Convex hedging of non-superreplicable claims in discrete-time market models

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Abstract All of the papers written so far deal with efficient hedging of contingent claims for which superhedging exists. The goal of this paper is to investigate the convex hedging of contingent claims for which superhedging does not exist. Without superhedging assumption it is still possible to prove the existence of a solution, but one cannot obtain structure of the solution using techniques known so far. Therefore, we develop a new approximative approach to deduce structure of the solution in case of non-superreplicable claims.

Keywords Discrete-time market model · Incomplete market · Contingent claim · Hedging · Efficient hedging · Convex measure of risk

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1 Introduction

We consider a hedging problem of nonnegative European contingent claims in discrete time arbitrage-free financial market models. In complete markets every contingent claim is attainable, i.e. it can be replicated by a self-financing trading strategy and its price is uniquely determined. In an incomplete market not every contingent claim is attainable. Therefore, the writer of an option may be faced with a problem of

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searching strategies, which minimize risk of a shortfall resulting from the difference between hedging and liability. For some contingent claims the seller may use the superhedging strategy. Superhedging can be quite expensive (e.g. [Rüschendorf 2002](#)), which motivates a search for the best hedge which the seller of a claim can achieve with the initial endowment strictly smaller than the superreplication cost. This always leads to a shortfall, the risk of which should be minimized. Problems of this type are referred to in the literature as the efficient hedging problems. [Föllmer and Leukert \(1999\)](#) (resp. [Föllmer and Leukert 2000](#)) consider a problem of finding a strategy which minimizes the probability of a loss (resp. expectation of a loss function). Efficient hedging with respect to coherent risk measures is investigated by [Nakano \(2003, 2004\)](#) and [Rudloff \(2009\)](#). [Rudloff \(2007\)](#) solves the *convex hedging* problem, i.e. the efficient hedging problem in which risk is quantified using convex lower semi-continuous measures of risk. All of the papers mentioned above deal with the efficient hedging of nonnegative contingent claims for which superhedging exists, which is equivalent to assumption that supremum of expectations of the claim with respect to all martingale measures is finite. We show that this assumption may be violated even for a standard plain vanilla call with payoff based on a basket of non-traded securities in a one-period model. We also give an example of a non-superreplicable instrument providing protection against insurance risk. These examples motivate the investigation of convex hedging problem of contingent claims for which superhedging does not exist.

The dynamic optimization problem of finding an admissible trading strategy which minimizes shortfall risk is solved in two steps. First an associated static problem is solved. Then the optimal strategy is obtained from the optional decomposition of a modified claim $\tilde{\phi}H$ where H is the original contingent claim and $\tilde{\phi}$ is a randomized test solving the static problem. Without the superhedging assumption one cannot obtain the structure of the solution of the static problem using techniques developed so far, since this assumption plays an essential role in the proof of the Theorem 4.8 in [Rudloff \(2007\)](#). Therefore, we develop a new approach by approximation. We define a sequence of 'approximating static problems' which can be solved using Fenchel duality methods presented in [Rudloff \(2007\)](#) and then construct a solution of the original static problem using the sequence of the solutions of the approximating problems.

This paper is organized as follows: in Sect. 2 we define a market model, recall the definition and representation result for convex risk measures, formulate the convex hedging problem and sketch the idea of the solution presented in [Rudloff \(2007\)](#), explaining the role of the superhedging assumption. In Sect. 3 we provide examples which motivate consideration of the convex hedging problem of contingent claims for which superhedging does not exist. This problem is solved in Sect. 4, where a new approximative technique is presented.

2 Formulation of the problem

Let $\mathbb{N}$ denote the set of positive integers. Fix $T \in \mathbb{N}$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\mathbb{F} = (\mathcal{F})_{t=0}^T$, where we assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. Let $L^0 = L^0(\Omega, \mathcal{F}, \mathbb{P})$ denote the space of all real valued random variables endowed

with the topology of convergence in probability. We also write L^1 (resp. L^∞) for $L^1(\Omega, \mathcal{F}, \mathbb{P})$ (resp. $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$). We endow L^1 and L^∞ with norm topologies. For $p \in [0, 1, \infty]$ let L_+^p denote the subset of nonnegative elements of L^p . Equations and inequalities between random variables are always understood as $\mathbb{P}$ -a.s. The random variable equal to 1 is denoted by $\mathbf{1}$. Let $\mathcal{P}$ denote the set of all probability measures absolutely continuous with respect to $\mathbb{P}$. For any $\mathbb{Q} \in \mathcal{P}$ let $Z_{\mathbb{Q}} := \frac{d\mathbb{Q}}{d\mathbb{P}}$ be a Radon–Nikodym density of $\mathbb{Q}$ with respect to $\mathbb{P}$ and let $E_{\mathbb{Q}}Y$ (resp. EY) denote the expectation of a random variable Y with respect to $\mathbb{Q}$ (resp. $\mathbb{P}$) (provided it exists in a generalized sense (e.g. Shiryaev 1996 II, §6, def. 2), i.e. if $\min(EY^+, EY^-) < \infty$, then $EY := EY^+ - EY^-$). Let $\mathcal{P}_b = \{\mathbb{Q} \in \mathcal{P} : Z_{\mathbb{Q}} \leq n \text{ for some } n \in \mathbb{N}\}$.

Fix $d \in \mathbb{N}$ and let Φ^d be a set of pairs (x, π) , where $x \in \mathbb{R}$ and $\pi = (\pi_t)_{t=1}^T$ is a d -dimensional $\mathbb{F}$ -predictable process. For any d -dimensional adapted process $X = (X_t)_{t=1}^T$ we define a market model $\mathcal{M} = (X, \Phi^d)$. X describes the evolution of the price processes of risky assets traded in the market and Φ^d is a set of trading strategies. Following Stettner (2000), the interpretation of $(x, \pi) \in \Phi^d$ is as follows. Starting with an initial capital x we invest in risky assets at each time t , leaving noninvested capital in a bank account that is assumed for simplicity to have a rate of return $r = 0$. Let π_t^i be the number of units of i -th asset we have in our portfolio after possible transactions at time $t - 1$, which is a real random variable (we allow short selling, so that π_t^i may be negative) adapted to the σ -field $\mathcal{F}_{t-1}$. Let $V^{x, \pi}$ be the value process of a trading strategy (x, π) and let $(\pi \cdot X)_t = \sum_{s=1}^t \pi_s \cdot \Delta X_s$ denote discrete time stochastic integral, where $a \cdot b$ denotes inner product of vectors $a, b \in \mathbb{R}^d$ and $\Delta X_s = X_s - X_{s-1}$. Then $V_0^{x, \pi} = x$ and $V_t^{x, \pi} = x + (\pi \cdot X)_t$ for any $(x, \pi) \in \Phi^d$ and $t = 1, \dots, T$. Obviously the value process of a strategy depends on X but we shall omit this to simplify notation. Strategy $(x, \pi) \in \Phi^d$ is an arbitrage opportunity if $x = 0$, $V_T^{x, \pi} \geq 0$ and $\mathbb{P}(V_T^{x, \pi} > 0) > 0$. Market model $\mathcal{M}$ is arbitrage-free if no arbitrage opportunity exists. Let $\mathcal{P}(\mathcal{M})$ denote the set of all martingale measures for a model $\mathcal{M}$. We shall use version of the fundamental theorem of asset pricing which follows from Tehranchi (2010), Theorem 1.2.

Theorem 2.1 *The following conditions are equivalent:*

- (1) $\mathcal{M}$ is arbitrage-free.
- (2) $\mathcal{P}(\mathcal{M}) \neq \emptyset$.
- (3) For any random variable Y there exists a martingale measure $\mathbb{Q}$ such that $\max\{1, |Y|\}Z_{\mathbb{Q}}$ is bounded.

Contingent claims considered in this paper are elements of L_+^1 .

From now on, we assume that, we have a fixed adapted d -dimensional process S , such that, the following condition holds:

$$(NA) \quad \text{Market model } \mathcal{M} = (S, \Phi^d) \text{ is arbitrage - free.}$$

For an arbitrary $\tilde{V}_0 > 0$ let $\Phi_{\tilde{V}_0}^d = \{(x, \pi) \in \Phi^d : V^{x, \pi} \geq 0, x \leq \tilde{V}_0\}$ be a set of all admissible trading strategies, i.e. trading strategies with nonnegative value process and initial endowment not exceeding $\tilde{V}_0$. For convenience we recall a definition and some properties of convex risk measures.

Definition 2.2 A function $\rho : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$ with $\rho(0) = 0$ is a convex risk measure if it satisfies for all $X_1, X_2 \in L^1$:

- (a) monotonicity: $X_1 \geq X_2 \Rightarrow \rho(X_1) \leq \rho(X_2)$,
- (b) translation property: $\rho(X_1 + c\mathbf{1}) = \rho(X_1) - c$ for any $c \in \mathbb{R}$,
- (c) convexity: $\rho(\lambda X_1 + (1 - \lambda)X_2) \leq \lambda\rho(X_1) + (1 - \lambda)\rho(X_2)$ for any $\lambda \in [0, 1]$.

Rudloff (2007) argues that assumption $\rho(0) = 0$ is reasonable, since under this assumption $\rho(X)$ can be interpreted as risk-adjusted capital requirement. A set $\mathcal{A}_\rho = \{X \in L^1 : \rho(X) \leq 0\}$ is called the acceptance set of the risk measure ρ . We shall use the following representation result for convex risk measures which follows from Kaina and Rüschendorf (2009), Theorem 2.4.

Theorem 2.3 A function $\rho : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$ such that $\rho(0) = 0$ is a convex, lower semi-continuous (abbrev. l.s.c.) risk measure if and only if

$$\rho(X) = \sup_{Q \in \mathcal{P}_b} [E_Q(-X) - \sup_{Y \in \mathcal{A}_\rho} E_Q(-Y)].$$

For a given convex l.s.c. measure of risk $\rho : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$, a contingent claim H and constant $\tilde{V}_0 > 0$ a convex hedging problem is the dynamic optimization problem of finding an admissible trading strategy solving

$$\min_{(x, \pi) \in \Phi_{\tilde{V}_0}^d} \rho(-(V_T^{x, \pi} - H)^-). \quad (2.1)$$

Random variable $-(V_T^{x, \pi} - H)^-$ represents shortfall resulting from hedging of the contingent claim H with an admissible trading strategy (x, π) .

We introduce some notation and then give an overview of the standard approach to solving problem (2.1). Let $\mathcal{R} = \{\phi : \Omega \rightarrow [0, 1] : \phi - \mathcal{F}_T - \text{measurable}\}$ be a set of randomized tests. For any $\mathcal{Q} \subseteq \mathcal{P}$ we define a constrained set $\mathcal{R}_\mathcal{Q} := \{\phi \in \mathcal{R} : \sup_{Q \in \mathcal{Q}} E_Q \phi H \leq \tilde{V}_0\}$. Obviously $\mathcal{R}_\mathcal{Q}$ depends on H and $\tilde{V}_0$ which are fixed throughout the remaining part of the paper so we suppress this dependence for simplicity of notation. The standard approach to solving the convex hedging problem proceeds as follows:

1. One shows existence of $\tilde{\phi}$ solving the static problem

$$\inf_{\phi \in \mathcal{R}_\mathcal{Q}} \rho((\phi - \mathbf{1})H), \quad (2.2)$$

with $\mathcal{Q} = \mathcal{P}(\mathcal{M})$ (Rudloff 2007, Theorem 4.3).

2. Structure of $\tilde{\phi}$ is derived using Fenchel duality methods (Theorem 4.5, Lemma 4.6 and Theorem 4.8 in Rudloff (2007)).
3. Optimal strategy in (2.1) is obtained from Theorem 2.4, which can be viewed as a discrete-time version of Theorem 3.1 in Rudloff (2007).

Theorem 2.4 Let $\tilde{\phi}$ be a solution of the optimization problem (2.2) with $\mathcal{Q} = \mathcal{P}(\mathcal{M})$ and let $(\tilde{V}_0, \tilde{\pi}) \in \Phi_{\tilde{V}_0}^d$ be a superhedging strategy of a modified claim $\tilde{\phi}H$. Then $(\tilde{V}_0, \tilde{\pi})$ solves convex the hedging problem (2.1).

The methods used in the steps 1 and 2 rely on the following assumptions.

(R1) $\rho : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$ is a convex, l.s.c. risk measure that is continuous and finite in some $H(\phi_0 - \mathbf{1})$ with $\phi_0 \in \mathcal{R}_{\mathcal{P}(\mathcal{M})}$ (Rudloff 2007, Assumption 4.1).

(R2) $\sup_{\mathbb{P}^* \in \mathcal{P}(\mathcal{M})} E_{\mathbb{P}^*} H < \infty$. [Rudloff 2007, Eq. (1)]

Remark 2.5 Analysis of the proofs of Theorems 4.3, 4.5, 4.8 and Lemma 4.6 in Rudloff (2007) leads to the conclusion that assertions of these theorems hold for the problem (2.2) with an arbitrary set $\mathcal{Q} \subseteq \mathcal{P}$, whenever the following conditions are satisfied:

(A1) $\rho : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$ is a convex, l.s.c. risk measure, that is continuous and finite in $H(\phi_0 - \mathbf{1})$ with some $\phi_0 \in \mathcal{R}_{\mathcal{Q}}$,

(A2) $\sup_{\mathbb{Q} \in \mathcal{Q}} E_{\mathbb{Q}} H < \infty$.

Indeed, Rudloff uses the fact that $\mathcal{Q} = \mathcal{P}(\mathcal{M})$ only in connection with the assumption **(R2)**, which:

- (a) implies that $H Z_{\mathbb{P}^*} \in L^1$ for every $\mathbb{P}^* \in \mathcal{P}(\mathcal{M})$, which is used to prove Theorem 4.3 Rudloff (2007).
- (b) guarantees continuity of the operator B defined in the proof of Theorem 4.8 (Rudloff 2007, p. 447)
- (c) allows application of Tonelli's theorem (Dunford and Schwartz 1988, Corollary III.11.15) to integral

$$\int_{\mathcal{P}(\mathcal{M})} \int_{\Omega} (H Z_{\mathbb{P}^*} \phi) d\mathbb{P} \lambda(d\mathbb{P}^*)$$

also in the proof of Theorem 4.8 (Rudloff 2007, p. 448).

One still obtains (a), (b) and (c) with $\mathcal{P}(\mathcal{M})$ replaced by an arbitrary set $\mathcal{Q} \subseteq \mathcal{P}$, for which **(A2)** is satisfied. Obviously **(A1)** for $\mathcal{Q}$ plays a role of **(R1)** for $\mathcal{P}(\mathcal{M})$.

We use these observations in Sect. 4.

Remark 2.6 Rudloff (2007) considers continuous time market satisfying ‘no free lunch with vanishing risk’ (NFLVR) condition, which is a mild strengthening of a concept of ‘no arbitrage’, that has to be used in general semimartingale models. Delbaen and Schachermayer (2006) proved that (NFLVR) $\Leftrightarrow \mathcal{P}_{\sigma} \neq \emptyset$ where $\mathcal{P}_{\sigma}$ is a set of probability measures $\mathbb{P}^*$ equivalent to $\mathbb{P}$ such that S is a $\mathbb{P}^*$ -sigma martingale. In a discrete time case (NFLVR) (resp. $\mathcal{P}_{\sigma}$) is replaced by (NA) (resp. $\mathcal{P}(\mathcal{M})$). Following Rudloff's approach, one easily proves Theorem 2.4 using discrete time versions of the superhedging characterization (Stettner 2000, Theorem 2.1) and the optional decomposition theorem (Föllmer and Kabanov 1998, Theorem 2).

The goal of this paper is to solve the convex hedging problem without assumption **(R2)**.

3 Motivating examples

Throughout this section we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space which supports an infinite sequence of independent standard Gaussian random variables

$X, X_1, X_2, \dots$ as well as an independent random variable U with uniform distribution on $[0, 1]$. Let $\mathcal{T} = \{0, 1\}$ and define risky asset price process by $S_0 = s_0 > 0$, $S_1 = s_0 \exp(X - \frac{1}{2})$. Let $\mathbb{F}$ be a filtration such that $\mathcal{F}_0$ is trivial, $\mathcal{F}_1 = \mathcal{F}$ and let Φ^1 denote the set of trading strategies. The market model $\mathcal{M} = (S, \Phi^1)$ is arbitrage-free, since $\mathbb{P} \in \mathcal{P}(\mathcal{M})$.

We begin with two examples of non-superreplicable contingent claims.

Example 3.1 For $n = 1, 2, \dots$ let R^n denote the value process of a non-traded security $R^n = (R_t^n)_{t \in \mathcal{T}}$ correlated with S , given by

$$R_0^n = 1, \quad R_1^n = \exp \left[\rho_n X + \sqrt{1 - \rho_n^2} X_n - \frac{1}{2} \left(\rho_n^2 + n^2 (1 - \rho_n^2) \right) \right],$$

with $\rho_n \in (-1, 1)$.

Assume correlation decays to 0 with n , i.e. $\lim_{n \rightarrow \infty} \rho_n = 0$.

Now consider a hedging problem of a call option with the payoff

$$H = \left(\sum_{n=1}^{\infty} \frac{1}{2^n} R_1^n - K \right)^+, \quad (3.1)$$

for some $K > 0$.

H is well-defined and integrable since $E(\sum_{n=1}^{\infty} \frac{1}{2^n} R_1^n) = 1$. We show that superhedging of H is impossible for any initial endowment $x \in \mathbb{R}$.

For $n \in \mathbb{N}$ let $\mathbb{Q}_n$ be a measure with density

$$\frac{d\mathbb{Q}_n}{d\mathbb{P}} := \sqrt{2\pi} \cdot (2^n)^2 \cdot \exp\left(\frac{X_n^2}{2}\right) \cdot \mathbf{1}_{\{|X_n - \sqrt{1 - \rho_n^2} n| \leq 2^{-2n-1}\}}.$$

One easily verifies that $\mathbb{Q}_n(\Omega) = 1$. Therefore, independence of X and X_n implies, that $\mathbb{Q}_n$ is a martingale measure for every $n \in \mathbb{N}$. Now we show that $\sup_{n \in \mathbb{N}} E_{\mathbb{Q}_n} H = \infty$. Denote $\alpha_n := \sqrt{1 - \rho_n^2} n$, $n \in \mathbb{N}$.

$$E_{\mathbb{Q}_n} H \geq -K + \sum_{k=1}^{\infty} \frac{1}{2^k} E_{\mathbb{Q}_n} R_1^k \geq -K + \frac{1}{2^n} \underbrace{E \left[\frac{d\mathbb{Q}_n}{d\mathbb{P}} \exp \left(\alpha_n X_n - \frac{1}{2} \alpha_n^2 \right) \right]}_{=: I_n} \quad (3.2)$$

Straightforward calculation yields,

$$\begin{aligned} I_n &= (2^n)^2 \int_{\alpha_n - 2^{-2n-1}}^{\alpha_n + 2^{-2n-1}} \left[\exp\left(\frac{y^2}{2}\right) \exp\left(\alpha_n y - \frac{1}{2} \alpha_n^2\right) \exp\left(\frac{-y^2}{2}\right) \right] dy \\ &= (2^n)^2 \exp\left(\frac{\alpha_n^2}{2}\right) \frac{1}{\alpha_n} \left[\exp\left(\frac{\alpha_n}{2^{2n+1}}\right) - \exp\left(\frac{-\alpha_n}{2^{2n+1}}\right) \right] = \exp\left(\frac{\alpha_n^2}{2}\right) \left[1 + \frac{o\left(\frac{\alpha_n}{2^{2n+1}}\right)}{\frac{\alpha_n}{2^{2n+1}}} \right], \end{aligned}$$

where the last equation follows from the Taylor's formula. This and (3.2) imply that

$$E_{\mathbb{Q}_n} H \geq -K + \frac{1}{2^n} I_n = -K + \exp\left(\frac{(1 - \rho_n^2) n^2}{2} - n \ln 2\right) \left[1 + \frac{o\left(\frac{\alpha_n}{2^{2n+1}}\right)}{\frac{\alpha_n}{2^{2n+1}}}\right].$$

Thus, $\sup_{n \in \mathbb{N}} E_{\mathbb{Q}_n} H = \infty$, since we have assumed that ρ_n converges to 0. Hence $\sup_{\mathbb{P}^* \in \mathcal{P}(\mathcal{M})} E_{\mathbb{P}^*} H = \infty$ and it follows from Theorem 2.1 in Stettner (2000) that the superhedging strategy of H does not exist for any initial endowment $x \in \mathbb{R}$.

Example 3.2 Consider an insurance company holding one unit of stocks S and exposed also to insurance risk modeled by a random variable $-U^{-\frac{4}{5}}$, where U is random variable independent of the stock with uniform distribution on $[0, 1]$. Thus, the position of the company at time 1 is given by $Y = -U^{-\frac{4}{5}} + S_1$. Insurance company transfers some risk resulting from position Y to the reinsurer, who agrees to cover losses resulting from Y above the level $K > 0$ for a price $\tilde{V}_0$ paid at time 0. More precisely, the insurance company paid $\tilde{V}_0$ for the European contract $H := (Y + K)^-$ with expiry $T = 1$. We observe that

$$\begin{aligned} H &= (Y + K)^- = (-U^{-\frac{4}{5}} + S_1 + K)^- = (U^{-\frac{4}{5}} - S_1 - K)^+ \\ &= (U^{-\frac{4}{5}} + Z - K)^+, \end{aligned} \quad (3.3)$$

where $Z = -S_1$. Formula (3.3) defines contract known in insurance under the name *financial stop loss* (Møller 2002, section 4.2.3).

Now consider the situation of the reinsurer, whose goal is to hedge the contingent claim H using the premium $\tilde{V}_0$.

We show that superhedging of H is impossible for any initial endowment $x \in \mathbb{R}$. For $n \in \mathbb{N}$ let $\mathbb{Q}_n$ be a measure with density

$$\frac{d\mathbb{Q}_n}{d\mathbb{P}} := c_n \left(U^{-\frac{1}{5}} \mathbf{1}_{\{U \geq \frac{1}{n+1}\}} + \mathbf{1}_{\{U < \frac{1}{n+1}\}} \right),$$

where c_n is a normalizing constant (observe that density of $\mathbb{Q}_n$ is independent of S_1). Obviously $\frac{d\mathbb{Q}_n}{d\mathbb{P}} \in L^\infty$ and $\mathbb{Q}_n \in \mathcal{P}(\mathcal{M})$ for $n \in \mathbb{N}$. Moreover,

$$\begin{aligned} E_{\mathbb{Q}_n} H &\geq -K + E_{\mathbb{Q}_n} Z + E_{\mathbb{Q}_n} U^{-\frac{4}{5}} = -K - s_0 + c_n \int_{[\frac{1}{n+1}, 1]} \frac{1}{y} dy + 5 \cdot \frac{1}{(n+1)^{\frac{1}{5}}} \\ &\geq -K - s_0 + c_n \ln(n+1). \end{aligned} \quad (3.4)$$

Straightforward calculation yields, $c_n = \frac{4}{5} \cdot \frac{1}{1 - (\frac{1}{n+1})^{\frac{4}{5}}} - \frac{1}{n+1} \geq \frac{3}{5}$ for sufficiently large n . This and (3.4) imply that $\sup_{n \in \mathbb{N}} E_{\mathbb{Q}_n} H = \infty$. Hence $\sup_{\mathbb{P}^* \in \mathcal{P}(\mathcal{M})} E_{\mathbb{P}^*} H = \infty$ and non-superreplicability of H follows. We see that the presence of the insurance risk $-U^{-\frac{4}{5}}$ makes superhedging impossible. One may think of $-U^{-\frac{4}{5}}$ as a risky position

resulting from selling protection against a catastrophic event, e.g. a nuclear meltdown. From the economic point of view it is reasonable to assume that in case of such event, losses may be impossible to cover.

Let H be any of the claims given in the examples above. Since superhedging of H is not possible, we consider efficient hedging with a convex risk measure called *shortfall risk* (e.g. [Kaina and Rüschendorf 2009](#), section 4). We recall definition of this risk measure. Let $l : \mathbb{R} \rightarrow \mathbb{R}$ be a convex, strictly increasing function such that $E[l(-X)] < \infty$ for every $X \in L^1$. Denote

$$\mathcal{A} := \{X \in L^1 \mid E[l(-X)] \leq l(0)\}.$$

Shortfall risk $SR_1(X)$ is defined by $SR_1(X) := \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}\}$ for $X \in L^1$.

It is known ([Kaina and Rüschendorf 2009](#), Proposition 4.2) that SR_1 is a finite, L^1 -continuous function which is convex, monotone and has translation property. To show that it is a convex measure of risk in a sense of Definition 2.2, we verify the condition $SR_1(0) = 0$. Observe that $SR_1(0) \leq 0$, since $0 \in \mathcal{A}$. Now we show the reverse inequality. Assume on the contrary that $SR_1(0) < 0$, which implies existence of $\varepsilon > 0$ for which $E[l(-(0-\varepsilon))] \leq l(0)$. It follows that $l(\varepsilon) = E[l(-(-\varepsilon))] \leq l(0)$, which contradicts assumption that l is strictly increasing.

Convex hedging problem of contingent claim H with respect to SR_1 with initial capital $\tilde{V}_0$, i.e.

$$\inf_{(x,\pi) \in \Phi_{\tilde{V}_0}^1} SR_1(-(V_1^{x,\pi} - H)^-) \quad (3.5)$$

cannot be solved using the results known so far, especially [Rudloff \(2007\)](#), since the assumption **(R2)** is not satisfied.

4 Convex hedging of non-superreplicable claims

In this section we present a new approach to solving the efficient hedging problem for non-superreplicable claims. The following assumption is used in the proofs of the main results.

Assumption 4.1 ([Rudloff 2007](#), Assumption 4.1) $\rho : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$ is a convex, l.s.c. risk measure that is continuous and finite in some $H(\phi_0 - \mathbf{1})$ with $\phi_0 \in \mathcal{R}_{\mathcal{P}(\mathcal{M})}$.

In other words we assume **(R1)** and do not impose **(R2)**. Without **(R2)** it is still possible to show existence of a randomized test $\tilde{\phi}$ solving the static problem and then to apply Theorem 2.4 to derive the solution of the dynamic problem (2.1). But structure of $\tilde{\phi}$ cannot be obtained from Theorem 4.8 in [Rudloff \(2007\)](#) which relies on **(R2)**. Therefore, we develop a new approximative approach.

Definition 4.2 We say that $(\gamma_n)_{n \in \mathbb{N}}$ is a *DS-sequence*¹ if for every $n \in \mathbb{N}$:

¹ ‘DS’ for ‘Delbaen and Schachermayer’ who used such sequences in a very useful Lemma A1.1 in [Delbaen and Schachermayer \(1994\)](#)

1. $\gamma_n = (\gamma_{n,n}, \gamma_{n,n+1}, \dots)$.
2. Vector γ_n has nonnegative coordinates with finitely many not equal to 0.
3. $\sum_{k \geq n} \gamma_{n,k} = 1$.

Let $\mathcal{P}_b(\mathcal{M})$ denote a set of martingale measures with bounded densities with respect to $\mathbb{P}$.

The approximative approach which leads to the solution of the static problem proceeds in the following steps.

- (i) Static problem (2.2) with $\mathcal{Q} = \mathcal{P}(\mathcal{M})$ is rewritten as

$$\inf_{\phi \in \mathcal{R}_{\mathcal{P}_b(\mathcal{M})}} \rho((\phi - \mathbf{1})H). \quad (4.1)$$

- (ii) For $r \in \mathbb{N}$ one defines $(\mathcal{P}_b(\mathcal{M}))^r := \{\mathbb{P}^* \in \mathcal{P}_b(\mathcal{M}) : Z_{\mathbb{P}^*} \leq r\}$ and considers the approximating static problems

$$\inf_{\phi \in \mathcal{R}_{(\mathcal{P}_b(\mathcal{M}))^r}} \rho((\phi - \mathbf{1})H). \quad (4.2)$$

For every $r \in \mathbb{N}$ existence and structure of a randomized test ϕ_r solving (4.2) is obtained using Theorems 4.3, 4.5, 4.8 and Lemma 4.6 in Rudloff (2007).

- (iii) One proves existence of a DS-sequence $(\beta_m)_{m=1}^\infty$ for which the sequence $(\sum_{r \geq m} \beta_{m,r} \phi_r)_{m \in \mathbb{N}}$ converges $\mathbb{P}$ -a.s. and in L^1 to a random variable $\tilde{\phi}$ solving the static optimization problem (2.2).

Remark 4.3 Rudloff (2007) gives necessary and sufficient conditions a solution of (2.2) has to satisfy and separately shows that the solution indeed exists (Rudloff 2007, Theorem 4.3). Our approach is slightly different. We construct randomized test $\tilde{\phi}$ and prove it solves the static problem.

Now we provide details of the procedure described above.

4.1 Rewriting the static problem (2.2) as (4.1).

The static problem (2.2) with $\mathcal{Q} = \mathcal{P}(\mathcal{M})$ can be rewritten as (4.1) due to the following lemma.

Lemma 4.4 *In an arbitrage-free market model $\mathcal{M} = (S, \Phi^d)$ the equality*

$$\sup_{\mathbb{P}^* \in \mathcal{P}(\mathcal{M})} E_{\mathbb{P}^*} Y = \sup_{\mathbb{P}^* \in \mathcal{P}_b(\mathcal{M})} E_{\mathbb{P}^*} Y \quad (4.3)$$

holds for every $Y \in L^0_+$.

Proof Step 1. Observe that both sides of Eq. (4.3) are well defined since Y is nonnegative. Denote the left (resp. right) hand side of (4.3) by L (resp. P). Obviously $L \geq P$, since $\mathcal{P}_b(\mathcal{M}) \subseteq \mathcal{P}(\mathcal{M})$.

Step 2. We show that for any measure $\mathbb{Q} \in \mathcal{P}(\mathcal{M})$ such that $Y \in L^1(\mathbb{Q})$, there exists a measure $\tilde{\mathbb{Q}} \in \mathcal{P}_b(\mathcal{M})$ for which $E_{\tilde{\mathbb{Q}}}Y = E_{\mathbb{Q}}Y$.

Let $\mathbb{Q} \in \mathcal{P}(\mathcal{M})$ be a measure such that Y is $\mathbb{Q}$ -integrable. For $t = 0, \dots, T$ define $M_t := E_{\mathbb{Q}}(Y|\mathcal{F}_t)$ and let $M = (M_t)_{t=0}^T$. Consider market model $\mathcal{M}' = (S', \Phi^{d+1})$, where $S' = (S, M)$. Since $\mathbb{Q} \in \mathcal{P}(\mathcal{M}')$, implication (2) $\Rightarrow$ (3) of Theorem 2.1 guarantees existence of a measure $\tilde{\mathbb{Q}} \in \mathcal{P}(\mathcal{M}')$ for which $\max\{1, Y\}Z_{\tilde{\mathbb{Q}}}$ is bounded.

In particular $\tilde{\mathbb{Q}} \in \mathcal{P}_b(\mathcal{M}') \subseteq \mathcal{P}_b(\mathcal{M})$, because every martingale measure for the model $\mathcal{M}'$ is also a martingale measure for $\mathcal{M}$. Since Y is attainable in the model $\mathcal{M}'$, equality $E_{\tilde{\mathbb{Q}}}Y = E_{\mathbb{Q}}Y$ follows from Theorem 1.1 in Tehranchi (2010).

Step 3. Consider the case: $L = \infty$ and $E_{\mathbb{P}_{\infty}}Y = \infty$ for some $\mathbb{P}_{\infty} \in \mathcal{P}(\mathcal{M})$. To prove $L = P = \infty$, it suffices to show that for any $c > 0$ there exist $\mathbb{P}_c \in \mathcal{P}_b(\mathcal{M})$ such that $Y \in L^1(\mathbb{P}_c)$ and $E_{\mathbb{P}_c}Y > c$. Fix arbitrary $c > 0$ and choose smallest n for which $E_{\mathbb{P}_{\infty}}(Y \wedge n) > c$. Applying step 2 to the payoff $Y \wedge n$ and measure $\mathbb{Q} = \mathbb{P}_{\infty}$, we obtain existence of a measure $\mathbb{P}_c \in \mathcal{P}_b(\mathcal{M})$ for which $E_{\mathbb{P}_c}(Y \wedge n) = E_{\mathbb{P}_{\infty}}(Y \wedge n) > c$.

Step 4. To complete the proof consider the case $L = \lim_{n \rightarrow \infty} E_{\mathbb{P}_n^*}Y$ for some $\mathbb{P}_n^* \in \mathcal{P}(\mathcal{M})$ such that $Y \in L^1(\mathbb{P}_n^*)$, $n \in \mathbb{N}$. By step 2, for every $n \in \mathbb{N}$ there exists a measure $\mathbb{Q}_n \in \mathcal{P}_b(\mathcal{M})$ such that $Y \in L^1(\mathbb{Q}_n)$ and $E_{\mathbb{P}_n^*}Y = E_{\mathbb{Q}_n}Y$. Thus, $L = \sup_{\mathbb{P}^* \in \mathcal{P}(\mathcal{M})} E_{\mathbb{P}^*}Y = \lim_{n \rightarrow \infty} E_{\mathbb{P}_n^*}Y = \lim_{n \rightarrow \infty} E_{\mathbb{Q}_n}Y \leq \sup_{\mathbb{P}^* \in \mathcal{P}_b(\mathcal{M})} E_{\mathbb{P}^*}Y = P$. $\square$

Corollary 4.5 *The static problem (2.2) with $\mathcal{Q} = \mathcal{P}(\mathcal{M})$ rewrites as (4.1).*

From now on, we consider the static problem in a form (4.1), which is convenient for approximation.

4.2 Approximating problems and their solution

For any $r \in \mathbb{N}$ consider the approximating static problem (4.2). To prove existence and derive the structure of the solution to (4.2) we slightly generalize assertions of Theorems 4.3, 4.5, 4.8 and Lemma 4.6 in Rudloff (2007), using Remark 2.5.

Theorem 4.6 *Let $\mathcal{Q}$ be a subset of $\mathcal{P}$ such that (A2) holds for a given claim $H \in L_+^1$ and let $\rho : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$ be a function which satisfies conditions (A1). Then:*

- (i) *There exists $\tilde{\phi} \in \mathcal{R}_{\mathcal{Q}}$ solving the problem*

$$\inf_{\phi \in \mathcal{R}_{\mathcal{Q}}} \rho((\phi - I)H), \quad (4.4)$$

and $\rho(H(\tilde{\phi} - I))$ is finite.

- (ii) *There exists $\tilde{\mathbb{Q}} \in \mathcal{P}_b$ solving*

$$\sup \left\{ \inf_{\mathbb{Q} \in \mathcal{P}_b} \{E_{\mathbb{Q}}[(I - \phi)H] - \sup_{Y \in \mathcal{A}_{\rho}} E_{\mathbb{Q}}(-Y)\} \right\}. \quad (4.5)$$

Let p (resp. d) denote the value of (4.4) [resp. (4.5)]. Problems (4.4) and (4.5) are Fenchel dual and strong duality holds, i.e. $p = d$.

Moreover $(\tilde{\phi}, Z_{\tilde{\mathbb{Q}}})$ is a saddle point of the function $(\phi, Z_{\mathbb{Q}}) \rightarrow E_{\mathbb{Q}}[(I - \phi)H] - \sup_{Y \in \mathcal{A}_p} E_{\mathbb{Q}}(-Y)$. Thus,

$$\begin{aligned} & \min_{\phi \in \mathcal{R}_{\mathbb{Q}}} \left\{ \max_{\mathbb{Q} \in \mathcal{P}_b} \left\{ E_{\mathbb{Q}}[(I - \phi)H] - \sup_{Y \in \mathcal{A}_p} E_{\mathbb{Q}}(-Y) \right\} \right\} \\ &= \max_{\mathbb{Q} \in \mathcal{P}_b} \left\{ \min_{\phi \in \mathcal{R}_{\mathbb{Q}}} \left\{ E_{\mathbb{Q}}[(I - \phi)H] - \sup_{Y \in \mathcal{A}_p} E_{\mathbb{Q}}(-Y) \right\} \right\}. \end{aligned}$$

(iii) For any $\mathbb{Q} \in \mathcal{P}_b$ there exists a randomized test $\tilde{\phi}_{\mathbb{Q}}$ solving the problem

$$\sup_{\phi \in \mathcal{R}_{\mathbb{Q}}} E_{\mathbb{Q}}(\phi H). \quad (4.6)$$

and its value $p^i(\mathbb{Q})$ is finite.

(iv) For any $\mathbb{Q} \in \mathcal{P}_b$ there exists $\tilde{\lambda}_{\mathbb{Q}}$ solving

$$\inf_{\lambda \in \Lambda_+(\mathcal{Q})} \left\{ E \left[H Z_{\mathbb{Q}} - H \int_{\mathcal{Q}} Z_{\mathbb{P}^*} \lambda(d\mathbb{P}^*) \right]^+ + \tilde{V}_0 \lambda(\mathcal{Q}) \right\}, \quad (4.7)$$

where $\Lambda_+(\mathcal{Q})$ denotes the set of all measures with bounded variation defined on a σ -field of all subsets of $\mathcal{Q}$. Let $d^i(\mathbb{Q})$ denote the value of the problem (4.7). Problems (4.6) and (4.7) are Fenchel dual and the strong duality holds, i.e. $p^i(\mathbb{Q}) = d^i(\mathbb{Q})$. Moreover, randomized test $\tilde{\phi}_{\mathbb{Q}}$ has the following structure:

$$\tilde{\phi}_{\mathbb{Q}}(\omega) = \begin{cases} 1 & : H(Z_{\mathbb{Q}} - \int_{\mathcal{Q}} Z_{\mathbb{P}^*} \tilde{\lambda}_{\mathbb{Q}}(d\mathbb{P}^*))(\omega) > 0, \\ 0 & : H(Z_{\mathbb{Q}} - \int_{\mathcal{Q}} Z_{\mathbb{P}^*} \tilde{\lambda}_{\mathbb{Q}}(d\mathbb{P}^*))(\omega) < 0 \end{cases}$$

and

$$E_{\mathbb{P}^*} H \tilde{\phi}_{\mathbb{Q}} = \tilde{V}_0 \quad \tilde{\lambda}_{\mathbb{Q}} - a.s.$$

Using this theorem one easily derives the structure of the solution of the static problem (4.4).

Proposition 4.7 Let $\mathcal{Q}$ be a subset of $\mathcal{P}$ such that (A2) holds for a given claim $H \in L_+^1$ and let $\rho : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$ be a function which satisfies conditions (A1). The following assertions hold:

- (i) There exists $\tilde{\mathbb{Q}}$ (resp. $\tilde{\lambda}_{\tilde{\mathbb{Q}}}$) solving (4.5) [resp. (4.7) with $\mathbb{Q} = \tilde{\mathbb{Q}}$].
- (ii) The solution $\tilde{\phi}$ of the static problem (4.4) exists and satisfies the following conditions:

$$\tilde{\phi}(\omega) = \begin{cases} 1 : H(Z_{\tilde{\mathbb{Q}}} - \int_{\tilde{\mathbb{Q}}} Z_{\mathbb{P}^*} \tilde{\lambda}_{\tilde{\mathbb{Q}}}(d\mathbb{P}^*))(\omega) > 0, \\ 0 : H(Z_{\tilde{\mathbb{Q}}} - \int_{\tilde{\mathbb{Q}}} Z_{\mathbb{P}^*} \tilde{\lambda}_{\tilde{\mathbb{Q}}}(d\mathbb{P}^*))(\omega) < 0 \end{cases}$$

and

$$E_{\mathbb{P}^*} H \tilde{\phi} = \tilde{V}_0 \tilde{\lambda}_{\tilde{\mathbb{Q}}} - a.s.$$

Proof Assertion (i) as well as existence of $\tilde{\phi}$ solving the static problem (4.4) follow immediately from claims (i), (ii) and (iv) of Theorem 4.6. The structure of the solution is deduced as follows. Let $\mathbb{Q}$ be a solution of (4.5). Using assertion (ii) of Theorem 4.6, we obtain

$$\max_{\mathbb{Q} \in \mathcal{P}_b} \left\{ \min_{\phi \in \mathcal{R}_{\mathbb{Q}}} \left\{ E_{\mathbb{Q}}[(1 - \phi)H] - \sup_{Y \in \mathcal{A}_p} E_{\mathbb{Q}}(-Y) \right\} \right\} = E_{\tilde{\mathbb{Q}}}[(1 - \tilde{\phi})H] - \sup_{Y \in \mathcal{A}_p} E_{\tilde{\mathbb{Q}}}(-Y).$$

This implies that $\tilde{\phi}$ solves (4.6) with $\mathbb{Q} = \tilde{\mathbb{Q}}$. Thus, structure of $\tilde{\phi}$ is obtained from assertion (iv) of Theorem 4.6. $\square$

Observe that (4.2) can be rewritten as (4.4) with $\mathcal{Q} = (\mathcal{P}_b(\mathcal{M}))^r$. We now show that Assumption 4.1 implies that conditions (A1) and (A2) are satisfied with $\mathcal{Q} = (\mathcal{P}_b(\mathcal{M}))^r$.

Lemma 4.8 *Under Assumption 4.1 the following assertions hold:*

- (i) $\mathcal{R}_{\mathcal{P}_b(\mathcal{M})} \subseteq \mathcal{R}_{(\mathcal{P}_b(\mathcal{M}))^{r_2}} \subseteq \mathcal{R}_{(\mathcal{P}_b(\mathcal{M}))^{r_1}}$ for any $r_1 \leq r_2 \in \mathbb{N}$.
- (ii) Conditions (A1) and (A2) hold for $\mathcal{Q} = (\mathcal{P}_b(\mathcal{M}))^r$, $r \in \mathbb{N}$.

Proof Assertion (i) follows immediately from inclusions

$$(\mathcal{P}_b(\mathcal{M}))^{r_1} \subseteq (\mathcal{P}_b(\mathcal{M}))^{r_2} \subseteq \mathcal{P}_b(\mathcal{M}).$$

We show assertion (ii). Fix $r \in \mathbb{N}$. Assumption 4.1 guarantees existence of $\phi_0 \in \mathcal{R}_{\mathcal{P}_b(\mathcal{M})}$ such that $\rho(H(\phi_0 - \mathbf{1})) < \infty$ and ρ is continuous at $H(\phi_0 - \mathbf{1})$ in the norm topology of L^1 . Since $\phi_0 \in \mathcal{R}_{(\mathcal{P}_b(\mathcal{M}))^r}$ by assertion (i), condition (A1) is satisfied with $\mathcal{Q} = (\mathcal{P}_b(\mathcal{M}))^r$.

Moreover, $E_{\mathbb{Q}} H \leq r E H$ for any $\mathbb{Q} \in (\mathcal{P}_b(\mathcal{M}))^r$, which implies condition (A2) for $\mathcal{Q} = (\mathcal{P}_b(\mathcal{M}))^r$. $\square$

Assertion (ii) of Lemma 4.8 allows using Proposition 4.7 from which one obtains existence and gives structure of the solution of the problem (4.2) for every $r \in \mathbb{N}$.

Corollary 4.9 *Under Assumption 4.1 for every $r \in \mathbb{N}$ the following assertions hold:*

- (i) There exists $\mathbb{Q}_r$ (resp. $\lambda_{\mathbb{Q}_r}$) a solution of (4.5) [resp. (4.7) with $\mathbb{Q} = \mathbb{Q}_r$ and $\mathcal{Q} = (\mathcal{P}_b(\mathcal{M}))^r$].
- (ii) The solution ϕ_r of the static problem (4.2) exists and satisfies the following conditions:

$$\phi_r(\omega) = \begin{cases} 1 : H(Z_{\mathbb{Q}_r} - \int_{(\mathcal{P}_b(\mathcal{M}))^r} Z_{\mathbb{P}^*} \lambda_{\mathbb{Q}_r}(d\mathbb{P}^*))(\omega) > 0, \\ 0 : H(Z_{\mathbb{Q}_r} - \int_{(\mathcal{P}_b(\mathcal{M}))^r} Z_{\mathbb{P}^*} \lambda_{\mathbb{Q}_r}(d\mathbb{P}^*))(\omega) < 0 \end{cases}$$

and

$$E_{\mathbb{P}^*} H \phi_r = \tilde{V}_0 \quad \lambda_{\mathbb{Q}_r} - a.s.$$

4.3 Solution of the static problem (4.1) and the convex hedging problem (2.1).

Lemma A1.1 in [Delbaen and Schachermayer \(1994\)](#) guarantees existence of a DS-sequence $(\beta_n)_{n=1}^\infty$ such that the sequence of random variables $\tilde{\phi}_n := \sum_{r \geq n} \beta_{n,r} \phi_r$ converges $\mathbb{P}$ -a.s. to some $[0, \infty]$ -valued random variable $\tilde{\phi}$. Clearly $\tilde{\phi}$ is a randomized test, since ϕ_r is a randomized test for every $r \in \mathbb{N}$. Also $\tilde{\phi}_n \xrightarrow{L^1} \tilde{\phi}$ by bounded convergence.

Theorem 4.10 *Under Assumption 4.1 the following assertions hold:*

- (i) $\tilde{\phi} \in \mathcal{R}_{\mathcal{P}_b(\mathcal{M})}$.
- (ii) $\tilde{\phi}$ solves the static problem (4.1).
- (iii) The strategy $(\tilde{V}_0, \tilde{\pi})$ superreplicating the contingent claim $\tilde{\phi}H$ is a solution of the convex hedging problem (2.1).

Proof To show assertion (i) we fix arbitrary $\mathbb{P}^* \in \mathcal{P}_b(\mathcal{M})$ and show $E_{\mathbb{P}^*}(\tilde{\phi}H) \leq \tilde{V}_0$. Since $E_{\mathbb{P}^*}(\tilde{\phi}H) \leq \liminf_{n \rightarrow \infty} E_{\mathbb{P}^*}(\tilde{\phi}_n H)$ by Fatou lemma, it suffices to show that

$$E_{\mathbb{P}^*}(\tilde{\phi}_n H) \leq \tilde{V}_0 \tag{4.8}$$

for $n \geq N_0$ with some $N_0 \in \mathbb{N}$. Let N_0 be the smallest positive integer for which $Z_{\mathbb{P}^*} \leq N_0$. $E_{\mathbb{P}^*}(\phi_r H) \leq \tilde{V}_0$ for every $r \geq N_0$, since $\mathbb{P}^* \in (\mathcal{P}_b(\mathcal{M}))^r$ and $\phi_r \in \mathcal{R}_{(\mathcal{P}_b(\mathcal{M}))^r}$. Hence, for arbitrary $n \geq N_0$ we have $E_{\mathbb{P}^*}(\tilde{\phi}_n H) = \sum_{r \geq n} \beta_{n,r} E_{\mathbb{P}^*} \phi_r H \leq \sum_{r \geq n} \beta_{n,r} \tilde{V}_0 = \tilde{V}_0$, which yields (4.8).

Now we prove assertion (ii). Let L (resp. L_r) denote value of the static problem (4.1) (resp. (4.2)). Obviously $L_r \leq L$, since $\mathcal{R}_{\mathcal{P}_b(\mathcal{M})} \subseteq \mathcal{R}_{(\mathcal{P}_b(\mathcal{M}))^r}$ by assertion (i) of Lemma 4.8. On the other hand, by lower-semicontinuity and convexity of ρ

$$\begin{aligned} L &\leq \rho((\tilde{\phi} - \mathbf{1})H) \leq \liminf_{n \rightarrow \infty} \rho((\tilde{\phi}_n - \mathbf{1})H) = \liminf_{n \rightarrow \infty} \rho\left(\sum_{r \geq n} \beta_{n,r}(\phi_r - \mathbf{1})H\right) \\ &\leq \liminf_{n \rightarrow \infty} \sum_{r \geq n} \beta_{n,r} \rho((\phi_r - \mathbf{1})H) \stackrel{(*)}{=} \liminf_{n \rightarrow \infty} \sum_{r \geq n} \beta_{n,r} L_r \leq L, \end{aligned}$$

where in $(*)$ we used the fact that ϕ_r solves (4.2). Thus, we have shown that a minimum of a function $\phi \rightarrow \rho(H(\phi - \mathbf{1}))$ over $\mathcal{R}_{\mathcal{P}_b(\mathcal{M})}$ is attained at $\tilde{\phi}$, which therefore solves

the static problem (4.1). This completes the proof of the assertion (ii). By corollary 4.5 the randomized test $\tilde{\phi}$ also solves the static problem (2.2) and the assertion (iii) follows immediately from Theorem 2.4. $\square$

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