Strong bounds on perturbations

Bernd Heidergott · Arie Hordijk · Haralambie Leahu

Received: 12 November 2007 / Accepted: 21 May 2008 / Published online: 1 July 2008
© The Author(s) 2008

Abstract This paper provides strong bounds on perturbations over a collection of independent random variables, where ‘strong’ has to be understood as *uniform w.r.t. some functional norm*. Our analysis is based on studying the concept of weak differentiability. By applying a fundamental result from the theory of Banach spaces, we show that weak differentiability implies norm Lipschitz continuity. This result leads to bounds on the sensitivity of finite products of probability measures, in norm sense. We apply our results to derive bounds on perturbations for the transient waiting times in a $G/G/1$ queue.

Keywords Weak differentiation · Banach space · Strong convergence · Perturbation analysis

0 Introduction

A wide range of probabilistic models in the area of manufacturing, transportation, finance and communication can be modeled by studying cost functions over a finite

This research is supported by the Technology Foundation STW, applied science division of NWO and the technology programme of the Ministry of Economic Affairs.

B. Heidergott (✉) · H. Leahu
Vrije Universiteit, Amsterdam, The Netherlands
e-mail: bheidergott@feweb.vu.nl

H. Leahu
e-mail: hleahu@feweb.vu.nl

A. Hordijk
Leiden Universiteit, Leiden, The Netherlands
e-mail: arie-hordijk@xs4all.nl

collection of random variables. More specifically, letting $\mu_{i,\theta}$ be a probability measure on some state space $\mathbb{S}_i$ (for $1 \leq i \leq n$) depending on some parameter θ , then this paper is concerned with the following type of models

$$J_g(\theta) := \mathbb{E}_\theta[g(X_n, \dots, X_1)] = \int \cdots \int g(s_n, \dots, s_1) \mu_{n,\theta}(ds_n) \cdots \mu_{1,\theta}(ds_1), \quad (1)$$

for a real-valued cost function g , where X_i is distributed according to $\mu_{i,\theta}$; and we assume, for ease of exposition, that $\theta \in \Theta = (a, b) \subset \mathbb{R}$, with $a < b$, as the extension to vector-valued parameters with $\Theta \subset \mathbb{R}^n$ is straightforward. This class of models contains, for example, transient waiting times in queueing networks or insurance models over a finite number of claims. If $\theta_1, \theta_2 \in \Theta$ then this paper studies the following type of bounds:

- (i) Bounds on $|J_g(\theta_2) - J_g(\theta_1)|$, for some specified cost function g .
- (ii) Uniform bounds with respect to a class $\mathcal{D}$ of cost functions, i.e., for

$$\sup_{g \in \mathcal{D}} |J_g(\theta_2) - J_g(\theta_1)|.$$

Starting point of the analysis put forward in this paper is that weak differentiability of the probability measures $\mu_{i,\theta}$, $1 \leq i \leq n$, in (1) (to be defined below) implies local Lipschitz continuity of $J_g(\theta)$ as a function of θ , which leads to type (i) bounds. For type (ii) bounds, we will work with a norm framework where we will use the weighted supremum norm or v -norm (to be defined below). Elaborating on a link between functional analysis and weak differentiability, we will show that $J_g(\theta)$ is v -norm Lipschitz, where the norm-Lipschitz constant can be obtained as a simple function of the norm-Lipschitz constant of the individual probability measures $\mu_{i,\theta}$. Moreover, the Lipschitz constant of a probability measure can be computed in a simple way from the weak derivative of the probability measure. Classical theory on v -norms assumes that the mapping v is bounded away from zero, i.e., $\inf_s v(s) > 0$. As it will turn out, this is not efficient when one establishes type (ii) bounds, and we will show how the definition of v -norm extends to the case where v may have zeros.

Norm bounds for Markov chains of the type developed in this paper have been intensively studied in the literature on perturbation analysis of Markov chains, which goes back to the pioneering paper of Schweitzer (1968). In this line of research one tries to bound the distance between the stationary distributions of two Markov chains by the difference between the corresponding Markov kernels, where the “distance” is measured by an appropriate norm. See Cho and Meyer (2001) for an overview on these results. The use of the weighted supremum norm for establishing norm bounds goes back to Lipman (1974) and the weighted supremum norm has been frequently used in Markov decision theory (Dekker and Hordijk 1988; Hordijk and Yushkevich 1999) and stability theory of Markov chains (Meyn and Tweedie 1993). A key element of the approach presented in this paper is that we assume that the function v is of exponential type, i.e., $v(s) = e^{\alpha s}$ for $s \geq 0$ and some $\alpha > 0$, which allows for a certain flexibility in choosing a convenient value for α . This approach is inspired by strong stable Markov chains (Aissani and Bouallouche-Medjkoune 2006; Kartashov

1996), where this technique is used to identify the stability neighborhood of a Markov chain. The paper deviates from the above approaches as it studies finite products of distributions, in contrast to stationary distributions of Markov chains, and in that it investigates to what extent a normed based bounding approach is of practical use in terms of actually computing bounds numerically.

The paper is organized as follows. In Sect. 1, we present basic notions for studying the limiting behavior of sequences of measures and weak differentiability is introduced. In addition, we show how can one establish bounds on perturbations using weak derivatives. In Sect. 2, we concentrate on perturbations of the waiting times in the $G/G/1$ queue and we establish bounds on perturbations for transient waiting times. The contributions of the paper are summarized in Sect. 3.

1 Weak differentiation of probability measures

In this section, we present an efficient method for deriving uniform bounds on perturbations for parameter-dependent stochastic systems. This is possible by means of weak derivatives. To this end, we start with an overview of the basic concepts in Sect. 1.1 and we present a rule for weak differentiation of product measures in Sect. 1.2. In Sect. 1.3, we properly define the concept of *strong convergence*, by means of the v -norm and establish a relation between weak differentiability and strong Lipschitz continuity that is essential for our analysis. Eventually, bounds on perturbations for some usual distributions are established in Sect. 1.4 and we present an application in Sect. 1.5.

1.1 Weak properties of measure-valued mappings

This section provides a short introduction to the theory of measure-valued differentiation. For the ease of exposure we list a few notations that we will use throughout this paper. Let $(\mathbb{S}, d)$ be a separable¹ metric space endowed with the Borel field $\mathcal{S}$:

- $\mathcal{C}(\mathbb{S})$ denotes the set of real-valued continuous mappings on $\mathbb{S}$.
- $\mathcal{C}_B(\mathbb{S}) \subset \mathcal{C}(\mathbb{S})$ denotes the subset of bounded mappings.
- $\mathcal{C}^+(\mathbb{S}) \subset \mathcal{C}(\mathbb{S})$ denotes the subset of non-negative mappings.
- $\mathcal{M}(\mathcal{S})$ denotes the space of finite, regular (signed) measures on $(\mathbb{S}, \mathcal{S})$.
- $\mathcal{M}^+(\mathcal{S}) \subset \mathcal{M}(\mathcal{S})$ denotes the cone of positive measures, i.e.,

$$\mathcal{M}^+(\mathcal{S}) := \{\mu \in \mathcal{M}(\mathcal{S}) : \mu(E) \geq 0, \forall E \in \mathcal{S}\}.$$

- $\mathcal{M}^1(\mathcal{S}) \subset \mathcal{M}^+(\mathcal{S})$ denotes the set of probability measures, i.e., $\mu(\mathbb{S}) = 1$.
- for $\mu \in \mathcal{M}$ we denote by $\mathcal{L}^1(\mu)$ the set of measurable mappings which are integrable w.r.t. μ and, for a family $\{\mu_i : i \in I\} \subset \mathcal{M}(\mathcal{S})$, let

$$\mathcal{L}^1(\mu_i : i \in I) := \bigcap_{i \in I} \mathcal{L}^1(\mu_i).$$

¹ A metric space $(\mathbb{S}, d)$ is called *separable* if it contains a countable dense subset.

In the following, we omit specifying the underlying space $\mathbb{S}$ (resp., Borel field $\mathcal{S}$) when no confusion occurs.

Let $\{\mu_n\}_n$ be a sequence of measures in $\mathcal{M}$. For $\mathcal{D} \subset \mathcal{L}^1(\mu_n : n \in \mathbb{N})$, we call μ_n weakly $\mathcal{D}$ -convergent if there exists $\mu \in \mathcal{M}$ such that

$$\forall g \in \mathcal{D} : \lim_{n \rightarrow \infty} \int g(s) \mu_n(ds) = \int g(s) \mu(ds). \quad (2)$$

We call μ in (2) a weak $\mathcal{D}$ -limit of $\{\mu_n\}_n$ and we note that, in general, the limit in (2) is not unique. Moreover, we can assume w.l.o.g. that $\mathcal{D}$ is a linear space, since the property in (2) extends in a straightforward way to the linear space generated by $\mathcal{D}$. Taking $\mathcal{D} = \mathcal{C}_B$ in (2), yields classical weak convergence of measures which defines the concept of *convergence in distribution*. Moreover, in his pioneering work on weak derivatives, Pflug used $\mathcal{D} = \mathcal{C}_B$ for defining weak derivatives; see Pflug (1996). The main reason behind this choice is that for $\mathcal{D} = \mathcal{C}_B$ the weak limit in (2) is uniquely determined.

Definition 1 For $v \in \mathcal{C}^+$, let $\mathcal{C}_v$ denote the set of continuous mappings that are bounded by a multiple of v ; in formula:

$$\mathcal{C}_v := \{g \in \mathcal{C} \mid \exists c \geq 0 : |g(s)| \leq c \cdot v(s), \forall s \in \mathbb{S}\}. \quad (3)$$

We call $\mathcal{C}_v$ the space of continuous v -bounded mappings on $\mathbb{S}$.

Note that, for $\mu \in \mathcal{M}$, we have $\mathcal{C}_v \subset \mathcal{L}^1(\mu)$ if and only if $v \in \mathcal{L}^1(\mu)$. It is straightforward that $\mathcal{C}_v$ is a linear subspace of $\mathcal{C}$ and for $v \equiv 1$ we have $\mathcal{C}_v = \mathcal{C}_B$. Moreover, as shown by Lemma 3 in the Appendix, the weak $\mathcal{C}_v$ -limit is uniquely determined up to an equivalence relation. Hence, the condition $\inf_{s \in \mathbb{S}} v(s) > 0$, which implies $\mathcal{C}_B \subset \mathcal{C}_v$ and consequently the uniqueness of the $\mathcal{C}_v$ -limit is not crucial for our analysis.

Example 1 Let $v \in \mathcal{C}^+([0, \infty))$ be defined as $v(x) = e^{\alpha x}$, for some $\alpha > 0$. Since for every polynomial P it holds that

$$\lim_{x \rightarrow \infty} e^{-\alpha x} P(x) = 0,$$

it turns out that the space $\mathcal{C}_v([0, \infty))$ contains all (finite) polynomials. However, the polynomials are not the only elements of $\mathcal{C}_v$ since, for instance, the mapping $x \mapsto \ln(1+x)$ also belongs to $\mathcal{C}_v$.

The next result relates general $\mathcal{C}_v$ -convergence to classical $\mathcal{C}_B$ -convergence.

Lemma 1 Let $(\mu_n : n \in \mathbb{N}) \subset \mathcal{M}^1$ and $v \in \mathcal{C}^+$ be s.t. $v \in \mathcal{L}^1(\mu_n : n \in \mathbb{N})$ and $\mu_n \xrightarrow{\mathcal{C}_B} \mu$, for some $\mu \in \mathcal{M}^1$, i.e., μ is the classical weak limit of the sequence $\{\mu_n\}_n$. Assume further that

$$\lim_{n \rightarrow \infty} \int v(x) \mu_n(dx) = \int v(x) \mu(dx). \quad (4)$$

Then, $\mu_n \xrightarrow{\mathcal{C}_v} \mu$, i.e., μ is also the weak $\mathcal{C}_v$ -limit of the sequence $\{\mu_n\}_n$.

Proof One has to show that the limit relation in (2) holds true for $\mathcal{D} = \mathcal{C}_v$. By hypothesis, the conclusion is equivalent to uniform integrability of each $g \in \mathcal{C}_v$ w.r.t. the family $\{\mu_n : n \in \mathbb{N}\}$; see, e.g., Billingsley (1966). That is

$$\forall g \in \mathcal{C}_v : \lim_{\alpha \rightarrow \infty} \sup_{n \in \mathbb{N}} \int |g(s)| \cdot \mathbb{I}_{\{|g| > \alpha\}}(s) \mu_n(ds) = 0, \quad (5)$$

where $\mathbb{I}_{\{|g| > \alpha\}}(s)$ denotes the indicator function of the set $\{s : |g(s)| > \alpha\}$. One can easily show that a sufficient condition for (5) is the uniform integrability of v w.r.t. the family $\{\mu_n : n \in \mathbb{N}\}$ which follows from (4). $\square$

Remark 1 Lemma 1 essentially says that uniform integrability of v , w.r.t. the sequence $\{\mu_n\}_n$ is a sufficient condition for $\mathcal{C}_v$ -convergence, provided that μ_n converges in classical sense.

As illustrated by Lemma 1, $\mathcal{C}_v$ -spaces are suitable choices for defining weak convergence because of their structure. In particular, applying Lemma 1 to $\mathcal{C}_v$ defined in Example 1, we conclude that if the sequence $\{\mu_n\}_n$ converges weakly to μ (in the classical sense) and (2) holds true for $v(x) = e^{\alpha x}$ then the moments of μ_n converge to those of μ .

Consider a family $\{\mu_\theta : \theta \in \Theta\} \subset \mathcal{M}$, with $\Theta := (a, b) \subset \mathbb{R}$, for some $a < b$. The correspondence $\theta \mapsto \mu_\theta$ defines a measure-valued mapping from Θ to $\mathcal{M}$. For a convenient notation, we agree to identify this mapping with its values, i.e., we use the notation μ_θ to illustrate the functional dependence w.r.t. θ , rather than specifying a particular value.

Definition 2 Let $\{\mu_\theta : \theta \in \Theta\} \subset \mathcal{M}^1$ and $v \in \mathcal{C}^+ \cap \mathcal{L}^1(\mu_\theta : \theta \in \Theta)$. We say that μ_θ is weakly $\mathcal{C}_v$ -continuous if the measure-valued mapping $\theta \mapsto \mu_\theta$ is weakly $\mathcal{C}_v$ -continuous at θ , i.e.,

$$\forall g \in \mathcal{C}_v : \lim_{\xi \rightarrow 0} \int g(s) \mu_{\theta+\xi}(ds) = \int g(s) \mu_\theta(ds).$$

We say that μ_θ is weakly $\mathcal{C}_v$ -continuous on Θ if, for each $\theta \in \Theta$, μ_θ is weakly $\mathcal{C}_v$ -continuous, i.e., the measure-valued mapping $\theta \mapsto \mu_\theta$ is weakly $\mathcal{C}_v$ -continuous at θ , for all $\theta \in \Theta$.

In the same vein, we say that μ_θ is weakly $\mathcal{C}_v$ -differentiable if there exists $\mu'_\theta \in \mathcal{M}$ such that

$$\forall g \in \mathcal{C}_v : \lim_{\xi \rightarrow 0} \frac{1}{\xi} \left(\int g(s) \mu_{\theta+\xi}(ds) - \int g(s) \mu_\theta(ds) \right) = \int g(s) \mu'_\theta(ds)$$

and we say that μ_θ is weakly $\mathcal{C}_v$ -differentiable on Θ if, for each $\theta \in \Theta$, μ_θ is weakly $\mathcal{C}_v$ -differentiable.

In the following, we omit specifying the set $\mathcal{C}_v$ when no confusion occurs and we use expressions such as “weakly convergent”, “weak limit”, “weakly continuous” and “weakly differentiable” instead.

Note that, if μ_θ is weakly differentiable, then μ'_θ is a finite signed measure. Any finite signed measure μ can be written in a unique way as the difference between two orthogonal² finite positive measures, denoted by $[\mu]^+$ and $[\mu]^-$. The fact is known as the *Hahn–Jordan decomposition*. Hence, it holds for μ'_θ that:

$$\forall E \in \mathcal{S} : \mu'_\theta(E) = [\mu'_\theta]^+(E) - [\mu'_\theta]^-(E). \quad (6)$$

Since $\mu_{\theta+\xi}(\mathbb{S}) = \mu_\theta(\mathbb{S}) = 1$, for all ξ , and since μ'_θ is the weak limit of the family $\{(\mu_{\theta+\xi} - \mu_\theta)/\xi\}_\xi$ as $\xi \rightarrow 0$, it follows that $\mu'_\theta(\mathbb{S}) = 0$; see the Appendix for details. This justifies the following definition:

$$\forall \theta \in \Theta : [\mu'_\theta]^+(\mathbb{S}) = [\mu'_\theta]^-(\mathbb{S}) := c_\theta.$$

This yields the following representation of μ'_θ :

$$\forall E \in \mathcal{S} : \mu'_\theta(E) = c_\theta \mu_\theta^+(E) - c_\theta \mu_\theta^-(E), \quad (7)$$

where $\mu_\theta^+ = c_\theta^{-1}[\mu'_\theta]^+ \in \mathcal{M}^1(\mathcal{S})$ and $\mu_\theta^- = c_\theta^{-1}[\mu'_\theta]^- \in \mathcal{M}^1(\mathcal{S})$.

The *variation* of μ'_θ , with decomposition as in (7), is defined by

$$\forall E \in \mathcal{S} : |\mu'_\theta|(E) := c_\theta \mu_\theta^+(E) + c_\theta \mu_\theta^-(E).$$

Letting $E = \mathbb{S}$ in the above definition, we obtain the *total variation* of μ'_θ , i.e.:

$$\|\mu'_\theta\| := |\mu'_\theta|(\mathbb{S}) = c_\theta \mu_\theta^+(\mathbb{S}) + c_\theta \mu_\theta^-(\mathbb{S}). \quad (8)$$

Note that the decomposition in (6) is by means of positive measures, whereas the decomposition in (7) is by means of probability measures. In accordance with (7), μ'_θ is completely characterized through the triple $(c_\theta, \mu_\theta^+, \mu_\theta^-)$, with $\mu_\theta^+, \mu_\theta^- \in \mathcal{M}^1(\mathcal{S})$. This representation has a clear advantage over (6) as it can be interpreted in terms of random variables and thus can be made fruitful for simulation. More specifically, let μ_θ be weakly differentiable with weak derivative $(c_\theta, \mu_\theta^+, \mu_\theta^-)$ satisfying (7). Then for any $g \in \mathcal{C}_v$ it holds that

$$\frac{d}{d\theta} \mathbb{E}[g(X_\theta)] = c_\theta \mathbb{E}[g(X_\theta^+) - g(X_\theta^-)],$$

where $X_\theta, X_\theta^+, X_\theta^-$ have distributions $\mu_\theta, \mu_\theta^+, \mu_\theta^-$, respectively. This leads to the following definition.

² Two measures μ, η defined on a common measurable space $(\mathbb{S}, \mathcal{S})$ are said to be orthogonal if there exists a set $A \in \mathcal{S}$ such that $\mu(\mathbb{S} \setminus A) = \eta(A) = 0$. That is to say that the two measures, μ and η have disjoint support.

Definition 3 Let $\mu_\theta \in \mathcal{M}^1$ be weakly differentiable. We call a triple $(c_\theta, \mu_\theta^+, \mu_\theta^-)$, with $\mu_\theta^\pm \in \mathcal{M}^1$ and $c_\theta \in \mathbb{R}$, a weak derivative of μ_θ if it holds that:

$$\forall g \in \mathcal{C}_v : \int g(s) \mu'_\theta(ds) = c_\theta \left(\int g(s) \mu_\theta^+(ds) - \int g(s) \mu_\theta^-(ds) \right).$$

A weak $\mathcal{C}_v$ -derivative, cf. Definition 3, is not unique and an instance of the derivative can always be found through the Hahn–Jordan decomposition. The weak derivative of a probability measure μ_θ becomes unique if we require that μ_θ^+ and μ_θ^- are orthogonal, in which case c_θ is minimized. However, in applications it is not always the case that the orthogonal representation is the most convenient one. This fact is illustrated by the following example.

Example 2 Let $\mathbb{S} = [0, \infty)$, $v(s) = s^p$, for some $p \geq 1$ and let

$$\forall s > 0 : f_\theta(s) := \theta e^{-\theta s},$$

be the density of the exponential distribution with rate θ . Take $\Theta = (a, b)$, for $0 < a < b < \infty$, and $\mu_\theta(ds) = f_\theta(s) \cdot ds$, for $\theta \in \Theta$. Then, μ_θ is $\mathcal{C}_v$ -differentiable, which can be easily seen by applying the Dominated Convergence Theorem. Indeed, one can directly differentiate the density $f_\theta(s)$ w.r.t. θ , which yields

$$\frac{d}{d\theta} f_\theta(s) = (1 - \theta s) e^{-\theta s}.$$

Letting $c_\theta = (\theta e)^{-1}$ and introducing densities

$$f_\theta^+(s) := \frac{1}{c_\theta} \mathbb{I}_{[0, 1/\theta]}(s) (1 - \theta s) e^{-\theta s}, \quad \forall s > 0, \quad (9)$$

and

$$f_\theta^-(s) := \frac{1}{c_\theta} \mathbb{I}_{[1/\theta, \infty)}(s) (\theta s - 1) e^{-\theta s}, \quad \forall s > 0, \quad (10)$$

we have $\frac{d}{d\theta} f_\theta(s) = c_\theta f_\theta^+(s) - c_\theta f_\theta^-(s)$ and we obtain:

$$\forall g \in \mathcal{C}_v : \frac{d}{d\theta} \int g(s) \mu_\theta(ds) = c_\theta \left(\int g(s) f_\theta^+(s) ds - \int g(s) f_\theta^-(s) ds \right)$$

and a weak derivative of μ_θ is given by

$$\mu'_\theta(ds) = (c_\theta, f_\theta^+(s) ds, f_\theta^-(s) ds).$$

Note that this is the orthogonal representation. However, one can also write

$$\frac{d}{d\theta} f_\theta(s) = \frac{1}{\theta} (\theta e^{-\theta s} - \theta^2 s e^{-\theta s}),$$

which yields another representation for the weak derivative. That is

$$\mu'_\theta(ds) = \left(\frac{1}{\theta}, \mu_\theta(ds), \gamma_{2,\theta}(ds) \right),$$

where $\gamma_{2,\theta}$ denotes the convolution of two exponential distributions with rate θ . Obviously, this representation is more meaningful in terms of random variables.

In the above example, μ_θ as well as μ_θ^+ and μ_θ^- have Lebesgue densities, that is, μ_θ as well as its weak derivative μ'_θ are absolutely continuous w.r.t. the same measure. The following example shows that this is not always the case.

Example 3 Let $v \in C^+([0, \infty))$ and ψ_θ denote the uniform distribution on the interval $[0, \theta]$. Then it holds that

$$\begin{aligned} \forall g \in C_v : \frac{d}{d\theta} \int g(x) \psi_\theta(dx) &= \frac{d}{d\theta} \left(\frac{1}{\theta} \int_0^\theta g(x) dx \right) = \frac{1}{\theta} g(\theta) - \frac{1}{\theta^2} \int_0^\theta g(x) dx \\ &= \frac{1}{\theta} \left(\int g(x) \delta_\theta(dx) - \int g(x) \psi_\theta(dx) \right), \end{aligned}$$

where δ_x denotes the Dirac measure in x . Hence, $(1/\theta, \delta_\theta, \psi_\theta)$ is a C_v -derivative of ψ_θ . Note that the C_v -derivative of ψ_θ has a continuous and a discrete part while ψ_θ is a continuous distribution, for a detailed discussion see [Heidergott and Hordijk \(2004\)](#).

Remark 2 In the literature, differentiation of a probability measure μ_θ has also been defined as differentiability of the corresponding set function, that is:

$$\mu'_\theta(A) = \frac{d}{d\theta} \mu_\theta(A), \quad \forall A \in \mathcal{S},$$

see, e.g., [Kushner and Vázquez-Abad \(1992\)](#). This definition can be recovered by extending C_v with the set of indicator functions. However, this is an unfortunate choice. Roughly speaking, there are too few probability measures that are differentiable according to this definition. The following example illustrates this. Taking ψ_θ to be the uniform distribution on $[0, \theta]$ and denoting the Lebesgue measure by λ it holds

$$\forall x > 0 : \psi_\theta([0, x]) = \frac{1}{\theta} \lambda([0, x] \cap [0, \theta]) = \frac{1}{\theta} \min(x, \theta).$$

At $\theta = x$, the left-hand side derivative of $\psi_\theta([0, x])$ equals 0 whereas the right-hand side derivative equals $-1/x$. Hence, ψ_θ fails to be weakly differentiable in the set-wise sense, whereas it is shown in [Example 3](#) that ψ_θ is C_v -differentiable.

1.2 Weak differentiation of product measures

Let $\{\mu_{i,\theta} : \theta \in \Theta\} \subset \mathcal{M}^1(\mathbb{S}_i)$, for $1 \leq i \leq n$, be a finite family of measure-valued mappings and let us consider the product mapping given by

$$\forall \theta \in \Theta : \Pi_\theta := \mu_{1,\theta} \times \cdots \times \mu_{n,\theta}. \quad (11)$$

If $v_i \in \mathcal{C}^+(\mathbb{S}_i)$, for $1 \leq i \leq n$, we define $V \in \mathcal{C}^+(\mathbb{S}_1 \times \cdots \times \mathbb{S}_n)$ as follows:

$$\forall (s_1, \dots, s_n) \in \mathbb{S}_1 \times \cdots \times \mathbb{S}_n : V(s_1, \dots, s_n) := \prod_{i=1}^n v_i(s_i). \quad (12)$$

Then, the following assertion holds true:

Theorem 1 *If $\mu_{i,\theta}$ is weakly $\mathcal{C}_{v_i}$ -differentiable, for $1 \leq i \leq n$, then the product measure Π_θ is weakly $\mathcal{C}_V$ -differentiable. Moreover, if $(c_{i,\theta}, \mu_{i,\theta}^+, \mu_{i,\theta}^-)$ is a weak derivative of $\mu_{i,\theta}$, for $1 \leq i \leq n$, then an instance of the weak derivative Π'_θ is given by $(C_\theta, \Pi_\theta^+, \Pi_\theta^-)$, where*

$$\begin{aligned} C_\theta &= \sum_{i=1}^n c_{i,\theta}, \\ \Pi_\theta^+ &= \sum_{i=1}^n \frac{c_{i,\theta}}{C_\theta} \cdot \mu_{1,\theta} \times \cdots \times \mu_{i,\theta}^+ \times \cdots \times \mu_{n,\theta}, \\ \Pi_\theta^- &= \sum_{i=1}^n \frac{c_{i,\theta}}{C_\theta} \cdot \mu_{1,\theta} \times \cdots \times \mu_{i,\theta}^- \times \cdots \times \mu_{n,\theta}. \end{aligned}$$

In particular, if $\mu_{1,\theta} = \cdots = \mu_{n,\theta} = \mu_\theta$ and $\mu'_\theta = (c_\theta, \mu_\theta^+, \mu_\theta^-)$ it holds that

$$(\mu_\theta^n)' = \left(nc_\theta, \frac{1}{n} \sum_{i=1}^n \mu_\theta^{i-1} \times \mu_\theta^+ \times \mu_\theta^{n-i}, \frac{1}{n} \sum_{i=1}^n \mu_\theta^{i-1} \times \mu_\theta^- \times \mu_\theta^{n-i} \right), \quad (13)$$

where, for $k \geq 1$, μ_θ^k denotes the k -fold product of μ_θ and we ignore the term μ_θ^0 .

For more details on rules of weak differentiation of product measures, including the proof of Theorem 1, we refer to [Heidergott and Leahu \(2008\)](#).

Example 4 Let us consider a $G/G/1$ queue where the service times have distribution μ and inter-arrival times have distribution η . If $\{W_n : n \geq 1\}$ denotes the sequence of waiting times, then Lindley's recursion yields:

$$\forall n \geq 1 : W_{n+1} = \max\{W_n + S_n - A_{n+1}, 0\},$$

where $\{S_n : n \geq 1\}$ and $\{A_n : n \geq 1\}$ denote the sequence of service and inter-arrival times, respectively. To apply Theorem 1 let us consider the sequence of mappings $\{g_n : n \geq 1\}$, $g_n \in \mathcal{C}(\mathbb{R}^{2n})$ defined as $g_1(s, t) = \max\{s - t, 0\}$ and

$$\forall n \geq 1, \sigma, \tau \in \mathbb{R}^n, s, t \in \mathbb{R} : g_{n+1}(\sigma, s, \tau, t) = \max\{g_n(\sigma, \tau) + s - t, 0\}.$$

It follows from Lindley's recursion that the waiting times satisfy

$$\forall n \geq 1 : W_{n+1} = g_n(S_1, \dots, S_n, A_2, \dots, A_{n+1}).$$

Hence, if the service times $S_1, \dots, S_n, \dots$ have distribution μ_θ , inter-arrival times have distribution η , independent of θ , and μ_θ is $\mathcal{C}_v$ -differentiable then Theorem 1 implies that for all $\bar{v} \in \mathcal{C}^+([0, \infty))$ satisfying

$$\sup_{\substack{s_1, \dots, s_n \in \mathbb{R} \\ t_1, \dots, t_n \in \mathbb{R}}} \frac{\bar{v}(g_n(s_1, \dots, s_n, t_1, \dots, t_n))}{\prod_{i=1}^n v(s_i) \prod_{i=1}^n v(t_i)} < \infty, \quad (14)$$

the distribution of W_{n+1} is weakly $\mathcal{C}_{\bar{v}}$ -differentiable and it holds that

$$\forall f \in \mathcal{C}_{\bar{v}} : \frac{d}{d\theta} \mathbb{E}_\theta[f(W_{n+1})] = c_\theta \sum_{k=1}^n \mathbb{E}_\theta \left[f(W_{n+1}^{k+}) - f(W_{n+1}^{k-}) \right], \quad (15)$$

where

$$W_{n+1}^{k\pm} = g_n(S_1, \dots, S_k^\pm, \dots, S_n, A_2, \dots, A_{n+1}) \quad (16)$$

and S_k^+ and S_k^- are S_k -measurable³ r.v. distributed according to μ_θ^+ and μ_θ^- , respectively, i.e., $W_{n+1}^{k\pm}$ denotes the $(n+1)^{\text{st}}$ waiting time in a modified queue, where the k th service time S_k has been replaced by S_k^+ and S_k^- , respectively.

Remark 3 Note that, (14) holds true if $\bar{v}$ is non-decreasing and it satisfies

$$\forall x, y \geq 0 : \bar{v}(x + y) \leq \gamma v(x)v(y), \quad (17)$$

for some $\gamma > 0$. Indeed, note that for all $n \geq 1$, g_n satisfies

$$\forall s_1, \dots, s_n, t_1, \dots, t_n \in \mathbb{R} : g_n(s_1, \dots, s_n, t_1, \dots, t_n) \leq s_1 + \dots + s_n + t_1 + \dots + t_n.$$

³ I.e., measurable w.r.t. the σ -field generated by S_k . In particular, this means that $S_k^\pm$ are independent of $\{S_i : i \geq 1, i \neq k\}$ and $\{T_j : j \geq 1\}$. However, note that it is not crucial that S_k^+ and S_k^- are mutually independent.

Using monotonicity of $\bar{v}$, from (17) we conclude that

$$\sup_{\substack{s_1, \dots, s_n \in \mathbb{R} \\ t_1, \dots, t_n \in \mathbb{R}}} \frac{\bar{v}(g_n(s_1, \dots, s_n, t_1, \dots, t_n))}{\prod_{i=1}^n v(s_i) \prod_{i=1}^n v(t_i)} \leq \gamma^{2n-1}.$$

For instance, if for all $x \geq 0$ $v(x) = \bar{v}(x) = e^{\alpha x}$, for some $\alpha \geq 0$, (17) is fulfilled and we conclude that if μ_θ is $\mathcal{C}_v$ -differentiable, then the distribution of W_{n+1} is $\mathcal{C}_v$ -differentiable, as well.

1.3 $\|\cdot\|_v$ -norm convergence

Throughout this section we let $v \in \mathcal{C}^+$ and $\{\mu_\theta : \theta \in \Theta\} \subset \mathcal{M}^1$ be such that $v \in \mathcal{L}^1(\mu_\theta : \theta \in \Theta)$. The “weak” definition of a limit of measures in (2) has a strong counterpart, where “strong” refers to a norm-setting. More specifically, let $\|\cdot\|$ be a norm on $\mathcal{M}$; we call μ_n $\|\cdot\|$ -convergent if $\mu \in \mathcal{M}$ exists such that

$$\lim_{n \rightarrow \infty} \|\mu_n - \mu\| = 0.$$

For example, $\|\mu\|$ may denote the total variation of μ , mentioned in (8). Accordingly, the mapping μ_θ is called *strongly continuous* if:

$$\lim_{\xi \rightarrow 0} \|\mu_{\theta+\xi} - \mu_\theta\| = 0, \quad (18)$$

and is called *strongly differentiable* if there exists $\mu'_\theta \in \mathcal{M}$, s.t:

$$\lim_{\xi \rightarrow 0} \left\| \frac{\mu_{\theta+\xi} - \mu_\theta}{\xi} - \mu'_\theta \right\| = 0. \quad (19)$$

Note that strong differentiability, as defined in (19), is also referred to as *Fréchet differentiability* on normed spaces.

For $v \in \mathcal{C}^+$, let

$$\mathbb{S}_v := \{s \in \mathbb{S} : v(s) > 0\}.$$

The v -norm of a mapping $g \in \mathcal{C}$ is defined as follows:

$$\|g\|_v := \begin{cases} \sup_{s \in \mathbb{S}_v} \frac{|g(s)|}{v(s)}, & \text{if } g(s) = 0 \text{ for each } s \notin \mathbb{S}_v, \\ \infty, & \text{if there is } s \notin \mathbb{S}_v \text{ with } g(s) \neq 0. \end{cases} \quad (20)$$

It follows that

$$\|g\|_v = \inf\{c > 0 : |g(s)| \leq c \cdot v(s), \forall s \in \mathbb{S}\},$$

where we agree that $\inf \emptyset = \infty$. Moreover,⁴

$$\forall s \in \mathbb{S} : |g(s)| \leq \|g\|_v \cdot v(s),$$

and $\mathcal{C}_v$ can be described in terms of the v -norm, as follows

$$\mathcal{C}_v = \{g \in \mathcal{C} : \|g\|_v < \infty\}.$$

The functional v -norm in (20) is extended to measures in the usual way:

$$\|\mu\|_v = \sup_{g \in \mathcal{C}_v, \|g\|_v \leq 1} \left| \int g(s) \mu(ds) \right| = \int v(s) |\mu|(ds), \quad (21)$$

where $|\mu| \in \mathcal{M}$ is the variation of μ . In the literature it is usually assumed that $\inf\{v(s) : s \in \mathbb{S}\} > 0$. However, as detailed in the Appendix, we may drop this assumption without any harm. Furthermore, $\mathcal{C}_v$ becomes a Banach space when endowed with the v -norm. Taking $v \equiv 1$ in (21), i.e., $\mathcal{C}_v = \mathcal{C}_B$ one recovers the total variation norm $\|\mu\|$ of μ .

Remark 4 The v -norm is also known as weighted supremum norm in the literature. An early reference is [Lipman \(1974\)](#). The v -norm is frequently used in Markov decision analysis. First traces date back to the early 1980s, see [Dekker and Hordijk \(1983\)](#) and the revised version which was published as [Dekker and Hordijk \(1988\)](#). It was originally used in analysis of Blackwell optimality; see [Dekker and Hordijk \(1988\)](#), and [Hordijk and Yushkevich \(1999\)](#) for a recent publication on this topic. Since then, it has been used in various forms under different names in many subsequent papers, see, for example, [Meyn and Tweedie \(1993\)](#) and [Kartashov \(1996\)](#). For the use of v -norm in the theory of measure-valued differentiation of Markov chains, see [Heidergott and Hordijk \(2003\)](#).

v -norm-convergence implies $\mathcal{C}_v$ -weak convergence. For example, it is known that convergence of μ_n towards μ in total variation norm implies that (2) holds for the set $\mathcal{D} = \mathcal{C}_B$. For general v this is a consequence of the *Cauchy–Schwartz Inequality*. More specifically, for each $\mu \in \mathcal{M}$ s.t. $v \in \mathcal{L}^1(\mu)$ it holds that:

$$\forall g \in \mathcal{C}_v : \left| \int g(s) \mu(ds) \right| \leq \int \frac{|g(s)|}{v(s)} \cdot v(s) |\mu|(ds) \leq \|g\|_v \cdot \|\mu\|_v. \quad (22)$$

If μ_n converges in v -norm to μ , replacing μ by $\mu_n - \mu$ in (22), it follows that (2) holds true for all $g \in \mathcal{C}_v$. In words, strong convergence implies weak convergence. The converse is, however, not true; see [Heidergott and Leahu \(2008\)](#) for details.

The following theorem shows that weak differentiability implies strong local Lipschitz continuity.

⁴ Note that the inequality still holds true if $\|g\|_v = \infty$.

Theorem 2 Let μ_θ be $\mathcal{C}_v$ -differentiable, with $v \in \mathcal{L}^1(\mu_\theta : \theta \in \Theta)$. Then, for any neighborhood $(\theta - \epsilon, \theta + \epsilon) \subset \Theta$ of θ exists $M_\epsilon > 0$ such that

$$\forall |\xi| \leq \epsilon : \|\mu_{\theta+\xi} - \mu_\theta\|_v \leq |\xi| M_\epsilon.$$

In words, μ_θ is locally v -norm Lipschitz continuous.

Proof Choose $\epsilon > 0$ such that $(\theta - \epsilon, \theta + \epsilon) \subset \Theta$. For $\xi \neq 0$, denote by T_ξ the linear continuous functional on $\mathcal{C}_v$ defined as:

$$\forall g \in \mathcal{C}_v : T_\xi g := \frac{1}{\xi} \int g(s)(\mu_{\theta+\xi} - \mu_\theta).$$

For $\xi \neq 0$ the operator norm of T_ξ satisfies:

$$\|T_\xi\|_v = \frac{\|\mu_{\theta+\xi} - \mu_\theta\|_v}{|\xi|}.$$

From $\mathcal{C}_v$ -differentiability of μ_θ it follows that

$$\forall g \in \mathcal{C}_v : \sup\{|T_\xi g| : |\xi| \leq \epsilon\} < \infty$$

and the Banach–Steinhaus Theorem applied to the family $\{T_\xi : |\xi| \leq \epsilon\}$ yields

$$\sup\{\|T_\xi\|_v : |\xi| \leq \epsilon\} = M_\epsilon < \infty.$$

Hence, for ξ such that $|\xi| < \epsilon$ it holds that

$$\|\mu_{\theta+\xi} - \mu_\theta\|_v \leq M_\epsilon \cdot |\xi|,$$

which concludes the proof. $\square$

As shown by Theorem 2, $\mathcal{C}_v$ -differentiability implies v -norm local Lipschitz continuity. This does, however, not imply that $\mathcal{C}_v$ -differentiability is equivalent to strong v -norm differentiability which stems from the fact that

$$\forall g \in \mathcal{C}_v : \lim_{\xi \rightarrow 0} \frac{1}{\xi} \left| \left(\int g(s)\mu_{\theta+\xi}(ds) - \int g(s)\mu_\theta(ds) \right) - \int g(s)\mu'_\theta(ds) \right| = 0 \quad (23)$$

does, in general, not imply that

$$\lim_{\xi \rightarrow 0} \sup_{|g| \leq v} \left| \frac{1}{\xi} \left(\int g(s)\mu_{\theta+\xi}(ds) - \int g(s)\mu_\theta(ds) \right) - \int g(s)\mu'_\theta(ds) \right| = 0. \quad (24)$$

The following example illustrates this fact.

Example 5 Consider the uniform distribution ψ_θ on $[0, \theta]$. In Example 3 we have shown that

$$\psi'_\theta = \frac{1}{\theta} \delta_\theta - \frac{1}{\theta} \psi_\theta.$$

Let $v(s) = 1$, for $s \in \mathbb{S} = [0, \infty)$, and consider the family of mappings $g_\xi \in \mathcal{C}_v$ defined as

$$\forall \xi > 0 : g_\xi(s) = \begin{cases} 0, & \text{if } s \leq \theta, \\ \frac{s-\theta}{\xi}, & \text{if } \theta < s \leq \theta + \xi, \\ 1, & \text{if } \theta + \xi < s. \end{cases}$$

It is immediate that for each $\xi > 0$ we have $\|g_\xi\|_v = 1$ and it holds that

$$\int g_\xi(s) \psi_\theta(ds) = \int g_\xi(s) \psi'_\theta(ds) = 0.$$

Therefore,

$$\left\| \frac{\psi_{\theta+\xi} - \psi_\theta}{\xi} - \psi'_\theta \right\|_v \geq \left| \frac{1}{\xi} \int g_\xi(s) \psi_{\theta+\xi}(ds) \right| = \frac{1}{2(\theta + \xi)} \rightarrow \frac{1}{2\theta} > 0.$$

Hence, $\mu_\theta = \psi_\theta$ satisfies (23) but, in general, violates (24) and we conclude that the uniform distribution is an example of a probability measure that is weakly differentiable but fails to be strongly differentiable.

In general, checking strong differentiability, as defined by (19), might be very demanding and it is desirable to have simpler verifiable sufficient conditions for strong differentiability. The following result shows that weak differentiability together with strong continuity of the weak derivative imply strong differentiability. The precise statement is as follows:

Proposition 1 *If μ_θ is weakly $\mathcal{C}_v$ -differentiable on Θ such that μ'_θ is v -norm continuous at θ , then μ_θ is strongly v -norm differentiable at θ .*

Proof Let $\epsilon > 0$ be arbitrary and choose $\varrho > 0$ such that

$$\forall \xi \in (-\varrho, \varrho) : \|\mu'_{\theta+\xi} - \mu'_\theta\|_v < \epsilon.$$

Applying the Mean Value Theorem for $\theta \mapsto \int g(s) \mu_\theta(ds)$ yields:

$$\forall \xi \in (-\varrho, \varrho), \forall g \in \mathcal{C}_v : \int g(s) \mu_{\theta+\xi}(ds) - \int g(s) \mu_\theta(ds) = \xi \int g(s) \mu'_{\theta+\xi_g}, \quad (25)$$

for some $\xi_g \in (-\varrho, \varrho)$ depending on g . Hence, for $\|g\|_v \leq 1$ it holds that

$$\begin{aligned} & \left| \int g(s) \mu_{\theta+\xi}(ds) - \int g(s) \mu_{\theta}(ds) - \xi \int g(s) \mu'_{\theta}(ds) \right| \\ &= |\xi| \cdot \left| \int g(s) \mu'_{\theta+\xi_g} - \int g(s) \mu'_{\theta} \right| \leq |\xi| \cdot \|\mu'_{\theta+\xi_g} - \mu'_{\theta}\|_v. \end{aligned}$$

Taking in the above inequality the supremum w.r.t. g , we conclude that

$$\|\mu_{\theta+\xi} - \mu_{\theta} - \xi \cdot \mu'_{\theta}\|_v \leq \epsilon |\xi|,$$

provided that $\xi \in (-\varrho, \varrho)$. Dividing both sides by $|\xi|$ and letting $\xi \rightarrow 0$, proves the claim. $\square$

Using Proposition 1, one can easily check, for instance, that for $v(s) = s^p$, for some $p \geq 1$, the exponential distribution is strongly differentiable w.r.t. v -norm.

1.4 Bounds on perturbations

Theorem 2 in Sect. 1.3 establishes strong Lipschitz continuity of weakly differentiable probability measures. For practical purposes one needs the actual Lipschitz constant. The following result shows how a Lipschitz constant can be obtained in explicit form, provided that

$$\sup_{\theta \in \Theta} \|\mu'_{\theta}\|_v < \infty.$$

Theorem 3 *Let μ_{θ} be C_v -differentiable on Θ with $v \in \mathcal{L}^1(\mu_{\theta} : \theta \in \Theta)$ and for $\theta_1, \theta_2 \in \Theta$ s.t. $\theta_1 < \theta_2$ set*

$$\mathbf{L}_{\mu}^v(\theta_1, \theta_2) := \sup_{\theta \in [\theta_1, \theta_2]} \|\mu'_{\theta}\|_v.$$

(i) *Then, for any $g \in C_v$ it holds that*

$$\left| \int g(s) \mu_{\theta_2}(ds) - \int g(s) \mu_{\theta_1}(ds) \right| \leq \mathbf{L}_{\mu}^v(\theta_1, \theta_2) \|g\|_v (\theta_2 - \theta_1). \quad (26)$$

(ii) *In particular, it holds that*

$$\|\mu_{\theta_2} - \mu_{\theta_1}\|_v \leq \mathbf{L}_{\mu}^v(\theta_1, \theta_2) (\theta_2 - \theta_1).$$

(iii) *Let $\mu'_{\theta} = (c_{\theta}, \mu_{\theta}^{+}, \mu_{\theta}^{-})$. If $g \geq 0$ then one can replace in (26) $\mathbf{L}_{\mu}^v(\theta_1, \theta_2)$ by*

$$\mathbf{M}_{\mu}^v(\theta_1, \theta_2) := \sup_{\theta \in [\theta_1, \theta_2]} (c_{\theta} \cdot \max\{\|\mu_{\theta}^{+}\|_v, \|\mu_{\theta}^{-}\|_v\}).$$

Proof Fix $g \in \mathcal{C}_v$. Applying the Mean Value Theorem yields

$$\left| \int g(s) \mu_{\theta_2}(ds) - \int g(s) \mu_{\theta_1}(ds) \right| = (\theta_2 - \theta_1) \left| \int g(s) \mu'_{\theta_g}(ds) \right|, \quad (27)$$

for some $\theta_g \in (\theta_1, \theta_2)$, depending on g . On the other hand,

$$\forall \theta \in (\theta_1, \theta_2) : \left| \int g(s) \mu'_\theta(ds) \right| \leq \|g\|_v \cdot \|\mu'_\theta\|_v \leq \mathbf{L}_\mu^v(\theta_1, \theta_2) \|g\|_v,$$

according to (22), which concludes (i). Taking in (26) the supremum w.r.t. $\|g\|_v \leq 1$, concludes (ii). Finally, for $g \geq 0$ we have

$$\begin{aligned} \left| \int g(s) \mu'_\theta(ds) \right| &= c_\theta \left| \int g(s) \mu_\theta^+(ds) - \int g(s) \mu_\theta^-(ds) \right| \\ &\leq c_\theta \cdot \max \left\{ \int g(s) \mu_\theta^+(ds), \int g(s) \mu_\theta^-(ds) \right\} \\ &\leq c_\theta \cdot \max \{ \|\mu_\theta^+\|_v, \|\mu_\theta^-\|_v \} \|g\|_v \end{aligned}$$

which, together with (27), concludes the proof of (iii). $\square$

In applications one is often interested in bounds of moments of certain random variables, i.e., one considers $v(s) = s^p$, for $s \geq 0$. The following example illustrates how Theorem 3 applies to two usual types of distributions.

Example 6 Let $\mathbb{S} = [0, \infty)$ and $v \in \mathcal{C}(\mathbb{S})$ defined as $v(s) = s^p$, for some $p \geq 0$.

- (i) Let μ_θ denote the exponential distribution with rate θ . Recall the definition of f_θ^+ and f_θ^- given in (9) and (10), respectively. It follows that

$$\begin{aligned} \|\mu'_\theta\|_v &= c_\theta \int v(x) f_\theta^+(x) dx + c_\theta \int v(x) f_\theta^-(x) dx \\ &= \frac{2e^{-1} + p\Gamma_{\sup}(p+1, 1) - p\Gamma_{\inf}(p+1, 1)}{\theta^{p+1}}, \end{aligned}$$

where $\Gamma_{\sup}$ and $\Gamma_{\inf}$ denote the superior (resp., inferior) Gamma functions, which are defined as follows

$$\forall p > 0, x \geq 0 : \Gamma_{\sup}(p, x) = \int_x^\infty s^{p-1} e^{-s} ds, \quad \Gamma_{\inf}(p, x) = \int_0^x s^{p-1} e^{-s} ds.$$

Therefore, the Lipschitz constant $\mathbf{L}_\mu^v(\theta_1, \theta_2)$, in Theorem 3 is given by

$$\mathbf{L}_\mu^v(\theta_1, \theta_2) = \frac{2e^{-1} + p\Gamma_{\sup}(p+1, 1) - p\Gamma_{\inf}(p+1, 1)}{\theta_1^{p+1}}$$

and the constant $\mathbf{M}_\mu^v(\theta_1, \theta_2)$ satisfies

$$\begin{aligned}\mathbf{M}_\mu^v(\theta_1, \theta_2) &= \sup_{\theta \in [\theta_1, \theta_2]} \max \left\{ \frac{e^{-1} - p\Gamma_{\inf}(p+1, 1)}{\theta^{p+1}}, \frac{e^{-1} + p\Gamma_{\sup}(p+1, 1)}{\theta^{p+1}} \right\} \\ &= \frac{e^{-1} + p\Gamma_{\sup}(p+1, 1)}{\theta_1^{p+1}}.\end{aligned}$$

In particular, if $p \geq 0$ is an integer it holds that

$$\mathbf{L}_\mu^v(\theta_1, \theta_2) = \frac{2(1 + pp! \sum_{k=0}^p (1/k!)) - pp!e}{\theta_1^{p+1}e}$$

and

$$\mathbf{M}_\mu^v(\theta_1, \theta_2) = \frac{1 + pp! \sum_{k=0}^p (1/k!)}{\theta_1^{p+1}e}.$$

- (ii) For the uniform distribution ψ_θ on $[0, \theta]$, in accordance with Example 3, we obtain the following Lipschitz constants:

$$\mathbf{L}_\psi^v(\theta_1, \theta_2) = \frac{p+2}{p+1} \theta_2^{p-1}$$

and

$$\mathbf{M}_\psi^v(\theta_1, \theta_2) = \theta_2^{p-1}.$$

Example 6 illustrates the fact that the Lipschitz constants very often depend on the values $\theta_1, \theta_2 \in \Theta$. Thus, from this point of view, notations such as $\mathbf{L}_\mu^v(\theta_1, \theta_2)$ and $\mathbf{M}_\mu^v(\theta_1, \theta_2)$ are justified. However, in what follows we omit specifying the values θ_1, θ_2 when not relevant.

We conclude this section by establishing bounds on perturbations for product measures. This is useful when considering performance measures which depend on a finite collection of random variables, as in (1). To this end, let us consider a finite family of measure-valued mappings $\{\mu_{i,\theta} : \theta \in \Theta\} \subset \mathcal{M}^+(\mathbb{S}_i)$, a family of non-negative, continuous mappings $v_i \in \mathcal{C}^+(\mathbb{S}_i)$ and recall the definitions of $J_g(\theta)$, Π_θ and V given in (1), (11) and (12), respectively.

Theorem 4 *Let $\mu_{i,\theta}$ be weakly $\mathcal{C}_{v_i}$ -differentiable on Θ , for each $1 \leq i \leq n$ and for arbitrary $\theta_1, \theta_2 \in \Theta$, such that $\theta_1 < \theta_2$ set*

$$\forall 1 \leq i \leq n : \mathbf{L}^i = \sup_{\theta \in [\theta_1, \theta_2]} \|\mu'_{\theta,i}\|_{v_i}.$$

(i) Then, for each $g \in \mathcal{C}_V$ it holds that

$$|J_g(\theta_2) - J_g(\theta_1)| \leq \mathbf{L}^* \|g\|_V (\theta_2 - \theta_1), \quad (28)$$

where

$$\mathbf{L}^* = \sum_{i=1}^n \left(\mathbf{L}^i \prod_{j=1}^{i-1} \|\mu_{j,\theta_2}\|_{v_j} \prod_{k=i+1}^n \|\mu_{k,\theta_1}\|_{v_k} \right) \quad (29)$$

and we agree that a void product, such as $\prod_{j=1}^0 \|\mu_{j,\theta_2}\|_{v_j}$, equals to 1.

(ii) In particular, it holds that

$$\|\Pi_{\theta_2} - \Pi_{\theta_1}\|_V \leq \mathbf{L}^* (\theta_2 - \theta_1).$$

(iii) If $g \geq 0$ and for $1 \leq i \leq n$, $\mu'_{i,\theta} = (c_{i,\theta}, \mu_{i,\theta}^+, \mu_{i,\theta}^-)$ then the constant $\mathbf{L}^*$ in (28) can be improved by replacing in (29) $\mathbf{L}^i$ by

$$\mathbf{M}^i = \sup_{\theta \in [\theta_1, \theta_2]} (c_{i,\theta} \cdot \max\{\|\mu_{i,\theta}^+\|_v, \|\mu_{i,\theta}^-\|_v\}).$$

Proof For arbitrary $g \in \mathcal{C}_V$, we have

$$\forall (s_1, \dots, s_n) : |g(s_1, \dots, s_n)| \leq \|g\|_V \cdot V(s_1, \dots, s_n) = \|g\|_V \cdot v(s_1) \cdot \dots \cdot v(s_n).$$

Therefore, for each $1 \leq i \leq n$, the mapping g_i defined as

$$g_i(s_i) = \int \dots \int g(s_1, \dots, s_n) \prod_{j=1}^{i-1} \mu_{j,\theta_2}(ds_j) \prod_{k=i+1}^n \mu_{k,\theta_1}(ds_k),$$

belongs to $\mathcal{C}_{v_i}$ since, according to Fubini's Theorem, it holds that

$$\|g_i\|_{v_i} \leq \|g\|_V \cdot \prod_{j=1}^{i-1} \|\mu_{j,\theta_2}\|_{v_j} \cdot \prod_{k=i+1}^n \|\mu_{k,\theta_1}\|_{v_k}. \quad (30)$$

Since $\mu_{i,\theta}$ is weakly $\mathcal{C}_{v_i}$ -differentiable, from Theorem 3 (i) we conclude that

$$\left| \int g_i(s_i) \mu_{i,\theta_2}(ds_i) - \int g_i(s_i) \mu_{i,\theta_1}(ds_i) \right| \leq \mathbf{L}^i \|g_i\|_{v_i} (\theta_2 - \theta_1). \quad (31)$$

On the other hand, simple algebraic calculations show that

$$J_g(\theta_2) - J_g(\theta_1) = \sum_{i=1}^n \left(\int g_i(s_i) \mu_{i,\theta_2}(ds_i) - \int g_i(s_i) \mu_{i,\theta_1}(ds_i) \right).$$

Hence, (31) and (30) imply that

$$|J_g(\theta_2) - J_g(\theta_1)| \leq \mathbf{L}^* \|g\|_V (\theta_2 - \theta_1)$$

holds true, for $\mathbf{L}^*$ defined by (29), which concludes (i). Finally, a similar reasoning as in Theorem 3 concludes the proof of (ii) and (iii). $\square$

Remark 5 If $\mu_{i,\theta}$ is $\mathcal{C}_{v_i}$ -differentiable, with derivative $\mu'_{i,\theta}$, we conclude from Theorem 1 that $J_g(\theta)$ is differentiable w.r.t. θ , for each $g \in \mathcal{C}_V$. Therefore, one could also derive a Lipschitz bound for $J_g(\theta)$ by bounding the derivative $J'_g(\theta)$ which, according to Theorem 3 can be written as

$$\sum_{i=1}^n \int \dots \int g(s_1, \dots, s_n) \mu'_{i,\theta}(ds_i) \prod_{j \neq i} \mu_{j,\theta}(ds_j).$$

Using a similar reasoning as in Theorem 4 one would obtain in (28) the following Lipschitz bound:

$$\mathbf{L}' = \sum_{i=1}^n \sup_{\theta \in [\theta_1, \theta_2]} \left(\|\mu'_{i,\theta}\|_{v_i} \prod_{j \neq i} \|\mu_{j,\theta}\|_{v_j} \right)$$

which, in general, is less accurate (larger) than $\mathbf{L}^*$ defined in (29).

Corollary 1 *Under the conditions put forward in Theorem 4, if $\mu_{i,\theta} = \mu_\theta$ and $v_i = v$, for each $1 \leq i \leq n$, then*

$$\mathbf{L}^* = \mathbf{L}_v^\mu \sum_{k=1}^n \|\mu_{\theta_2}\|_v^{k-1} \|\mu_{\theta_1}\|_v^{n-k}, \quad \mathbf{M}^* = \mathbf{M}_v^\mu \sum_{k=1}^n \|\mu_{\theta_2}\|_v^{k-1} \|\mu_{\theta_1}\|_v^{n-k}.$$

1.5 Application: sensitivity of the average price

In this section, we consider the following simple example from finance. Assume that an investor is purchasing $S > 0$ units worth of stock, each month, for $n \geq 1$ months in a row. If we denote by X_k the spot price per share in month k , for $1 \leq k \leq n$, then the amount purchased in month k equals to S/X_k . Hence, the average price per share X_a , he or she pays over the n months, is obtained by dividing the total amount of wealth spent divided by the total number of shares purchased. That is

$$X_a = \frac{n \cdot S}{\frac{S}{X_1} + \dots + \frac{S}{X_n}} = \frac{n}{\frac{1}{X_1} + \dots + \frac{1}{X_n}},$$

i.e., the average price is just the harmonic mean of the spot prices $X_1, \dots, X_n$.

Let us fix $n \geq 1$ and assume that $\{X_i : 1 \leq i \leq n\}$ are i.i.d. random variables with distribution μ_θ , depending on some parameter θ . One is interested in studying

the sensitivity of the expected average price w.r.t. θ , i.e., to obtain a bound for the perturbation

$$\Delta(\theta_1, \theta_2) := |\mathbb{E}_{\theta_2}[X_a] - \mathbb{E}_{\theta_1}[X_a]|,$$

for some $\theta_1, \theta_2 \in \Theta$, s.t. $\theta_1 < \theta_2$. To this end, note that the expected average price can be written as

$$\mathbb{E}_{\theta}[X_a] = J_h(\theta) = \int \dots \int h(x_1, \dots, x_n) \mu_{\theta}(dx_1) \dots \mu_{\theta}(dx_n), \quad (32)$$

where h is defined as

$$\forall x_1, \dots, x_n > 0 : h(x_1, \dots, x_n) = \frac{n}{\frac{1}{x_1} + \dots + \frac{1}{x_n}}.$$

Letting $v \in \mathcal{C}((0, \infty))$, $v(x) = \sqrt[n]{x}$, it holds that

$$\forall x_1, \dots, x_n > 0 : h(x_1, \dots, x_n) \leq \sqrt[n]{x_1 \cdots x_n} = V(x_1, \dots, x_n),$$

where the inequality follows from the well known fact that the harmonic mean of a collection of positive numbers $x_1, \dots, x_n$ is smaller than or equal to the geometric mean. Therefore, provided that μ_{θ} is weakly $\mathcal{C}_v$ -differentiable on Θ , Corollary 1 applies and since $h \geq 0$ it holds that; see Theorem 4, (iii)

$$\begin{aligned} \Delta(\theta_1, \theta_2) &\leq \mathbf{M}^*(\theta_2 - \theta_1) \\ \mathbf{M}^* &= \mathbf{M}_v^{\mu} \sum_{k=1}^n \|\mu_{\theta_2}\|_v^{k-1} \|\mu_{\theta_1}\|_v^{n-k}. \end{aligned} \quad (33)$$

Example 7 If, for instance, μ_{θ} is the exponential distribution with rate θ , introduced in Example 2 then we have

$$\forall \theta \in \Theta : \|\mu_{\theta}\|_v = \frac{1}{\sqrt[n]{\theta}} \Gamma\left(\frac{n+1}{n}\right),$$

for $v(x) = \sqrt[n]{x}$, where Γ denotes the usual Gamma function. Therefore, in accordance with Example 6, we obtain the following Lipschitz constant in (33):

$$\mathbf{M}^* = \frac{\frac{1}{e} + \frac{1}{n} \Gamma_{\sup}\left(\frac{n+1}{n}, 1\right)}{\sqrt[n]{\theta_1^{n+1}}} \cdot \frac{\theta_2 - \theta_1}{\sqrt[n]{\theta_2} - \sqrt[n]{\theta_1}} \cdot \left(\frac{\Gamma\left(\frac{n+1}{n}\right)}{\sqrt[n]{\theta_1 \theta_2}}\right)^{n-1}.$$

2 Perturbation of waiting times in $G/G/1$ queue

Throughout this section we aim to derive bounds on perturbations for the waiting times in the $G/G/1$ queue. To this end we use the Markov chain description of the sequence of waiting times. A brief overview of Markov operators is provided in Sect. 2.1. In Sect. 2.2, we extend the concept of weak differentiation to Markov operators and we apply the results to derive bounds on perturbations for the transient waiting time in Sect. 2.3.

2.1 Markov chain description

Recall that a *transition kernel* on $\mathbb{S}$ is a mapping $Q : \mathbb{S} \times \mathcal{S} \rightarrow \mathbb{R}$ satisfying:

- (i) $\forall E \in \mathcal{S}$, the mapping $Q(\cdot, E)$ is measurable,
- (ii) $\forall s \in \mathbb{S}$, $Q(s, \cdot) \in \mathcal{M}(\mathcal{S})$.

If $Q(s, \cdot) \in \mathcal{M}^1$, for all $s \in \mathbb{S}$, we call Q a *probability transition kernel* (or Markov operator). A transition kernel Q can be identified with a linear operator (denoted also by Q) on the set of measurable mappings on $\mathbb{S}$, defined as:

$$\forall s \in \mathbb{S} : (Qf)(s) := \int f(x) Q(s, dx),$$

for all f for which the right-hand side integral makes sense.

Example 8 We revisit Example 4. From Lindley's recursion we conclude that, for all measurable f it holds that

$$\mathbb{E}[f(W_{n+1}) | W_n] = \mathbb{E}[f(\max\{W_n + S_n - A_{n+1}, 0\}) | W_n].$$

The Markov operator attached to the sequence of waiting times is thus given by:

$$\forall x \in \mathbb{S} : (Qf)(x) = \int \int f((x + s - t)_+) \eta(dt) \mu(ds),$$

for suitable f where, we set $y_+ := \max\{y, 0\}$, for $y \in \mathbb{R}$.

For $v \in C^+$ we introduce the v -norm of Q as follows:

$$\|Q\|_v := \sup_{\|f\|_v \leq 1} \|Qf\|_v = \sup_{\|f\|_v \leq 1} \sup_{s \in \mathbb{S}} \frac{|(Qf)(s)|}{v(s)}.$$

Note that, if $\|Q\|_v < \infty$, then it holds that:

$$\forall f \in C_v : \|Qf\|_v \leq \|Q\|_v \cdot \|f\|_v. \quad (34)$$

Example 9 Let us consider the Markov operator Q introduced in Example 8 and let $v(x) = e^{\alpha x}$, for some $\alpha \geq 0$. Then

$$\|Q\|_v = \sup_{x \geq 0} e^{-\alpha x} \cdot \mathbb{E} \left[e^{\alpha(x+S-A)_+} \right] = \sup_{x \geq 0} \mathbb{E} \left[e^{\alpha[(x+S-A)_+ - x]} \right],$$

where S and A are r.v. distributed according to μ and η , respectively. Since the mapping $x \mapsto \alpha[(x+S-A)_+ - x]$ is non-increasing, a simple application of the Dominated Convergence Theorem yields:

$$\|Q\|_v = \mathbb{E} \left[e^{\sup_{x \geq 0} \alpha[(x+S-A)_+ - x]} \right] = \mathbb{E} \left[e^{\alpha(S-A)_+} \right],$$

provided that the right-hand side expectation above is finite.

We conclude this section by presenting some facts related to composition of kernels. Namely, let Q_1, Q_2 be transition kernels on $\mathbb{S}$ and for $s \in \mathbb{S}$ let:

$$(Q_2 Q_1)(s, \cdot) := \int Q_2(x, \cdot) Q_1(s, dx).$$

It is straightforward that $Q_2 Q_1$ is a transition kernel on $\mathbb{S}$. Consequently, the induced operator is given by:

$$(Q_2 Q_1 f)(s) = \int f(y) (Q_2 Q_1)(s, dy) = \int \int f(y) Q_2(x, dy) Q_1(s, dx).$$

One can easily check that $Q_2 Q_1 f = Q_2(Q_1 f)$ and according to (34) we have:

$$\forall f : \|Q_2 Q_1 f\|_v \leq \|Q_2\|_v \cdot \|Q_1 f\|_v \leq \|Q_2\|_v \|Q_1\|_v \|f\|_v.$$

Taking the supremum w.r.t. $\|f\|_v \leq 1$ yields:

$$\|Q_2 Q_1\|_v \leq \|Q_2\|_v \cdot \|Q_1\|_v.$$

In particular, if we set $Q^n = \underbrace{Q \dots Q}_n$, for $n \geq 1$, it holds that $\|Q^n\|_v \leq \|Q\|_v^n$.

2.2 Weak differentiation of Markov operators

Let $\{Q_\theta : \theta \in \Theta\}$ be a family of Markov operators on $\mathbb{S}$, for some $\Theta \subset \mathbb{R}$. We say that Q_θ is weakly $\mathcal{C}_v$ -differentiable if there exist a transition kernel Q'_θ , s.t.

$$\forall f \in \mathcal{C}_v, \forall x \in \mathbb{S} : \frac{d}{d\theta} (Q_\theta f)(x) = (Q'_\theta f)(x).$$

For instance, let $\mathbb{S} = [0, \infty)$, $\{\mu_\theta : \theta \in \Theta\} \subset \mathcal{M}^1(\mathbb{S})$ and denote by Q_θ the Markov operator introduced in Example 8, for $\mu = \mu_\theta$. That is:

$$\forall f, \forall x \in \mathbb{S} : (Q_\theta f)(x) = \int \int f((x+s-t)_+) \eta(dt) \mu_\theta(ds). \quad (35)$$

It turns out that weak differentiability of Q_θ is related to that of μ_θ and one can differentiate Q_θ as follows:

$$\forall f \in \mathcal{C}_v, \forall x \in \mathbb{S} : (Q'_\theta f)(x) = \int \int f((x+s-t)_+) \eta(dt) \mu'_\theta(ds). \quad (36)$$

This relation will be established by our next result. Specifically, we present a class of mappings v for which $\mathcal{C}_v$ -differentiability of μ_θ implies that of Q_θ .

Lemma 2 *Let $\{\mu_\theta : \theta \in \Theta\} \subset \mathcal{M}^1(\mathbb{S})$, $v \in \mathcal{C}(\mathbb{S})$ be a positive mapping and let Q_θ be defined as in (35). If:*

- (i) μ_θ is weakly $\mathcal{C}_v$ -differentiable,
- (ii) for each $x \in \mathbb{S}$ the mapping v satisfies

$$\sup_{s \in \mathbb{S}} \frac{\int v((x+s-t)_+) \eta(dt)}{v(s)} < \infty,$$

then Q_θ is weakly $\mathcal{C}_v$ -differentiable and (36) holds true.

Proof For $s, x \in \mathbb{S}$ and $f \in \mathcal{C}_v$ let

$$H_f(s, x) = \int f((x+s-t)_+) \eta(dt).$$

Condition (ii) essentially says that $\|H_f(\cdot, x)\|_v < \infty$ provided that $\|f\|_v < \infty$, for all $x \in \mathbb{S}$. To conclude the proof one has to show that $H_f(\cdot, x) \in \mathcal{C}_v$ for $f \in \mathcal{C}_v$, i.e., continuity of f implies continuity of $H_f(\cdot, x)$, for each $x \in \mathbb{S}$. To this end, assume that f is continuous, let $s \geq 0$ be fixed and let $\epsilon > 0$ be arbitrary. Since continuity of f implies uniform continuity on each compact set (see, e.g., Rudin 1976) it follows that there exists some $\varrho_\epsilon > 0$ such that for each $s_1, s_2 \in [0, x+s+1]$ it holds that

$$|s_2 - s_1| < \varrho_\epsilon \Rightarrow |f(s_2) - f(s_1)| < \epsilon.$$

Therefore, it follows that for any $x \geq 0$ and $|r| < \min\{1, \varrho_\epsilon\}$ we have

$$\begin{aligned} |H_f(s+r, x) - H_f(s, x)| &= \left| \int f((x+s+r-t)_+) - f((x+s-t)_+) \eta(dt) \right| \\ &\leq \int |f((x+s+r-t)_+) - f((x+s-t)_+)| \eta(dt) \\ &\leq \int \epsilon \mathbb{I}_{[0, x+s+1]}(t) \eta(dt) = \epsilon \eta([0, x+s+1]), \end{aligned}$$

where we used the fact that for $t \geq x + s + 1$ and $|r| < 1$ we have

$$(x + s + r - t)_+ = (x + s - t)_+ = 0.$$

Since ϵ was arbitrary chosen it follows that $H_f(\cdot, x)$ is continuous. Hence,

$$\forall x \in \mathbb{S}, \forall f \in \mathcal{C}_v : \frac{d}{d\theta}(Q_\theta f)(x) = \frac{d}{d\theta} \int H_f(s, x) \mu_\theta(ds) \stackrel{(i)}{=} \int H_f(s, x) \mu'_\theta(ds),$$

which concludes (36). $\square$

Example 10 Let $v(x) = e^{\alpha x}$, for some $\alpha \geq 0$. Since for $s, t, x \geq 0$ it holds that

$$e^{\alpha(x+s-t)_+} \leq e^{\alpha(x+s)},$$

we obtain

$$\forall x \geq 0 : \sup_{s \in \mathbb{S}} \frac{\int e^{\alpha(x+s-t)_+} \eta(dt)}{e^{\alpha s}} \leq e^{\alpha x} < \infty.$$

Hence, provided that μ_θ is weakly $\mathcal{C}_v$ -differentiable, Lemma 2 applies and we conclude that Q_θ is weakly $\mathcal{C}_v$ -differentiable, as well. Moreover, it holds that:

$$\begin{aligned} \|Q'_\theta f\|_v &= \sup_{x \geq 0} e^{-\alpha x} \left| \int \int f((x + s - t)_+) \eta(dt) \mu'_\theta(ds) \right| \\ &\leq \sup_{x \geq 0} \int \int e^{\alpha[(x+s-t)_+ - x]} |\mu'_\theta|(ds) \eta(dt) \\ &= \int \int e^{\alpha(s-t)_+} |\mu'_\theta|(ds) \eta(dt) \\ &= c_\theta \mathbb{E}_\theta \left[e^{\alpha(S^+ - A)_+} + e^{\alpha(S^- - A)_+} \right], \end{aligned}$$

provided that $\|f\|_v \leq 1$, where $\mathbb{E}_\theta$ denotes the expectation operator when S^+ and S^- are distributed according to μ_θ^+ and μ_θ^- , respectively.

Taking the supremum w.r.t. $\|f\|_v \leq 1$, yields

$$\|Q'_\theta\|_v \leq c_\theta \mathbb{E}_\theta \left[e^{\alpha(S^+ - A)_+} + e^{\alpha(S^- - A)_+} \right]. \quad (37)$$

2.3 The transient waiting time

In what follows we apply the results from previous section in order to derive bounds on perturbations for the expected waiting times in a $G/G/1$ queue.

Let us consider the $G/G/1$ queue, as introduced in Example 8 and let us assume that the service time distribution μ_θ depends on some design parameter $\theta \in \Theta$. Assuming

that $\theta_1 < \theta_2$, we aim to establish bounds for

$$\forall x, f, n : \Delta_n^{x,f}(\theta_1, \theta_2) := \left| \mathbb{E}_{\theta_2} [f(W_{n+1}^x)] - \mathbb{E}_{\theta_1} [f(W_{n+1}^x)] \right|, \quad (38)$$

for arbitrary $n \geq 1$, $x \geq 0$ and $f \in \mathcal{C}_v$, where W_n^x denotes the n th waiting time, when $W_1^x = x$ and $\mathbb{E}_\theta$ denotes the expectation operator, when the service times follow distribution μ_θ . To do so, let us consider the Markov operator Q_θ as introduced in (35) and note that, for any initial state $x \geq 0$, it holds that:

$$\forall n \geq 1, \forall g : \mathbb{E}_\theta [g(W_{n+1}^x)] = \mathbb{E} [\mathbb{E}_\theta [g(W_{n+1}^x) | W_1^x]] = (Q_\theta^n g)(x).$$

Hence, the expression in (38) can be re-written and bounded as follows:

$$\left| (Q_{\theta_2}^n f)(x) - (Q_{\theta_1}^n f)(x) \right| \leq \|Q_{\theta_2}^n - Q_{\theta_1}^n\|_v \cdot \|f\|_v \cdot v(x). \quad (39)$$

Elementary algebraic calculations show that:

$$Q_{\theta_2}^n - Q_{\theta_1}^n = \sum_{k=1}^n Q_{\theta_2}^{n-k} (Q_{\theta_2} - Q_{\theta_1}) Q_{\theta_1}^{k-1},$$

where we agree that Q^0 is the identity operator⁵. Hence:

$$\|Q_{\theta_2}^n - Q_{\theta_1}^n\|_v \leq \sum_{k=1}^n \|Q_{\theta_2}\|_v^{n-k} \|Q_{\theta_2} - Q_{\theta_1}\|_v \|Q_{\theta_1}\|_v^{k-1}. \quad (40)$$

Provided that Q_θ is $\mathcal{C}_v$ -differentiable on Θ , one can apply the Mean Value Theorem to the mapping $\theta \mapsto (Q_\theta g)(x)$, which yields

$$\forall x \in \mathbb{S}, g \in \mathcal{C}_v : (Q_{\theta_2} g)(x) - (Q_{\theta_1} g)(x) = (\theta_2 - \theta_1) \cdot (Q'_\theta g)(x),$$

for some $\theta \in (\theta_1, \theta_2)$, depending on g and x . Thus, for $\|g\|_v \leq 1$ we have:

$$\|(Q_{\theta_2} g) - (Q_{\theta_1} g)\|_v \leq (\theta_2 - \theta_1) \|Q'_\theta\|_v \leq (\theta_2 - \theta_1) \sup_{\theta \in [\theta_1, \theta_2]} \|Q'_\theta\|_v.$$

Consequently, taking the supremum w.r.t. $\|g\|_v \leq 1$ yields:

$$\|Q_{\theta_2} - Q_{\theta_1}\|_v \leq (\theta_2 - \theta_1) \sup_{\theta \in [\theta_1, \theta_2]} \|Q'_\theta\|_v. \quad (41)$$

Therefore, from (39) together with (40) and (41) we conclude that

$$\left| (Q_{\theta_2}^n f)(x) - (Q_{\theta_1}^n f)(x) \right| \leq \mathbf{C}_n \mathbf{L} \|f\|_v (\theta_2 - \theta_1) v(x), \quad (42)$$

⁵ The identity operator corresponds to ‘Dirac’ transition kernel $\mathbf{1}(x, A) = \delta_x(A)$. In particular, $\mathbf{1}f = f$, for all measurable f .

where

$$\mathbf{C}_n = \sum_{k=1}^n \|Q_{\theta_2}\|_v^{n-k} \|Q_{\theta_1}\|_v^{k-1}, \quad \mathbf{L} = \sup_{\theta \in [\theta_1, \theta_2]} \|Q'_\theta\|_v.$$

Finally, we obtain the following bound for the expression in (38):

$$\Delta_n^{x,f}(\theta_1, \theta_2) \leq \mathbf{C}_n \mathbf{L} \|f\|_v (\theta_2 - \theta_1) v(x). \quad (43)$$

Example 11 Let us consider a $M/U/1$ queue where service times have uniform distribution ψ_θ , on $[0, \theta]$ and inter-arrival times are exponentially distributed with rate λ , i.e., the Markov operator Q_θ is given by

$$\forall f, \forall x \in \mathbb{S} : (Q_\theta f)(x) = \frac{1}{\theta} \int_0^\theta \int_0^\infty f((x+s-t)_+) \lambda e^{-\lambda t} dt ds.$$

Then, according to Example 9, for $v(x) = e^{\alpha x}$, for some $\alpha \geq 0$, it holds that

$$\forall \lambda, \theta, \alpha : \|Q_\theta\|_v = \mathbb{E}_\theta \left[e^{\alpha(S-A)_+} \right] = \frac{\lambda^2 [e^{\alpha\theta} - 1] + \alpha^2 [1 - e^{-\lambda\theta}]}{\alpha\lambda(\alpha + \lambda)\theta}.$$

Similarly, according to (37) we conclude that

$$\forall \lambda, \theta, \alpha : \|Q'_\theta\|_v \leq \frac{\lambda^2 [(1 + \alpha\theta)e^{\alpha\theta} - 1] + \alpha^2 [1 - (1 + \lambda\theta)e^{-\lambda\theta}]}{\alpha\lambda(\alpha + \lambda)\theta}$$

and a bound for the perturbation in (38) can be obtained according to (43).

To illustrate the above findings we let $\alpha = \lambda$ and we obtain

$$\forall \lambda, \theta : \|Q_\theta\|_v = \frac{1}{\lambda\theta} \sinh(\lambda\theta), \quad \|Q'_\theta\|_v \leq \frac{1 + \lambda\theta}{\lambda\theta} \sinh(\lambda\theta),$$

where $\sinh$ denotes the hyperbolic sine function. Consequently, we have

$$\mathbf{C}_n = \sum_{k=1}^n \left(\frac{\sinh(\lambda\theta_2)}{\lambda\theta_2} \right)^{n-k} \left(\frac{\sinh(\lambda\theta_1)}{\lambda\theta_1} \right)^{k-1}, \quad \mathbf{L} \leq \frac{1 + \lambda\theta_2}{\lambda\theta_2} \sinh(\lambda\theta_2),$$

where we use the fact that the function $x \mapsto \frac{1+x}{x} \sinh(x)$ is non-decreasing in x . Substituting the above constants in (43) yields

$$\frac{\Delta_n^{x,f}(\theta_1, \theta_2)}{(\theta_2 - \theta_1)} \leq \|f\|_v (1 + \lambda\theta_2) e^{\lambda x} \sum_{k=1}^n \left(\frac{\sinh(\lambda\theta_2)}{\lambda\theta_2} \right)^k \left(\frac{\sinh(\lambda\theta_1)}{\lambda\theta_1} \right)^{n-k}.$$

3 Summary

We studied bounds on cost functions over finite products of probability measures. The main contribution of our analysis are the following. We show that the classical v -norm can be extended to mappings v such that $\inf_{s \in \mathbb{S}} v(s) = 0$. We provide a thorough study of the relation between weak differentiability and Fréchet differentiability with respect to the v -norm. In particular, by applying a fundamental result from the theory of Banach spaces, we show that weak differentiability implies v -norm Lipschitz continuity. This result leads to bounds on the sensitivity of finite products of probability measures in norm sense. We illustrate the application of these results by providing bounds for perturbations of the n th waiting time in an G/G/1 queue. The extension of our bound to the stationary waiting time is topic of further research.

Open Access This article is distributed under the terms of the Creative Commons Attribution Noncommercial License which permits any noncommercial use, distribution, and reproduction in any medium, provided the original author(s) and source are credited.

Appendix

I. Uniqueness of the $\mathcal{C}_v$ -limit

The $\mathcal{C}_v$ -limit in (2) is in general not unique. Indeed, let us consider $\mathbb{S} = [0, \infty)$ endowed with the usual metric, and $v(s) = s$, for all $s \in \mathbb{S}$, and denote by δ_0 the Dirac measure, i.e., δ_0 assigns mass 1 to point 0. Assume that μ is a $\mathcal{C}_v$ -limit of the sequence $\{\mu_n\}_n \subset \mathcal{M}$. Since for $g \in \mathcal{C}_v$ we have $g(0) = 0$, $\mu + \alpha \cdot \delta_0 \in \mathcal{M}$ is also a $\mathcal{C}_v$ -limit of the sequence $\{\mu_n\}_n$, for each $\alpha \in \mathbb{R}$ and the $\mathcal{C}_v$ -limit fails to be unique. In words, (2) still holds true if one assigns a different mass on the ‘zero set’ of v . Our next result will elucidate this uniqueness issue. In particular it shows that the set $\mathcal{C}_v$, likewise $\mathcal{C}_B$, is appropriate for introducing weak convergence.

Lemma 3 *Let $v \in \mathcal{C}$ and $v \geq 0$. Then the set $\mathcal{C}_v$ is a Banach space, when endowed with the v -norm and the $\mathcal{C}_v$ -limit is uniquely determined on $\mathbb{S}_v$.*

Proof Let $\Phi : \mathcal{C}_v \rightarrow \mathcal{C}_B$ be defined by

$$(\Phi g)(s) = \begin{cases} \frac{g(s)}{v(s)} & s \in \mathbb{S}_v; \\ \lim_{t \rightarrow s} \frac{g(t)}{v(t)} & \text{otherwise.} \end{cases}$$

Then, Φ is an isometry⁶, i.e., a bijection such that

$$\|\Phi g\|_\infty = \|g\|_v.$$

Since, $\mathcal{C}_B$ equipped with the $\|\cdot\|_\infty$ -norm is a Banach space, it follows that $\mathcal{C}_v$ equipped with the $\|\cdot\|_v$ -norm is a Banach space as well; For details, see Semadeni (1971).

⁶ The condition $v \in \mathcal{C}$ ensures that $\Phi g \in \mathcal{C}$, for $g \in \mathcal{C}$.

Let now $\{\mu_n\}_n \subset \mathcal{M}$ be such that $v \in \mathcal{L}^1(\mu_n : n \in \mathbb{N})$ and $\mu, \eta \in \mathcal{M}$ be $\mathcal{C}_v$ -limits of the sequence $\{\mu_n\}_n$. We will show that:

$$\forall E \in \mathcal{S} : E \subset \mathbb{S}_v \Rightarrow \mu(E) = \eta(E). \quad (44)$$

First, since μ and η are $\mathcal{C}_v$ -limits of the same sequence, (2) yields:

$$\forall g \in \mathcal{C}_v : \int g(s) \mu(ds) = \int g(s) \eta(ds). \quad (45)$$

Next, for arbitrary $E \in \mathcal{S}$ let $\mathbb{I}_E$ denote its indicator function and set, for $\gamma > 0$:

$$h_\gamma^E(s) = \min\{\gamma \cdot v(s), \mathbb{I}_E(s)\}, \forall s \in \mathbb{S}.$$

The fact that $|h_\gamma^E(s)| \leq \gamma \cdot v(s)$, for $s \in \mathbb{S}$, together with $E \subset \mathbb{S}_v$ implies $h_\gamma^E \in \mathcal{C}_v$. For $v \in \mathcal{L}^1(\mu, \eta)$ it follows that $h_\gamma^E \in \mathcal{L}^1(\mu, \eta)$, for all $\gamma > 0$. Moreover, it holds that:

$$\forall s \in \mathbb{S}, \forall E \in \mathcal{S} : \lim_{\gamma \rightarrow \infty} h_\gamma^E(s) = \begin{cases} \mathbb{I}_E(s) & s \in \mathbb{S}_v \\ 0 & \text{otherwise.} \end{cases}$$

Hence $h_\gamma^E \uparrow \mathbb{I}_{E \cap \mathbb{S}_v}$, for $\gamma \uparrow \infty$, and the Monotone Convergence Theorem together with (45) yields:

$$\forall E \subset \mathbb{S}_v : \mu(E) = \lim_{\gamma \rightarrow \infty} \int h_\gamma^E(s) \mu(ds) = \lim_{\gamma \rightarrow \infty} \int h_\gamma^E(s) \eta(ds) = \eta(E),$$

which concludes (44). $\square$

The above result can be phrased as follows: the trace of the $\mathcal{C}_v$ -limit μ is unique on the set $\mathbb{S}_v$. More formally, if we denote by $\equiv$ the equivalence relation on $\mathcal{M}$ given by

$$\mu \equiv \eta \quad \text{iff} \quad \forall E \subset \mathbb{S}_v : \mu(E) = \eta(E),$$

then the $\mathcal{C}_v$ -limit is uniquely determined up to this equivalence relation and the precise definition of the $\mathcal{C}_v$ -limit would be in terms of the equivalence class of μ , denoted by $[\mu]$. Note that since $g \in \mathcal{C}_v$ implies $g(s) = 0$ for $s \notin \mathbb{S}_v$, the behavior of μ outside $\mathbb{S}_v$ is not relevant for our analysis and we may, with slight abuse of notation, identify μ and $[\mu]$. Lemma 3 shows that:

$$[\mu_n](\mathbb{S}) = 0, \forall n \in \mathbb{N} \Rightarrow [\mu](\mathbb{S}) = 0. \quad (46)$$

In fact, the algebraic dual space $\mathcal{C}_v^*$ of $\mathcal{C}_v$ (i.e., the set of all linear functionals on $\mathcal{C}_v$) is the quotient space of $\mathcal{M}$ with respect to equivalence relation $\equiv$.

References

- Aissani D, Bouallouche-Medjkoune L (2006) Performance analysis approximation in a queueing system of type M/G/1. *Math Methods Oper Res* 63(2):341–356
- Billingsley B (1966) Weak convergence of probability measures. Wiley, New York
- Cho GE, Meyer CD (2001) Comparison of perturbation bounds for the stationary distribution of a Markov chain. *Linear Algebra Appl* 335:137–150
- Dekker R, Hordijk A (1983) Average, sensitive and Blackwell optimal policies in denumerable Markov decision chains with unbounded rewards. Report no.83-36, Institute of Applied Mathematics and Computing Science
- Dekker R, Hordijk A (1988) Average, sensitive and Blackwell optimal policies in denumerable Markov decision chains with unbounded rewards. *Math Oper Res* 13:395–421
- Heidergott B, Hordijk A (2003) Taylor series expansions for stationary Markov chains. *Adv Appl Probab* 23:1046–1070
- Heidergott B, Hordijk A (2004) Single-run gradient estimation via measure-valued differentiation. *IEEE Trans Automat Control* 49:1843–1846
- Heidergott B, Leahu H (2008) Differentiability of product measures. FEWEB Research Memorandum 2008-5, Vrije Universiteit Amsterdam, The Netherlands
- Hordijk A, Yushkevich AA (1999) Blackwell optimality in the class of all policies in Markov decision chains with a Borel state space and unbounded rewards. *Math Oper Res* 50:421–448
- Kartashov N (1996) Strong stable Markov chains. VSP, Utrecht
- Kushner H, Vázquez-Abad F (1992) Estimation of the derivative of a stationary measure with respect to a control parameter. *J Appl Probab* 29:343–352
- Lipman S (1974) On dynamic programming with unbounded rewards. *Mange Sci* 21:1225–1233
- Meyn SP, Tweedie RL (1993) Markov chains and stochastic stability. Springer, London
- Pflug G (1996) Optimization of stochastic models. Kluwer, Boston
- Rudin W (1976) Principles of mathematical analysis, 3rd edn. McGraw-Hill, Tokyo
- Schweitzer PJ (1968) Perturbation theory and finite Markov chains. *J Appl Prob* 5:401–413
- Semadeni Z (1971) Banach spaces of continuous functions. Polish scientific publishers, Warszawa