Note that the right hand sides of these equations are roughly of the form of the Evans–Honkapohja iterative scheme, but the gains parameter $1 - \vartheta^{-2}$ of equation (31) is constant, and both right hand sides depend not only on $\tilde{\beta}_{1,t}$ and $\tilde{\beta}_{2,t}$, but also on earlier time lags of these variables. These two points prevent us from applying Theorem 8 directly. However, by modifying the ideas behind Theorem 8, we obtain a result that can be applied to the evolutionary system given above. This result will

be derived in the remainder of this appendix. Consider, to that end, a system of the form

$$\xi_t = \xi_{t-1} + \alpha_t (f(\xi_{t-1}, \dots, \xi_{t-k}, x_t, \alpha_t) - \xi_{t-1}), \quad (35)$$

$$x_t = g(\xi_{t-1}, \dots, \xi_{t-k}, x_{t-1}), \quad (36)$$

where $\xi_t \in \mathbb{R}^n$, $x_t \in \mathbb{R}^m$, and where α_t satisfies the conditions given at the beginning of this section. It is assumed that $f(0, \dots, 0, 0, 0) = 0$ and that $g(0, \dots, 0, 0) = 0$; moreover the functions f and g are assumed to be continuously differentiable. Note that the system (31)–(34) is indeed of the form (35) and (36). To see this observe that we can write (34) as

$$\tilde{n}_{t+1} = \tilde{n}_t + \frac{1}{t} \left(\left(1 - \left(1 - \frac{1}{t} \right)^{-1} \right) \tilde{n}_t - \tilde{n}_t \right),$$

which is of the required form. For this system we then have $\xi_t = (\tilde{\beta}_{2,t}, \tilde{n}_t) \in \mathbb{R}^2$, $x_t = (\tilde{\beta}_{1,t}, \tilde{\beta}_{1,t-1}, \tilde{g}_t) \in \mathbb{R}^3$ and $\alpha_t = \frac{1}{t}$.

Recall “small-o” and “big-O” notation: $\phi(x) = o(\psi(x))$ and $\phi(x) = O(\psi(x))$ as $x \rightarrow 0$ indicate respectively that $|\phi(x)| / |\psi(x)| \rightarrow 0$ and $|\phi(x)| < C |\psi(x)|$ as $x \rightarrow 0$; we shall write $o(x, y)$ short for $o((x, y))$. We set $\xi = (\xi_1, \dots, \xi_k)$ and introduce matrices A_j, C_j, B, D by

$$f(\xi, x, \alpha) = A_1 \xi_1 + \dots + A_k \xi_k + Bx + o(\xi, x) + O(\alpha) O(\xi, x), \quad (37)$$

$$g(\xi, x) = C_1 \xi_1 + \dots + C_k \xi_k + Dx + o(\xi, x); \quad (38)$$

we introduce moreover $A = \sum_j A_j$ and $C = \sum_j C_j$.

Recall that by a linear coordinate transformation, any matrix A can be put into Jordan normal form; that is, there is a block diagonal matrix $J = \text{diag}(J_1, \dots, J_L)$ and an invertible matrix U such that $A = U^{-1}JU$. The coordinate transformation can be chosen in such a way that for a given $\delta > 0$, the block matrices J_l are all of the form $J_l = \lambda_l I + \delta N$, with I the identity matrix and with $N = (n_{ij})$ satisfying $n_{ij} = 1$ if $j = i + 1$ and 0 otherwise. It can be shown that the choices involving the transformations U can be fully specified; then the U are functions of the matrix A . If $\lambda_* = \max_l |\lambda_l| + \delta$, then $|Jy| \leq \lambda_* |y|$, where $|\cdot|$ denotes the Euclidean norm.

Recall also that to any vector norm $\|\cdot\|$ there is associated a matrix norm, also denoted $\|\cdot\|$, by setting $\|A\| = \max_{\|x\|=1} \|Ax\|$. More generally, if A is a linear map between spaces with different norms, say $A : (\mathbb{R}^m, \|\cdot\|_1) \rightarrow (\mathbb{R}^n, \|\cdot\|_2)$, then the matrix norm $\|A\|_{1,2} = \max_{\|x\|_1=1} \|Ax\|_2$ is induced. Define the norm $\|x\|_A \equiv |Ux|$ if $A = U^{-1}JU$. The virtue of the norm $\|x\|_A$ is the inequality

$$\|Ax\|_A = |UU^{-1}JUx| = |JUx| \leq \lambda_* |Ux| = \lambda_* \|x\|_A.$$

Proposition 9 Assume that all eigenvalues of A have real part strictly smaller than $a < 1$ and all eigenvalues of D are strictly smaller than $d < 1$ in absolute value; moreover, assume that

$$(1 - a)(1 - d) - \|B\|_{D,A} \|C\|_{A,D} > 0.$$

Then the equilibrium $(\xi, x) = (0, 0)$ of (35) and (36) is locally asymptotically stable.

Theorem 8 is a special case of this result, which is obtained by adding an equation of the form $x_t = 0$ to equation (30); this yields $D = 0$, $C = 0$. Note moreover that if the theorem is applied to the system given by equations (31)–(34) then the matrices A and C in equations (37) and (38) satisfy $A = 0$, $C = 0$. The condition of Proposition 9 is then satisfied if the eigenvalues of the matrix D in (38) are less than 1 in absolute value, as has been verified in the proof of Proposition 7.

Proof Let $\bar{\alpha}_t = \sup_{s \geq t} \{\alpha_s\}$. Note that $\{\bar{\alpha}_t\}$ is a nonincreasing sequence and $\bar{\alpha}_t \rightarrow 0$ as $t \rightarrow \infty$. Using (35) and substituting ξ_{t-1} for ξ_{t-2} up to ξ_{t-k} we obtain

$$f(\xi_{t-1}, \dots, \xi_{t-k}, x_{t-1}, \alpha_t) = A\xi_{t-1} + Bx_{t-1} + o(\xi_{t-1}, \dots, \xi_{t-k}, x_{t-1}) \\ + O(\bar{\alpha}_{t-k+1}) O(\xi_{t-1}, \dots, \xi_{t-k}, x_{t-1}) \quad (39)$$

$$g(\xi_{t-1}, \dots, \xi_{t-k}, x_{t-1}) = C\xi_{t-1} + Dx_{t-1} \\ + o(\xi_{t-1}, \dots, \xi_{t-k}, x_{t-1}). \quad (40)$$

We claim that the local stability of the system (35) and (36) is governed by the terms f and g that are linear in ξ and x .

The proof uses induction on t . In order to formulate the induction hypothesis, vector norms and matrix norms have to be specified. Choose therefore a^* and d^* such that $a < a^* < 1$, $0 < d < d^* < 1$, and $(1 - a^*)(1 - d^*) - \|B\|_{D,A} \|C\|_{A,D} > 0$; this is possible by hypothesis. Let the matrix norms $\|\cdot\|_A$ and $\|\cdot\|_D$ be such that

$$\|I + \alpha(A - I)\|_A \leq 1 + \alpha(a - 1), \quad \|D\|_D \leq d,$$

for all $\alpha > 0$ sufficiently small. Also, choose $b^*, c^* > 0$ such that $\|B\|_{D,A} < b^*$, $\|C\|_{A,D} < c^*$ and

$$(1 - a^*)(1 - d^*) - b^*c^* > 0. \quad (41)$$

Note that the conditions $a^* + \gamma^{-1}b^* < 1$ and $\gamma c^* + d^* < 1$ can only be satisfied both if $\gamma \in (\frac{b^*}{1-a^*}, \frac{1-d^*}{c^*})$. Equation (41) ensures that the open interval on the right hand side is not empty. Choose γ to be the middle of this interval. Introduce $\delta = \max\{a^* + \gamma^{-1}b^*, \gamma c^* + d^*\}$ and note that $0 < \delta < 1$ by construction.

Given $\eta_0 > 0$ define the sequence $\{\eta_t\}$ as follows. Since $\alpha_t \rightarrow 0$, there exists a $t_0 > 0$ such that $\alpha_t < (a^* - a)/2$ if $t \geq t_0$. Set

$$\eta_t = \begin{cases} c\eta_{t-1} & \text{if } t \leq t_0, \\ (1 - \delta\alpha_t)\eta_{t-1} & \text{if } t > t_0. \end{cases}$$

As induction hypothesis, it is assumed that

$$\|\xi_s\|_A, \gamma\|x_s\|_D \leq \eta_s, \quad 0 \leq s < t.$$

Choose ε such that $\varepsilon < (a^* - a)/2$ and $\varepsilon < d^* - d$. From the induction hypothesis and equations (39) and (40) we obtain that there is $\eta_0 > 0$ such that for every value of t

$$\|\xi_t\|_A \leq (1 + \|A\|_A + \gamma\|B\|_{D,A} + \varepsilon + \kappa\bar{\alpha}_{t-k+1})\eta_{t-1} \\ \gamma\|x_t\|_D \leq (\gamma c^* + d + \varepsilon)\eta_{t-1},$$

where κ is a positive constant. Taking $c = 2 + \kappa + \|A\|_A + \gamma \|B\|_{D,A}$, it follows that the induction hypothesis is satisfied for $t \leq t_0$. For $t > t_0$, the following stronger inequalities hold true

$$\begin{aligned}\|\xi_t\|_A &\leq \eta_{t-1} - \alpha_t (1 - a - \gamma^{-1}b^* - \varepsilon - \kappa\bar{\alpha}_{t-k+1}) \eta_{t-1}, \\ &\leq \eta_{t-1} - \alpha_t (1 - a^* - \gamma^{-1}b^*) \eta_{t-1}, \\ \gamma \|x_t\|_D &\leq \eta_{t-1} - \alpha_t (1 - \gamma c^* - d^*) \eta_{t-1}.\end{aligned}$$

We obtain

$$\|\xi_t\|_A, \gamma \|x_t\|_D \leq (1 - \delta\alpha_t) \eta_{t-1}.$$

It is clear that the induction hypothesis is satisfied for all remaining values of t .

Consequently, for all $t > t_0$

$$\eta_t = c^{t_0} \eta_0 \prod_{s=t_0+1}^t (1 - \delta\alpha_s) < c^{t_0} \eta_0 e^{-\delta \sum_{s=t_0+1}^t \alpha_s};$$

the last expression tends to 0 as $t \rightarrow \infty$ since $\sum_{t=0}^{\infty} \alpha_t = \infty$. $\square$

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