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ORIGINAL ARTICLE

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# Tandem Brownian queues

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**Abstract** We analyze a two-node tandem queue with Brownian input. We first derive an explicit expression for the joint distribution function of the workloads of the first and second queue, which also allows us to calculate their exact large-buffer asymptotics. The nature of these asymptotics depends on the model parameters, i.e., there are different regimes. By using sample-path large-deviations (Schilder's theorem) these regimes can be interpreted: we explicitly characterize the most likely way the buffers fill.

## 1 Introduction

Consider $\{B(t) - ct, t \geq 0\}$, where $B(t)$ is a standard Brownian motion, and $c > 0$ is a scalar. The distribution of the supremum $\overline{B}_c$ of such a Brownian motion with drift is known: $\mathbb{P}(\overline{B}_c > b) = \exp(-2bc)$. The reflection of $\{B(t) - ct, t \geq 0\}$ at 0 could be called a *Brownian queue*. It can be argued (see Reich (1958)) that the steady-state workload $Q$ of such a Brownian queue is distributed as $\overline{B}_c$, i.e., also exponentially with mean $1/(2c)$.

The case of *networks* of Brownian queues is considerably less studied. In Mandjes (2004) and Dębicki et al. (2007a) a two-node tandem queue is analyzed: Mandjes (2004) derives the joint distribution function of the first and total queue length,

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whereas Dębicki et al. (2007a) focuses on the distribution function of the second queue. Also, several papers consider the more general case of tandem systems with Lévy input, i.e., arrival processes with stationary, independent increments (this class comprises, besides Brownian motion, also compound Poisson input). We remark that the solution presented in Kella and Whitt (1992) and Dębicki et al. (2007b) is in terms of a joint Laplace transform; no explicit expression for the joint distribution function is given.

We note that Brownian motions also appear in the analysis of queueing models, where the input process is no Brownian motion. Multi-dimensional reflected Brownian motions are often used to approximate the behavior of open networks, i.e., the joint queue-length or joint workload processes, under heavy traffic conditions, e.g., see Harrison and Williams (1992); Majewski (1998).

In this paper we analyze a two-node tandem queue with Brownian input. Building on the work of Mandjes (2004), we explicitly derive the joint distribution function $\mathbb{P}(Q_1 > b_1, Q_2 > b_2)$, where $Q_i$ is the steady-state workload of node $i$. By setting $b_1 = \alpha b$, $b_2 = (1-\alpha)b$, with $\alpha \in [0, 1]$, and letting $b \to \infty$, we also obtain exact large-buffer asymptotics, i.e., we find a function $f(\cdot)$ such that $\mathbb{P}(Q_1 > \alpha b, Q_2 > (1-\alpha)b)/f(b) \to 1$ as $b \to \infty$. It turns out that the nature of the asymptotics depends on the value of $\alpha$ and the service rates of both queues, i.e., there are different regimes. These regimes can be further interpreted relying on Schilder's sample-path large-deviations theorem. In particular, we obtain the so-called most probable path, i.e., the most likely way for the buffers to fill. Interestingly, each regime has its own type of most likely path; either (a) queue 2 starts to fill earlier than queue 1, but they reach $b_1$ and $b_2$ at the same time, or (b) both queues start to grow at the same time, and reach $b_1$ and $b_2$ at the same time, or (c) both queues start to grow at the same time, but at the time queue 1 reaches $b_1$, queue 2 is strictly larger than $b_2$.

The remainder of the paper is organized as follows. In Sect. 2 we present a detailed description of the two-node tandem queue, as well as a closely related two-node parallel queue. We also give formal implicit expressions for the overflow probabilities, and we briefly discuss Schilder's sample-path large-deviations theorem. In Sect. 3 the two-node parallel queue is analyzed: we derive an exact expression of the joint distribution function, large-buffer asymptotics, and the most probable path. Then we argue that the two-node parallel queue is closely related to the two-node tandem queue. Exploiting this property we obtain in Sect. 4 the desired results for the tandem system. Finally, in Sect. 5 we further discuss our results, and identify some open research questions.

## 2 Preliminaries

In this section we first describe our queueing models: the two-node parallel queue and the two-node tandem queue. Next we by briefly discuss some large-deviations results, which will be needed in the next sections. We conclude by presenting an implicit expression for the joint overflow probability in each of the two models.

### 2.1 Queueing models

Section 3 considers a two-node parallel queue with service rate $c_{\mathrm{I}}$ at queue I, and $c_{\mathrm{II}}$ at queue II. Traffic that enters the system has to be served at both queue I and

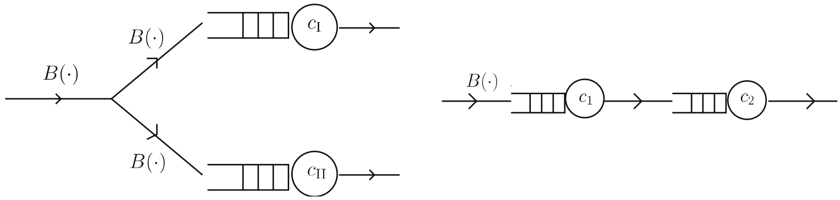


**Fig. 1** Left: Two-node parallel queue. Right: two-node tandem queue

II, which is done in parallel; see Fig. 1 for an illustration. The case $c_{\mathrm{I}} = c_{\mathrm{II}}$ being trivial, we assume without loss of generality that $c_{\mathrm{I}} > c_{\mathrm{II}} > 0$.

We assume that the input process is a standard Brownian motion $\{B(t), t \in \mathbb{R}\}$. It can be verified that $\Gamma(s,t) := \mathbb{C}\mathrm{ov}(B(s), B(t)) = \min\{|s|, |t|\}$ if $s, t \geq 0$ or $s, t < 0$, and $\Gamma(s,t) = 0$ otherwise.

In Sect. 4 we consider a two-node tandem queue, again with standard Brownian input. Thus, the output of the first queue is fed into the second queue; see Fig. 1. Assume constant service rates $c_1$ and $c_2$, respectively. To avoid the trivial situation of the second queue remaining empty, it is assumed that $c_1 > c_2 > 0$. We note that this model corresponds with the heavy-traffic limit of the two-node tandem queue with Poisson arrivals, see Mandjes (2004).

## 2.2 Large deviations

In this subsection we recall two key large-deviations theorems, which are needed in the analysis of Sect. 3.3 and 4.3.

**Theorem 2.1** *Let* $(X, Y) \sim Norm(0, \Sigma)$*, for a non-degenerate 2-dimensional covariance-matrix* $\Sigma$*. Then,*

(i) $-\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq x\right) = \frac{1}{2}x^2/(\Sigma_{11})^2$;

(ii) $-\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq x, \frac{1}{n}\sum_{i=1}^{n} Y_i \geq y\right) = \inf_{a\geq x} \inf_{b\geq y} \Lambda(a,b)$,

*where* $\Lambda(a,b) := \frac{1}{2}(a\ b)\Sigma^{-1}(a\ b)^{\mathrm{T}}$.

We continue with a description of the framework of Schilder's sample-path LDP see Bahadur and Zabell (1979), and also Theorem 1.3.27 of Deuschel and Stroock (1989) for a more detailed treatment). Define the path space $\Omega$ as

$$\Omega := \left\{\omega : \mathbb{R} \to \mathbb{R},\ \text{continuous},\ \ \omega(0) = 0,\ \ \lim_{t\to\infty} \frac{\omega(t)}{1+|t|} = \lim_{t\to-\infty} \frac{\omega(t)}{1+|t|} = 0\right\}.$$

We note that in Addie et al. (2000) it was pointed out that $B(\cdot)$ can be realized on $\Omega$. Then one can construct a reproducing kernel Hilbert space $R \subseteq \Omega$, consisting of elements that are roughly as smooth as the covariance function $\Gamma(s,\cdot)$; for details, see Adler (1990). We start from a 'smaller' space $R^*$, defined by

$$R^* := \left\{\omega : \mathbb{R} \to \mathbb{R},\ \ \omega(\cdot) = \sum_{i=1}^{n} a_i \Gamma(s_i, \cdot),\ \ a_i, s_i \in \mathbb{R}, n \in \mathbb{N}\right\}.$$

The inner product on this space $R^*$ is, for $\omega_a, \omega_b \in R^*$, defined as

$$\langle \omega_a, \omega_b \rangle_R := \left\langle \sum_{i=1}^n a_i \Gamma(s_i, \cdot), \sum_{j=1}^n b_j \Gamma(s_j, \cdot) \right\rangle_R = \sum_{i=1}^n \sum_{j=1}^n a_i b_j \Gamma(s_i, s_j); \quad (1)$$

notice that this implies $\langle \Gamma(s, \cdot), \Gamma(\cdot, t) \rangle_R = \Gamma(s, t)$. This inner product has the following useful property, which is known as the *reproducing kernel* property,

$$\omega(t) = \sum_{i=1}^n a_i \Gamma(s_i, t) = \left\langle \sum_{i=1}^n a_i \Gamma(s_i, \cdot), \Gamma(t, \cdot) \right\rangle_R = \langle \omega(\cdot), \Gamma(t, \cdot) \rangle_R.$$

From this we introduce the norm $||\omega||_R := \sqrt{\langle \omega, \omega \rangle_R}$. The closure of $R^*$ under this norm is defined as space $R$. Now we can define the rate function:

$$I(\omega) := \begin{cases} \frac{1}{2}||\omega||_R^2 & \text{if } \omega \in R; \\ \infty & \text{otherwise.} \end{cases} \quad (2)$$

As a side remark we mention that the above framework in fact holds for a general and versatile class of input processes, covering a broad range of correlation structures, viz. the class of centered Gaussian inputs $(A(t), t \in \mathbb{R})$ (which obviously covers standard Brownian input). In that case one should set $\Gamma(s, t) = \mathbb{C}\text{ov}(A(s), A(t)), s \le t$. Using (1) and the definition of $\Gamma(s, t)$ in case of standard Brownian inputs (see Sect. 2.1), we find that, for $\omega(t) = \sum_{i=1}^n a_i \Gamma(s_i, t)$, with $s_1 < \cdots < s_n$,

$$\begin{aligned} \frac{1}{2}||\omega||_R^2 &= \frac{1}{2}\sum_{i=1}^{k-1}\sum_{j=1}^{k-1} a_i a_j \min\{|s_i|, |s_j|\} + \frac{1}{2}\sum_{i=k}^{n}\sum_{j=k}^{n} a_i a_j \min\{s_i, s_j\} \\ &= \frac{1}{2}\int_{-\infty}^{0} (\omega'(t))^2 \mathrm{d}t + \frac{1}{2}\int_{0}^{\infty} (\omega'(t))^2 \mathrm{d}t, \end{aligned}$$

where $k := \min\{i \in \{1, \dots, n\} : s_i \ge 0\}$ if defined, and $k := n + 1$ otherwise. It turns out that (2) is equivalent to

$$I(\omega) = \begin{cases} \frac{1}{2}\int_{-\infty}^{\infty} (\omega'(t))^2 \mathrm{d}t & \text{if } \omega \in R; \\ \infty & \text{otherwise,} \end{cases} \quad (3)$$

in case of standard Brownian inputs (see Theorem. 5.2.3 of Dembo and Zeitouni (1998)).

**Theorem 2.2 [Schilder]** *For standard Brownian inputs the following sample-path large deviations principle (LDP) holds:*
(a) *For any closed set $F \subset \Omega$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n B_i(\cdot) \in F\right) \le -\inf_{\omega\in F} I(\omega);$$

(b) *For any open set $G \subset \Omega$,*

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n B_i(\cdot) \in G\right) \ge -\inf_{\omega\in G} I(\omega).$$

*Remark* Theorem 2.2 shows that the LDP consists of an upper and lower bound, which apply to closed and open sets, respectively. We will use Theorem 2.2 for the open sets $U$, $S$ and $T$, to be defined in the next subsection. It can be verified that

$$\inf_{\omega \in U} I(\omega) = \inf_{\omega \in \overline{U}} I(\omega),$$

where $\overline{U}$ is the closure of $U$. The way to prove this is to show that an arbitrarily chosen path in $\overline{U}$ can be approximated by a path in $U$. This proof is completely analogously to (Norros 1999) and Appendix A of Mandjes and van Uitert (2005). The same holds for $S$ and $T$.

## 2.3 Joint overflow probabilities

In this subsection we present an implicit expression for the joint overflow probability in each of the two queueing models.

Let $Q_{\rm I}$ and $Q_{\rm II}$ denote the steady-state workload of queue I and queue II, respectively, in the two-node parallel queue We study the joint distribution of the steady-state workloads of queue I and queue II:

$$\mathbb{P}(Q_{\rm I} > b_{\rm I}, Q_{\rm II} > b_{\rm II}). \tag{4}$$

Note that if $b_{\rm II} < b_{\rm I}$, then (due to $c_{\rm I} > c_{\rm II}$) the event $\{Q_{\rm I} > b_{\rm I}\}$ automatically implies $\{Q_{\rm II} > b_{\rm II}\}$. Hence, we concentrate on $b_{\rm II} \geq b_{\rm I}$. Reich's formula (see Reich (1958)) states that

$$Q_{\rm I} = \sup_{s \geq 0}\{-B(-s) - c_{\rm I}s\} \quad \text{and} \quad Q_{\rm II} = \sup_{t \geq 0}\{-B(-t) - c_{\rm II}t\}. \tag{5}$$

Let $s^*$ and $t^*$ denote an optimizing $s$ and $t$ in (5). Now, $-s^*$ $(-t^*)$ can be interpreted as the beginning of the busy period of queue I (queue II) containing time 0. Hence, $c_{\rm I} > c_{\rm II}$ implies that $s^* \leq t^*$, and therefore (4) can be rewritten as $\mathbb{P}(B(\cdot) \in S)$, with

$$S := \{f \in \Omega | \exists t \geq 0 : \exists s \in [0,t] : -f(-s) > b_{\rm I} + c_{\rm I}s, -f(-t) > b_{\rm II} + c_{\rm II}t\}. \tag{6}$$

In the two-node tandem queue we focus on the joint probability that the stationary workloads of the first and second queue, $Q_1$ and $Q_2$, respectively, exceed thresholds $b_1$ and $b_2$, with $b_1, b_2 \geq 0$. For any queue in which traffic leaves the first queue as fluid, the steady-state *total* workload $Q_{\rm T}$ in the two-node tandem queue behaves as single queue emptied at rate $c_2$, see e.g. Mandjes and van Uitert (2005) and references therein. As a consequence,

$$Q_1 = \sup_{s \geq 0}\{-B(-s) - c_1 s\} \quad \text{and} \quad Q_{\rm T} = \sup_{t \geq 0}\{-B(-t) - c_2 t\}. \tag{7}$$

As for the parallel system, we have that the optimizing $s$ is not larger than the optimizing $t$ in (7). Hence, for $b_{\rm T} \geq b_1 \geq 0$, $\mathbb{P}(Q_1 > b_1, Q_{\rm T} > b_{\rm T})$ equals $\mathbb{P}(B(\cdot) \in T)$, with

$$T := \{f \in \Omega | \exists t \geq 0 : \exists s \in [0,t] : -f(-s) > b_1 + c_1 s, -f(-t) > b_{\rm T} + c_2 t\}. \tag{8}$$

Note that (6) and (8) coincide if $c_1 = c_{\mathrm{I}}$, $c_2 = c_{\mathrm{II}}$, $b_1 = b_{\mathrm{I}}$, and $b_{\mathrm{T}} = b_{\mathrm{II}}$. We will exploit this property in Sect. 4. Evidently, the distribution of $(Q_1, Q_{\mathrm{T}})$ uniquely determines the distribution of $(Q_1, Q_2)$. Using that $Q_2 = Q_{\mathrm{T}} - Q_1$, we obtain that $\mathbb{P}(Q_1 > b_1, Q_2 > b_2)$, with $b_1, b_2 \geq 0$, equals $\mathbb{P}(B(\cdot) \in U)$, where

$$U := \left\{ f \in \Omega \,\middle|\, \exists t \geq 0 : \exists s \in [0, t] : \forall u \in [0, t] : \begin{array}{l} -f(-s) > b_1 + c_1 s, \\ f(-u) - f(-t) > b_2 + c_2 t - c_1 u \end{array} \right\}. \tag{9}$$

## 3 Analysis of the two-node parallel queue

In this section we focus on the two-node parallel queue. We derive the joint distribution function of queue I and queue II, large-buffer asymptotics, and the most probable path leading to overflow.

### 3.1 Joint distribution function

In this subsection we derive an exact expression for $p(\overline{b}) := \mathbb{P}(Q_{\mathrm{I}} > b_{\mathrm{I}}, Q_{\mathrm{II}} > b_{\mathrm{II}})$, with $\overline{b} \equiv (b_{\mathrm{I}}, b_{\mathrm{II}})$. For the sake of brevity, write $\chi \equiv \chi(\overline{b}) := (b_{\mathrm{II}} - b_{\mathrm{I}})/(c_{\mathrm{I}} - c_{\mathrm{II}})$. Furthermore, let $\Phi(\cdot)$ denote the distribution function of a standard Normal random variable, $\phi(\cdot) := \Phi'(\cdot)$, and $\Psi(\cdot) := 1 - \Phi(\cdot)$. We first present the main theorem of this subsection.

**Theorem 3.1** *For each $b_{\mathrm{II}} \geq b_{\mathrm{I}} \geq 0$,*

$$p(\overline{b}) = -\Psi(k_1(\overline{b})) + \Psi(k_2(\overline{b}))e^{-2b_{\mathrm{I}}c_{\mathrm{I}}} + \Psi(k_3(\overline{b}))e^{-2b_{\mathrm{II}}c_{\mathrm{II}}} + (1 - \Psi(k_4(\overline{b})))e^{-2(b_{\mathrm{I}}(c_{\mathrm{I}} - 2c_{\mathrm{II}}) + b_{\mathrm{II}}c_{\mathrm{II}})},$$

*where*

$$k_1(\overline{b}) := \frac{b_{\mathrm{I}} + c_{\mathrm{I}}\chi}{\sqrt{\chi}};\ k_2(\overline{b}) := \frac{-b_{\mathrm{I}} + c_{\mathrm{I}}\chi}{\sqrt{\chi}};$$
$$k_3(\overline{b}) := \frac{b_{\mathrm{I}} + (c_{\mathrm{I}} - 2c_{\mathrm{II}})\chi}{\sqrt{\chi}};\ k_4(\overline{b}) := \frac{-b_{\mathrm{I}} + (c_{\mathrm{I}} - 2c_{\mathrm{II}})\chi}{\sqrt{\chi}}.$$

*Proof* In Mandjes (2004) an expression was derived for $\mathbb{P}(Q_{\mathrm{I}} \leq b_{\mathrm{I}}, Q_{\mathrm{II}} \leq b_{\mathrm{II}})$ in case of standard Brownian input. We give a short sketch of the proof. First note that, due to time-reversibility arguments,

$$\mathbb{P}(Q_{\mathrm{I}} \leq b_{\mathrm{I}}, Q_{\mathrm{II}} \leq b_{\mathrm{II}}) = \mathbb{P}(\forall t \geq 0 : B(t) \leq \min\{b_{\mathrm{I}} + c_{\mathrm{I}}t, b_{\mathrm{II}} + c_{\mathrm{II}}t\}).$$

Let $y \equiv y(\overline{b}) := b_{\mathrm{I}} + c_{\mathrm{I}}\chi$. Hence, $(\chi, y)$ is the point where $b_{\mathrm{I}} + c_{\mathrm{I}}t$ and $b_{\mathrm{II}} + c_{\mathrm{II}}t$ intersect. For $t \in [0, \chi]$ the minimum is given by $b_{\mathrm{I}} + c_{\mathrm{I}}t$, whereas for $t \in [\chi, \infty)$ the minimum is $b_{\mathrm{II}} + c_{\mathrm{II}}t$. Now, conditioning on the value of $B(\chi)$, being normally distributed with mean 0 and variance $\chi$, one obtains that $\mathbb{P}(Q_{\mathrm{I}} \leq b_{\mathrm{I}}, Q_{\mathrm{II}} \leq b_{\mathrm{II}})$ equals

$$\int_{-\infty}^{y} \frac{1}{\sqrt{\chi}}\phi\left(\frac{x}{\sqrt{\chi}}\right)\mathbb{P}(\forall t \in [0, \chi] : B(t) \leq b_{\mathrm{I}} + c_{\mathrm{I}}t | B(\chi) = x) \times \mathbb{P}(\forall t \geq 0 : B(t) \leq y - x + c_{\mathrm{II}}t)\mathrm{d}x.$$

The first probability can be expressed (after some rescaling) in terms of the Brownian bridge:

$$\mathbb{P}\left(\forall t \in [0,1] : B(t) \leq b + ct | B(1) = 0\right) = 1 - \exp(-2b(b+c)),$$

whereas the second translates into the supremum of a Brownian motion: $1 - \exp(-2(y-x)c_{\text{II}})$. After substantial calculus we obtain that $\mathbb{P}(Q_{\text{I}} \leq b_{\text{I}}, Q_{\text{II}} \leq b_{\text{II}})$ equals

$$\Phi(k_1(\overline{b})) - \Phi(k_2(\overline{b}))e^{-2b_{\text{I}}c_{\text{I}}} - \Phi(k_3(\overline{b}))e^{-2b_{\text{II}}c_{\text{II}}} + \Phi(k_4(\overline{b}))e^{-2(b_{\text{I}}(c_{\text{I}}-2c_{\text{II}})+b_{\text{II}}c_{\text{II}})}.$$

Furthermore, it is well known that $\mathbb{P}(Q_i > b_i) = e^{-2b_i c_i}$, $i = \text{I}, \text{II}$. The stated follows from

$$p(\overline{b}) = 1 - \mathbb{P}(Q_{\text{I}} \leq b_{\text{I}}) - \mathbb{P}(Q_{\text{II}} \leq b_{\text{II}}) + \mathbb{P}(Q_{\text{I}} \leq b_{\text{I}}, Q_{\text{II}} \leq b_{\text{II}}).$$

□

### 3.2 Exact large-buffer asymptotics

In this subsection we derive the exact asymptotics of the joint buffer content distribution. We write $f(u) \sim g(u)$ when $f(u)/g(u) \to 1$ if $u \to \infty$. Define $\zeta(x) := \left(\sqrt{2\pi}x\right)^{-1} \exp(-x^2/2)$. Also,

$$\alpha_+ := \frac{c_{\text{I}}}{2c_{\text{I}} - c_{\text{II}}};\ \alpha_0 := \frac{2c_{\text{II}} - c_{\text{I}}}{c_{\text{II}}};\ \alpha_- := \frac{c_{\text{I}} - 2c_{\text{II}}}{2c_{\text{I}} - 3c_{\text{II}}}.$$

It can be verified that $\alpha_0 < 0 < \alpha_- < \alpha_+ < 1$ if $c_{\text{I}} > 2c_{\text{II}}$, whereas $0 \leq \alpha_0 < \alpha_+ < 1$ if $c_{\text{I}} \leq 2c_{\text{II}}$. Let us first present the following useful lemma.

**Lemma 3.2** *Let $b_{\text{I}} = \alpha b$ and $b_{\text{II}} = b$, with $\alpha \in [0,1]$. If $b \to \infty$, then*

$$\Psi(k_1(\overline{b})) \sim \zeta(k_1(\overline{b}));$$

$$\Psi(k_2(\overline{b})) \sim \begin{cases} \zeta(k_2(\overline{b})) & \text{if } \alpha < \alpha_+; \\ 1/2 & \text{if } \alpha = \alpha_+; \\ 1 & \text{otherwise}; \end{cases}$$

$$\Psi(k_3(\overline{b})) \sim \begin{cases} \zeta(k_3(\overline{b})) & \text{if } \alpha > \alpha_0; \\ 1/2 & \text{if } \alpha = \alpha_0; \\ 1 & \text{otherwise}; \end{cases}$$

$$1 - \Psi(k_4(\overline{b})) \sim \begin{cases} 1 & \text{if } \alpha < \alpha_- \text{ and } c_{\text{I}} > 2c_{\text{II}}; \\ 1/2 & \text{if } \alpha = \alpha_- \text{ and } c_{\text{I}} \geq 2c_{\text{II}}; \\ -\zeta(k_4(\overline{b})) & \text{otherwise}. \end{cases}$$

*Proof* First determine for which values of $b_{\text{I}}/b_{\text{II}} = \alpha$, $k_i(\overline{b})$, $i \in \{1,2,3,4\}$, is positive or negative. Note that $k_1(\overline{b})$ is always positive, given that $b_{\text{II}} \geq b_{\text{I}} \geq 0$. Also, $k_4(\overline{b})$ is always negative if $c_{\text{I}} \leq 2c_{\text{II}}$ and $b_{\text{I}} > 0$. Hence, we obtain $\alpha_+, \alpha_0$ and $\alpha_-$ as critical values from $k_i(\overline{b})$, $i = 2,3,4$, respectively. Next use the fact that $\Psi(u) \sim \zeta(u)$ and $\Psi(-u) \sim 1$ as $u \to \infty$. Observe that $\Psi(0) = 1/2$. □

Define

$$\beta(\overline{b}) := \frac{1}{\sqrt{2\pi}}\left(-\frac{1}{k_1(\overline{b})}+\frac{1}{k_2(\overline{b})}+\frac{1}{k_3(\overline{b})}-\frac{1}{k_4(\overline{b})}\right) \quad \text{and}$$
$$\gamma(\overline{b}) := \frac{(b_{\mathrm{II}}c_{\mathrm{I}}-b_{\mathrm{I}}c_{\mathrm{II}})^2}{2(b_{\mathrm{II}}-b_{\mathrm{I}})(c_{\mathrm{I}}-c_{\mathrm{II}})}.$$

Straightforward calculus also shows the following equalities:

$$\exp\left(-\frac{k_1(\overline{b})^2}{2}\right)=\exp\left(-\frac{k_2(\overline{b})^2}{2}\right)\exp(-2b_{\mathrm{I}}c_{\mathrm{I}})=\exp\left(-\frac{k_3(\overline{b})^2}{2}\right)\exp(-2b_{\mathrm{II}}c_{\mathrm{II}})$$
$$=\exp\left(-\frac{k_4(\overline{b})^2}{2}\right)\exp(-2(b_{\mathrm{I}}(c_{\mathrm{I}}-2c_{\mathrm{II}})+b_{\mathrm{II}}c_{\mathrm{II}}))=\exp\left(-\gamma(\overline{b})\right). \tag{10}$$

**Theorem 3.3** *Let* $b_{\mathrm{I}} = \alpha b$ *and* $b_{\mathrm{II}} = b$*, with* $\alpha \in [0, 1]$. *Suppose* $c_{\mathrm{I}} > 2c_{\mathrm{II}}$. *For* $b \to \infty$,

$$p(\overline{b}) \sim \begin{cases} e^{-2(b_{\mathrm{I}}(c_{\mathrm{I}}-2c_{\mathrm{II}})+b_{\mathrm{II}}c_{\mathrm{II}})} & \text{if } \alpha \in [0,\alpha_-); \\ \frac{1}{2}e^{-2(b_{\mathrm{I}}(c_{\mathrm{I}}-2c_{\mathrm{II}})+b_{\mathrm{II}}c_{\mathrm{II}})} & \text{if } \alpha = \alpha_-; \\ \beta(\overline{b})e^{-\gamma(\overline{b})} & \text{if } \alpha \in (\alpha_-,\alpha_+); \\ \frac{1}{2}e^{-2b_{\mathrm{I}}c_{\mathrm{I}}} & \text{if } \alpha = \alpha_+; \\ e^{-2b_{\mathrm{I}}c_{\mathrm{I}}} & \text{if } \alpha \in (\alpha_+, 1]. \end{cases}$$

*Proof* We only prove the first statement, as the other four statements follow in a similar way. We have to prove that

$$p(\overline{b})\exp(2(b_{\mathrm{I}}(c_{\mathrm{I}}-2c_{\mathrm{II}})+b_{\mathrm{II}}c_{\mathrm{II}})) \to 1 \text{ as } b \to \infty, \quad \text{for } \alpha \in [0,\alpha_-).$$

From Lemma 3.2 we obtain that for $\alpha \in [0, \alpha_-)$,

$$\Psi(k_1(\overline{b})) \sim \zeta(k_1(\overline{b})); \quad \Psi(k_2(\overline{b})) \sim \zeta(k_2(\overline{b})); \ \Psi(k_3(\overline{b})) \sim \zeta(k_3(\overline{b}));$$
$$1-\Psi(k_4(\overline{b})) \sim 1-\zeta(k_4(\overline{b})).$$

Now it can be checked from (10) that, as $b \to \infty$,

$$\Psi(k_1(\overline{b})) = o\left(e^{-2(b_{\mathrm{I}}(c_{\mathrm{I}}-2c_{\mathrm{II}})+b_{\mathrm{II}}c_{\mathrm{II}}))}\right),$$

and the same applies for $\Psi(k_2(\overline{b}))e^{-2b_{\mathrm{I}}c_{\mathrm{I}}}$ and $\Psi(k_3(\overline{b}))e^{-2b_{\mathrm{II}}c_{\mathrm{II}}}$. With $1-\Psi(k_4(\overline{b})) \sim 1$, Theorem 3.1 implies the stated. □

**Theorem 3.4** *Let* $b_{\mathrm{I}} = \alpha b$ *and* $b_{\mathrm{II}} = b$*, with* $\alpha \in [0, 1]$. *Suppose* $c_{\mathrm{I}} < 2c_{\mathrm{II}}$. *For* $b \to \infty$,

$$p(\overline{b}) \sim \begin{cases} e^{-2b_{\mathrm{II}}c_{\mathrm{II}}} & \text{if } \alpha \in [0,\alpha_0); \\ \frac{1}{2}e^{-2b_{\mathrm{II}}c_{\mathrm{II}}} & \text{if } \alpha = \alpha_0; \\ \beta(\overline{b})e^{-\gamma(\overline{b})} & \text{if } \alpha \in (\alpha_0,\alpha_+); \\ \frac{1}{2}e^{-2b_{\mathrm{I}}c_{\mathrm{I}}} & \text{if } \alpha = \alpha_+; \\ e^{-2b_{\mathrm{I}}c_{\mathrm{I}}} & \text{if } \alpha \in (\alpha_+, 1]. \end{cases}$$

*Proof* The proof is similar to that of Theorem 3.3. □

*Remark* Note that for $c_{\mathrm{I}} = 2c_{\mathrm{II}}$, one obtains $\alpha_0 = 0$. It can be verified that in this special case Theorem 3.4 reduces to

$$p(\bar{b}) \sim \begin{cases} e^{-2b_{\mathrm{II}}c_{\mathrm{II}}} & \text{if } \alpha = 0; \\ \beta(\bar{b})e^{-\gamma(\bar{b})} & \text{if } \alpha \in (0, \alpha_+); \\ \frac{1}{2}e^{-2b_{\mathrm{I}}c_{\mathrm{I}}} & \text{if } \alpha = \alpha_+; \\ e^{-2b_{\mathrm{I}}c_{\mathrm{I}}} & \text{if } \alpha \in (\alpha_+, 1]. \end{cases}$$

### 3.3 Most probable path

In the previous subsection it was shown that the nature of the large-buffer asymptotics strongly depends on the model parameters $\alpha$, $c_{\mathrm{I}}$ and $c_{\mathrm{II}}$, i.e., there are different regimes. In this subsection we will interpret and explain these regimes by using sample-path large deviations. In particular, by using Schilder's theorem (Theorem 2.2) we show that in each of these regimes the system has a typical (most likely) behavior, and we characterize this behavior for each regime.

Schilder's theorem implies that the exponential decay rate of the joint overflow probability in the parallel system is characterized by the path in $S$ that minimizes the decay rate. Among all paths such that queue I exceeds $b_{\mathrm{I}}$ and queue II exceeds $b_{\mathrm{II}}$, this is the so-called most probable path (MPP): informally speaking, given that this rare event occurs, with overwhelming probability $(b_{\mathrm{I}}, b_{\mathrm{II}})$ is reached by a path 'close to' the MPP. The goal of this subsection is to find the MPP in $S$, and to relate its form to the regimes identified in Sect. 3.2.

Consider the two-node parallel queue as described before. Now, in order to apply 'Schilder', we feed this network by $n$ i.i.d. standard Brownian sources. The link rates and buffer thresholds are also scaled by $n$: $nc_{\mathrm{I}}$, $nc_{\mathrm{II}}$, $nb_{\mathrm{I}}$ and $nb_{\mathrm{II}}$, respectively. Now, $p_n(\bar{b}) := \mathbb{P}(Q_{\mathrm{I,n}} > nb_{\mathrm{I}}, Q_{\mathrm{II,n}} > nb_{\mathrm{II}})$ can be expressed as

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} B_i(\cdot) \in S\right).$$

From 'Schilder' it follows that

$$J(\bar{b}) := -\lim_{n\to\infty}\frac{1}{n}\log p_n(\bar{b}) = \inf_{f\in S} I(f) = \inf_{t\geq 0}\inf_{s\in[0,t]} \Upsilon(s,t), \tag{11}$$

with

$$\Upsilon(s,t) := \inf_{f\in S^{s,t}} I(f) \quad \text{and}$$
$$S^{s,t} := \{f \in \Omega \mid -f(-s) > b_{\mathrm{I}} + c_{\mathrm{I}}s, -f(-t) > b_{\mathrm{II}} + c_{\mathrm{II}}t\},$$

using the fact that the decay rate of a union of events is the minimum of the decay rates of the individual events. As mentioned in the remark of Sect. 2.2, we can replace '>' by '≥' in $S^{s,t}$, without any impact on the decay rate.

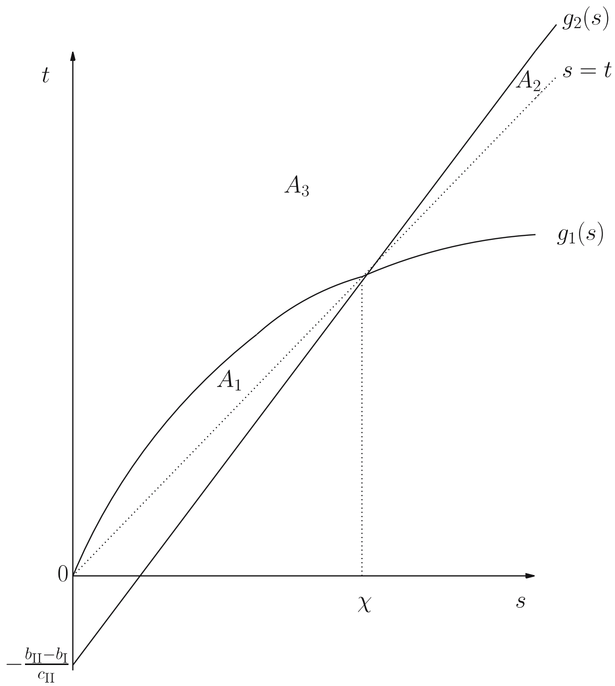


**Fig. 2** The partitioning of $A$

We first show how, for fixed $s, t$, with $0 \le s \le t$, the nfimum of $\Upsilon(s,t)$ over $S^{s,t}$ can be computed. Define

$$g_1(s) := \frac{b_{II}s}{b_I + (c_I - c_{II})s} \quad \text{and} \quad g_2(s) := s\frac{c_I}{c_{II}} + \frac{b_I - b_{II}}{c_{II}}, \quad s \ge 0.$$

Note that $g_1(\cdot)$ is a concave function, whereas $g_2(\cdot)$ is a linear function. Furthermore, $g_1(s) > g_2(s)$ if $s<\chi$, $g_1(s)=g_2(s)$ if $s=\chi$, and otherwise $g_1(s) < g_2(s)$. Also, define

$$A_1 := \{(s,t)|0 \le s \le t \le g_1(s)\};$$
$$A_2 := \{(s,t)|0 \le s \le t \le g_2(s)\};$$
$$A_3 := \{(s,t)|t > \max\{g_1(s), g_2(s)\}, s \ge 0\}.$$

Note that $A := \{(s,t)|0 \le s \le t\} = A_1 \cup A_2 \cup A_3$, for disjoint $A_1$, $A_2$ and $A_3$, as illustrated in Fig. 2.

**Lemma 3.5** *For $t \ge 0$, and $s \in [0,t]$,*

$$\Upsilon(s,t) = \begin{cases} h_1(t) := \frac{(b_{II}+c_{II}t)^2}{2t} & \text{if } (s,t) \in A_1; \\ h_2(s) := \frac{(b_I+c_Is)^2}{2s} & \text{if } (s,t) \in A_2; \\ h_3(s,t) := \frac{(b_I+c_Is)^2}{2s} + \frac{(b_{II}+c_{II}t-b_I-c_Is)^2}{2(t-s)} & \text{if } (s,t) \in A_3. \end{cases}$$

*Proof* The proof is analogous to Lemma 3.4 of (Mandjes and van Uitert 2005). First note that the values of the Brownian input at times $-s$ and $-t$ are bivariate Normally distributed, i.e., as $(-B(-s), -B(-t))$. Now, by using Theorem 2.1, we find for $y, z \in \mathbb{R}$ and $t \geq 0$, $s \in [0, t]$,

$$\Upsilon(s,t) = \inf_{y \geq b_{\mathrm{I}}+c_{\mathrm{I}}s} \inf_{z \geq b_{\mathrm{II}}+c_{\mathrm{II}}t} \Lambda(y,z), \quad \text{with } \Lambda(y,z) = \frac{1}{2} \begin{pmatrix} y & z \end{pmatrix} \begin{pmatrix} s & s \\ s & t \end{pmatrix}^{-1} \begin{pmatrix} y \\ z \end{pmatrix}. \tag{12}$$

One can show that if

$$y_0 := \mathbb{E}(-B(-s) \mid -B(-t) = b_{\mathrm{II}} + c_{\mathrm{II}}t) \geq b_{\mathrm{I}} + c_{\mathrm{I}}s,$$

or, equivalently, $t \leq g_1(s)$, then the optimum in (12) is attained at $(y^*, z^*) = (y_0, b_{\mathrm{II}} + c_{\mathrm{II}}t)$. Hence, the rate function is independent of $s$, and given by $\Lambda(y_0, b_{\mathrm{II}} + c_{\mathrm{II}}t) = h_1(t)$.

In a similar way, if

$$z_0 := \mathbb{E}(-B(-t) \mid -B(-s) = b_{\mathrm{I}} + c_{\mathrm{I}}s) \geq b_{\mathrm{II}} + c_{\mathrm{II}}t,$$

or, after rewriting, $t \leq g_2(s)$, then the optimum in (12) is attained at $(y^*, z^*) = (b_{\mathrm{I}} + c_{\mathrm{I}}s, z_0)$. The rate function is then given by $\Lambda(b_{\mathrm{I}} + c_{\mathrm{I}}s, z_0) = h_2(s)$ (independently of $t$).

If $y_0 < b_{\mathrm{I}} + c_{\mathrm{I}}s$ and $z_0 < b_{\mathrm{II}} + c_{\mathrm{II}}t$, then the optimum in (12) is attained at $(y^*, z^*) = (b_{\mathrm{I}} + c_{\mathrm{I}}s, b_{\mathrm{II}} + c_{\mathrm{II}}t)$. It is readily verified that this yields $h_3(s,t)$ for $t > \max\{g_1(s), g_2(s)\}$. □

In order to obtain $J(\bar{b})$, it follows from (11) that we have to compute

$$\inf_{(s,t)\in A} \Upsilon(s,t). \tag{13}$$

We will obtain (13) by first deriving

$$\inf_{(s,t)\in A_1} \Upsilon(s,t) = \inf_{(s,t)\in A_1} h_1(t); \tag{14}$$

$$\inf_{(s,t)\in A_2} \Upsilon(s,t) = \inf_{(s,t)\in A_2} h_2(s); \tag{15}$$

$$\inf_{(s,t)\in A_3} \Upsilon(s,t) = \inf_{(s,t)\in A_3} h_3(s,t), \tag{16}$$

and subsequently taking the minimum of (14)–(16) (recall that $A = A_1 \cup A_2 \cup A_3$). We start by computing (14).

### 3.3.1 Area $A_1$

The optimization over $A_1$ reduces to

$$\inf_{(s,t)\in A_1} \Upsilon(s,t) = \inf_{(s,t)\in A_1} h_1(t) = \inf_{t\in[0,\chi]} h_1(t). \tag{17}$$

It can be verified that $h_1(t)$ is strictly decreasing on the interval $[0, b_{\mathrm{II}}/c_{\mathrm{II}})$, and strictly increasing on the interval $(b_{\mathrm{II}}/c_{\mathrm{II}}, \infty)$. Therefore, if $b_{\mathrm{II}}/c_{\mathrm{II}} \leq \chi$ then $t^* = b_{\mathrm{II}}/c_{\mathrm{II}}$ and $s^* \in [g_1^{-1}(t^*), t^*]$, whereas otherwise $t^* = s^* = \chi$.

**Lemma 3.6** *Expression* (17) *equals*

$$\begin{cases} 2b_{\mathrm{II}}c_{\mathrm{II}} & \text{if } c_{\mathrm{I}} \leq 2c_{\mathrm{II}} \text{ and } b_{\mathrm{I}}/b_{\mathrm{II}} \in [0, \alpha_0]; \\ \gamma(\overline{b}) & \text{if } c_{\mathrm{I}} \leq 2c_{\mathrm{II}} \text{ and } b_{\mathrm{I}}/b_{\mathrm{II}} \in (\alpha_0, 1]; \\ \gamma(\overline{b}) & \text{if } c_{\mathrm{I}} > 2c_{\mathrm{II}}. \end{cases}$$

*Proof* The condition $b_{\mathrm{II}}/c_{\mathrm{II}} \leq \chi$ is equivalent to $b_{\mathrm{I}}/b_{\mathrm{II}} \leq (2c_{\mathrm{II}} - c_{\mathrm{I}})/c_{\mathrm{II}} = \alpha_0$. Note that $\alpha_0$ is only non-negative if $c_{\mathrm{I}} \leq 2c_{\mathrm{II}}$. Hence, evaluation of (17) for $t^* = b_{\mathrm{II}}/c_{\mathrm{II}}$ proves the first statement. Similarly, evaluation of (17) for $t^* = \chi$ proves the second statement. □

*3.3.2 Area $A_2$*

The approach is very similar to above. We are to solve the following optimization problem:

$$\inf_{(s,t)\in A_2} \Upsilon(s,t) = \inf_{(s,t)\in A_2} h_2(s) = \inf_{s\in[\chi,\infty)} h_2(s). \tag{18}$$

The function $h_2(s)$ has a global minimum that is attained at $s = b_{\mathrm{I}}/c_{\mathrm{I}}$. Thus, if $b_{\mathrm{I}}/c_{\mathrm{I}} \geq \chi$, then $s^* = b_{\mathrm{I}}/c_{\mathrm{I}}$ and $t^* \in [s^*, g_2(s^*)]$, whereas otherwise $s^* = t^* = \chi$. The following lemma is proven analogously to Lemma 3.6.

**Lemma 3.7** *Expression* (18) *equals*

$$\begin{cases} \gamma(\overline{b}) & \text{if } b_{\mathrm{I}}/b_{\mathrm{II}} \in [0, \alpha_+); \\ 2b_{\mathrm{I}}c_{\mathrm{I}} & \text{if } b_{\mathrm{I}}/b_{\mathrm{II}} \in [\alpha_+, 1]. \end{cases}$$

*3.3.3 Area $A_3$*

Now we are to solve the following optimization problem:

$$\inf_{(s,t)\in A_3} \Upsilon(s,t) = \inf_{(s,t)\in A_3} h_3(s,t) = \inf_{s\geq 0} \inf_{t>\max\{g_1(s),g_2(s)\}} h_3(s,t).$$

We can divide area $A_3$ in two parts, namely: $s \in [0, \chi]$ and $t \in (g_1(s), \infty)$, and $s \in (\chi, \infty)$ and $t \in (g_2(s), \infty)$ (see Fig. 2). Let us start with the second part:

$$\inf_{s\in(\chi,\infty)} \inf_{t\in(g_2(s),\infty)} h_3(s,t). \tag{19}$$

Clearly, (19) is bounded from below by

$$\inf_{s\in(\chi,\infty)} \inf_{t\in(g_2(s),\infty)} h_2(s).$$

One can show that $h_3(s,t)$ reduces to $h_2(s)$ if $t = g_2(s)$ $(s \in [\chi, \infty))$. Therefore, analogously to area $A_2$, if $b_{\mathrm{I}}/c_{\mathrm{I}} \geq \chi$, then $s^* = b_{\mathrm{I}}/c_{\mathrm{I}}$ and $t^* = g_2(s^*) = (2b_{\mathrm{I}} - b_{\mathrm{II}})/c_{\mathrm{II}}$, whereas otherwise $s^* = t^* = \chi$. We thus obtain the following result.

**Lemma 3.8** *Expression* (19) *equals*

$$\begin{cases} \gamma(\overline{b}) & \text{if } b_{\mathrm{I}}/b_{\mathrm{II}} \in [0, \alpha_+); \\ 2b_{\mathrm{I}}c_{\mathrm{I}} & \text{if } b_{\mathrm{I}}/b_{\mathrm{II}} \in [\alpha_+, 1]. \end{cases}$$

We now turn to the first part:

$$\inf_{s\in[0,\chi]}\ \inf_{t\in(g_1(s),\infty)} h_3(s,t). \tag{20}$$

First concentrate on the minimum of $h_3(s,t)$ over $t \geq 0$, which is attained at

$$t = \frac{b_{\mathrm{II}} - b_{\mathrm{I}}}{c_{\mathrm{II}}} + s\frac{2c_{\mathrm{II}} - c_{\mathrm{I}}}{c_{\mathrm{II}}} =: g_3(s)$$

if $s \in [0,\chi]$ (for $s > \chi$ it is attained at $t = g_2(s)$, but this case is irrelevant here). Note that $g_3(s)$ is linearly decreasing (increasing) if $c_{\mathrm{I}} > 2c_{\mathrm{II}}$ ($c_{\mathrm{I}} < 2c_{\mathrm{II}}$). Also, $g_3(\chi) = \chi$. Hence, we have to distinguish between two cases:

- First concentrate on $c_{\mathrm{I}} > 2c_{\mathrm{II}}$. Then $g_3(s) > g_1(s)$ for all $s \in [0,\chi)$ (as $g_3(s)$ is non-increasing and $g_3(\chi) = \chi$). Substituting $t = g_3(s)$ in (20) gives

$$\inf_{s\in[0,\chi]} \frac{b_{\mathrm{I}}^2 + 2b_{\mathrm{I}}(c_{\mathrm{I}} - 2c_{\mathrm{II}})s + 4b_{\mathrm{II}}c_{\mathrm{II}}s + (c_{\mathrm{I}} - 2c_{\mathrm{II}})^2 s^2}{2s}. \tag{21}$$

  This is minimized for $s^* = b_{\mathrm{I}}/(c_{\mathrm{I}} - 2c_{\mathrm{II}})$ and $t^* = g_3(s^*) = (b_{\mathrm{II}} - 2b_{\mathrm{I}})/c_{\mathrm{II}}$ if $b_{\mathrm{I}}/(c_{\mathrm{I}} - 2c_{\mathrm{II}}) \leq \chi$, whereas otherwise $s^* = \chi = t^*$.
- Next consider $c_{\mathrm{I}} \leq 2c_{\mathrm{II}}$. In this case it is not clear a priori whether $g_3(s) \geq g_1(s)$ for all $s \in [0,\chi)$. For the moment assume that this is true. Then (21) is again appropriate, and this is minimized for $s^* = b_{\mathrm{I}}/(2c_{\mathrm{II}} - c_{\mathrm{I}})$ and $t^* = g_3(s) = b_{\mathrm{II}}/c_{\mathrm{II}}$ if $b_{\mathrm{I}}/(2c_{\mathrm{II}} - c_{\mathrm{I}}) \leq \chi$, whereas otherwise $s^* = \chi = t^*$. Now, in the former case it can be checked that $g_3(s^*) = g_1(s^*) = b_{\mathrm{II}}/c_{\mathrm{II}}$, and in the latter case we find $g_3(s^*) = g_1(s^*) = \chi$, i.e., the minimizers satisfy $g_3(s^*) \geq g_1(s^*)$, and hence we are done.

This reasoning leads to the following result.

**Lemma 3.9** *Expression* (20) *equals*

$$\begin{cases} 2(b_{\mathrm{I}}(c_{\mathrm{I}} - 2c_{\mathrm{II}}) + b_{\mathrm{II}}c_{\mathrm{II}}) & \text{if } c_{\mathrm{I}} > 2c_{\mathrm{II}} \text{ and } b_{\mathrm{I}}/b_{\mathrm{II}} \in [0,\alpha_-];\\ \gamma(\bar{b}) & \text{if } c_{\mathrm{I}} > 2c_{\mathrm{II}} \text{ and } b_{\mathrm{I}}/b_{\mathrm{II}} \in (\alpha_-,1];\\ 2b_{\mathrm{II}}c_{\mathrm{II}} & \text{if } c_{\mathrm{I}} \leq 2c_{\mathrm{II}} \text{ and } b_{\mathrm{I}}/b_{\mathrm{II}} \in [0,\alpha_0];\\ \gamma(\bar{b}) & \text{if } c_{\mathrm{I}} \leq 2c_{\mathrm{II}} \text{ and } b_{\mathrm{I}}/b_{\mathrm{II}} \in (\alpha_0,1]. \end{cases}$$

*3.3.4 Exponential decay rate*

In order to find $J(\bar{b})$, we have to determine the minimum of (14), (15) and (16). This minimum can be obtained by combining Lemmas 3.6–3.9. From this, we already see that the minimum depends on the value of $b_{\mathrm{I}}/b_{\mathrm{II}} \in [0,1]$ and the sign of $c_{\mathrm{I}} - 2c_{\mathrm{II}}$. We now present an exact expression for the rate function $J(\bar{b})$. We start with the case $c_{\mathrm{I}} > 2c_{\mathrm{II}}$.

**Theorem 3.10** *Suppose* $c_{\mathrm{I}} > 2c_{\mathrm{II}}$. *Then it holds that*

$$J(\bar{b}) = \begin{cases} 2(b_{\mathrm{I}}(c_{\mathrm{I}} - 2c_{\mathrm{II}}) + b_{\mathrm{II}}c_{\mathrm{II}}) & \text{if } b_{\mathrm{I}}/b_{\mathrm{II}} \in [0,\alpha_-];\\ \gamma(\bar{b}) & \text{if } b_{\mathrm{I}}/b_{\mathrm{II}} \in (\alpha_-,\alpha_+);\\ 2b_{\mathrm{I}}c_{\mathrm{I}} & \text{if } b_{\mathrm{I}}/b_{\mathrm{II}} \in [\alpha_+,1]. \end{cases}$$

*Proof* By combining Lemmas 3.6–3.9, we find that there exist two critical values of $b_{\rm I}/b_{\rm II} \in [0,1]$, given that $c_{\rm I}>2c_{\rm II}$: $\alpha_-$ and $\alpha_+$. Recall from Sect. 3.2 that $0<\alpha_-<\alpha_+<1$ if $c_{\rm I}>2c_{\rm II}$. Now, if $b_{\rm I}/b_{\rm II} \in [0,\alpha_-]$, then it follows from Lemmas 3.6–3.9 that $J(\bar{b}) = \min\left\{2(b_{\rm I}(c_{\rm I}-2c_{\rm II})+b_{\rm II}c_{\rm II}), \gamma(\bar{b})\right\}$. Straightforward calculus shows that the first argument is smaller for these values of $b_{\rm I}/b_{\rm II}$. Similarly, if $b_{\rm I}/b_{\rm II} \in (\alpha_-,\alpha_+)$, then $J(\bar{b}) = \gamma(\bar{b})$. Finally, if $b_{\rm I}/b_{\rm II} \in [\alpha_+,1]$, then $J(\bar{b}) = \min\left\{2b_{\rm I}c_{\rm I}, \gamma(\bar{b})\right\}$. Applying straightforward calculus yields that the first argument is smaller if $b_{\rm I}/b_{\rm II} \in (\alpha_+,1]$. □

Schilder's theorem says that knowledge of the MPP $f^*$ for the buffers to fill, also implies that the exponential decay rate is known: $J(\bar{b}) = I(f^*)$. Luckily, we do not have to derive the MPPs corresponding to the three decay rates of Theorem 3.10, because we have already implicitly obtained them. The values of $-s^*$ and $-t^*$, where $s^*$ and $t^*$ are the optimizers in Sects. 3.3.1–3.3.3 associated with the three decay rates of Theorem 3.10, can be interpreted as the time where the first and second queue, respectively, start to build up in the corresponding MPP.

The $s^*$ and $t^*$ associated with the decay rate of the first regime in Theorem 3.10 are $s^* = b_{\rm I}/(c_{\rm I}-2c_{\rm II})$ and $t^* = (b_{\rm II}-2b_{\rm I})/c_{\rm II}$, see Sect. 3.3.3. Hence, in the MPP of the first regime, queue I starts to build up at $-s^*$, whereas queue II starts to build up at $-t^*$. The MPP is given by, for $r \in [-t^*, 0]$,

$$f^*(r) = \mathbb{E}\left(B(r) \mid -B(-s^*) = b_{\rm I} + c_{\rm I}s^*, -B(-t^*) = b_{\rm II} + c_{\rm II}t^*\right).$$

Let $(Y,Z)$ be $(1+d)$-variate Normally distributed ($Y$ is 1-dimensional and $Z$ is $d$-dimensional). Now, using that the random variable $(Y|Z=z)$, for some $z \in \mathbb{R}^d$, is normally distributed with mean

$$\mathbb{E}(Y|Z=z) = \mathbb{E}Y + \begin{pmatrix} \mathbb{C}\mathrm{ov}(Y,Z_1) \\ \vdots \\ \mathbb{C}\mathrm{ov}(Y,Z_d) \end{pmatrix}^{\rm T} \Sigma^{-1} \begin{pmatrix} z_1 - \mathbb{E}Z_1 \\ \vdots \\ z_d - \mathbb{E}Z_d \end{pmatrix}, \tag{22}$$

where

$$\Sigma = \begin{pmatrix} \mathbb{V}\mathrm{ar}Z_1 & \mathbb{C}\mathrm{ov}(Z_1,Z_2) & \dots & \mathbb{C}\mathrm{ov}(Z_1,Z_d) \\ \mathbb{C}\mathrm{ov}(Z_1,Z_2) & \mathbb{V}\mathrm{ar}Z_2 & \dots & \mathbb{C}\mathrm{ov}(Z_2,Z_d) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{C}\mathrm{ov}(Z_1,Z_d) & \mathbb{C}\mathrm{ov}(Z_2,Z_d) & \dots & \mathbb{V}\mathrm{ar}Z_d \end{pmatrix},$$

it can be verified that

$$\begin{aligned} (f^*)'(r) &= 2c_{\rm II} && \text{if } r \in [-t^*, -s^*); \\ (f^*)'(r) &= 2(c_{\rm I}-c_{\rm II}) && \text{if } r \in [-s^*, 0]. \end{aligned}$$

Applying 'Schilder', i.e., using (3), one can verify that, as expected,

$$I(f^*) = \frac{1}{2}\left((2c_{\rm II})^2(t^*-s^*) + (2(c_{\rm I}-c_{\rm II}))^2 s^*\right) = 2(b_{\rm I}(c_{\rm I}-2c_{\rm II}) + b_{\rm II}c_{\rm II}),$$

so $f^*$ is indeed the MPP. Note that given service rates $c_{\rm I}$ and $c_{\rm II}$ at queue I and queue II, respectively, with $c_{\rm I} > 2c_{\rm II}$, the MPP yields $Q_{\rm I}(0) = b_{\rm I}$ and $Q_{\rm II}(0) = b_{\rm II}$.

Also note that we have not specified the MPP outside $[-t^*, 0]$, because outside this interval the MPP produces traffic according to the average rate $\mathbb{E}B(1)$, which equals 0 (we are dealing with standard Brownian input), and therefore this does not affect $I(f^*)$. Below we will therefore also not specify the MPPs outside $[-t^*, 0]$ (for different values of $t^*$).

The $s^*$ and $t^*$ associated with the decay rate of the second regime in Theorem 3.10 are $s^* = t^* = (b_{\rm II} - b_{\rm I})/(c_{\rm I} - c_{\rm II}) = \chi$, see Sects. 3.3.1–3.3.3, i.e., in the second regime, both queue I and queue II start to build up at $-t^*$. The MPP is given by, for $r \in [-t^*, 0]$,

$$f^*(r) = \mathbb{E}\left(B(r) | - B(-t^*) = b_{\rm I} + c_{\rm I} t^*\right). \tag{23}$$

Using (22), it can be verified that that this MPP is such that traffic enters the network with constant rate $(b_{\rm I}/(b_{\rm II} - b_{\rm I}))(c_{\rm I} - c_{\rm II}) + c_{\rm I}$ in the interval $[-t^*, 0]$, and this yields $Q_{\rm I}(0) = b_{\rm I}$ and $Q_{\rm II}(0) = b_{\rm II}$. Using (3), we find

$$I(f^*) = \frac{1}{2}\left(\frac{b_{\rm I}}{b_{\rm II} - b_{\rm I}}(c_{\rm I} - c_{\rm II}) + c_{\rm I}\right)^2 t^* = \gamma(\bar{b}),$$

so $f^*$ is indeed the MPP.

The $s^*$ and $t^*$ associated with the decay rate of the third regime in Theorem 3.10 are $s^* = t^* = b_{\rm I}/c_{\rm I}$, see Sect. 3.3.2, i.e., in the third regime, both queues start to build up at $-t^*$. The MPP is given by, for $r \in [-t^*, 0]$,

$$f^*(r) = \mathbb{E}\left(B(r) | - B(-t^*) = b_{\rm I} + c_{\rm I} t^*\right).$$

Again, using (22), we find that this MPP is such that traffic is produced at constant rate $2c_{\rm I}$ in the interval $[-t^*, 0]$, and this gives $Q_{\rm I}(0) = b_{\rm I}$ and $Q_{\rm II}(0) = (b_{\rm I}/c_{\rm I})(2c_{\rm I} - c_{\rm II})$. Note that $Q_{\rm II}(0)$ is larger than $b_{\rm II}$ if $b_{\rm I}/b_{\rm II} \in (\alpha_+, 1]$, so there is indeed exceedance of $b_{\rm II}$. From (3), it follows that

$$I(f^*) = \frac{1}{2}(2c_{\rm I})^2 t^* = 2b_{\rm I}c_{\rm I},$$

as required.

**Theorem 3.11** *Suppose* $c_{\rm I} \le 2c_{\rm II}$. *Then it holds that*

$$J(\bar{b}) = \begin{cases} 2b_{\rm II}c_{\rm II} & \text{if } b_{\rm I}/b_{\rm II} \in [0, \alpha_0]; \\ \gamma(\bar{b}) & \text{if } b_{\rm I}/b_{\rm II} \in (\alpha_0, \alpha_+); \\ 2b_{\rm I}c_{\rm I} & \text{if } b_{\rm I}/b_{\rm II} \in [\alpha_+, 1]. \end{cases}$$

*Proof* The proof is similar to that of Theorem 3.10. □

The $s^*$ and $t^*$ associated with the decay rate of the first regime in Theorem 3.11 are $s^* = t^* = b_{\rm II}/c_{\rm II}$, see Sect. 3.3.1. Hence, in the MPP corresponding to the first regime of Theorem 3.11, both queues start to build up at $-t^*$. The MPP is given by, for $r \in [-t^*, 0]$,

$$f^*(r) = \mathbb{E}\left(B(r) | - B(-t^*) = b_{\rm II} + c_{\rm II} t^*\right).$$

Using (22), we find that traffic is generated at a constant rate $2c_{\mathrm{II}}$ in the interval $[-t^*, 0]$, and this results in $Q_{\mathrm{I}}(0) = (b_{\mathrm{II}}/c_{\mathrm{II}})(2c_{\mathrm{II}} - c_{\mathrm{I}}) > b_{\mathrm{I}}$ and $Q_{\mathrm{II}}(0) = b_{\mathrm{II}}$. Applying 'Schilder', yields

$$I(f^*) = \frac{1}{2}(2c_{\mathrm{II}})^2 t^* = 2b_{\mathrm{II}}c_{\mathrm{II}}.$$

The MPPs corresponding to the second and third regime are similar to the MPPs corresponding to the second and third regime of Theorem 3.10.

### 3.4 Discussion

Using Theorems 3.3 and 3.4, also the logarithmic large-buffer asymptotics follow directly. To this end, define

$$J^*(\overline{b}_\alpha) := -\lim_{b\to\infty}\frac{1}{b}\log\mathbb{P}\left(Q_{\mathrm{I}} > \alpha b, Q_{\mathrm{II}} > b\right) \quad \text{with} \quad \alpha\in[0,1],\ b\geq 0,$$

where $\overline{b}_\alpha \equiv (\alpha b, b)$. With $\alpha b = b_{\mathrm{I}}$ and $b = b_{\mathrm{II}}$, i.e., $\overline{b}_\alpha = \overline{b}$, it is not hard to see that $J^*(\overline{b}_\alpha)$ equals $J(\overline{b})$; compare Theorems 3.10 and 3.11 with Theorems 3.3 and 3.4, respectively. Indeed, since we assumed that in the many-sources framework the standard Brownian sources are i.i.d., and because a standard Brownian motion is characterized by independent increments, $J^*(\overline{b}_\alpha)$ and $J(\overline{b})$ should match, see for instance Example 7.4 of Ganesh et al. (2004).

In the analysis of the two-node parallel queue we assumed that the input process was a standard Brownian motion, i.e., no drift and $v(t) = t$. We now show how the results can be extended to general Brownian input, which have drift $\mu > 0$ and variance $v(t) = \lambda t$, $\lambda > 0$. Clearly, we should have that $c_{\mathrm{I}} > c_{\mathrm{II}} > \mu > 0$ to ensure stability. We denote the input process of a general Brownian motion by $\{B^*(t), t\in\mathbb{R}\}$. Then, analogously to (6), $p(\overline{b}) = \mathbb{P}(B^*(\cdot)\in S) = \mathbb{P}(B(\cdot)\in S^*)$, with

$$S^* := \left\{ f\in\Omega \,\middle|\, \exists t\geq 0 : \exists s\in[0,t] : -f(-s) > \frac{b_{\mathrm{I}} + (c_{\mathrm{I}}-\mu)s}{\sqrt{\lambda}}, \right.$$
$$\left. -f(-t) > \frac{b_{\mathrm{II}} + (c_{\mathrm{II}}-\mu)t}{\sqrt{\lambda}} \right\}.$$

Hence in order to generalize the results of Sect. 3 to general Brownian input, we have to set $c_i \leftarrow (c_i-\mu)/\sqrt{\lambda}$ and $b_i \leftarrow b_i/\sqrt{\lambda}$, $i = \mathrm{I}, \mathrm{II}$.

## 4 Analysis of the two-node tandem queue

In this section we focus on the two-node tandem queue. Exploiting the results of the two-node parallel queue in Sect. 3, we derive similar results for the two-node tandem queue.

### 4.1 Joint distribution function

In this subsection we derive an exact expression for $q(\overline{b}) := \mathbb{P}(Q_1 > b_1, Q_2 > b_2)$, with $\overline{b} \equiv (b_1, b_2)$. In Sect. 2.3 we argued that $p(b_{\mathrm{I}}, b_{\mathrm{II}})$ equals $q(\overline{b}_{\mathrm{T}}) := \mathbb{P}(Q_1 > b_1, Q_{\mathrm{T}} > b_{\mathrm{T}})$, with $\overline{b}_{\mathrm{T}} \equiv (b_1, b_{\mathrm{T}})$, given that $b_{\mathrm{I}} = b_1$, $b_{\mathrm{II}} = b_{\mathrm{T}}$, $c_{\mathrm{I}} = c_1$ and $c_{\mathrm{II}} = c_2$. In a first step to obtain $q(\overline{b})$, we derive $q_f(\overline{b}_{\mathrm{T}}) := -\partial q(\overline{b}_{\mathrm{T}})/\partial b_1$. With mild abuse of notation, we also write $q_f(\overline{b}_{\mathrm{T}}) = \mathbb{P}(Q_1 = b_1, Q_{\mathrm{T}} > b_{\mathrm{T}})$. Define $\tau_{\mathrm{T}} \equiv \tau(\overline{b}_{\mathrm{T}}) := (b_{\mathrm{T}} - b_1)/(c_1 - c_2)$ and $\tau \equiv \tau(b_2) := b_2/(c_1 - c_2)$.

**Lemma 4.1** *For each* $b_{\mathrm{T}} \geq b_1 \geq 0$*,*

$$
\begin{aligned}
q_f(\overline{b}_{\mathrm{T}}) = &-\frac{\partial \ell_1(\overline{b}_{\mathrm{T}})}{\partial b_1}\phi(\ell_1(\overline{b}_{\mathrm{T}})) + 2c_1\Psi(\ell_2(\overline{b}_{\mathrm{T}}))e^{-2b_1c_1}\\
&+\frac{\partial \ell_2(\overline{b}_{\mathrm{T}})}{\partial b_1}\phi(\ell_2(\overline{b}_{\mathrm{T}}))e^{-2b_1c_1} + \frac{\partial \ell_3(\overline{b}_{\mathrm{T}})}{\partial b_1}\phi(\ell_3(\overline{b}_{\mathrm{T}}))e^{-2b_{\mathrm{T}}c_2}\\
&+2(c_1-2c_2)(1-\Psi(\ell_4(\overline{b}_{\mathrm{T}})))e^{-2(b_1(c_1-2c_2)+b_{\mathrm{T}}c_2)}\\
&-\frac{\partial \ell_4(\overline{b}_{\mathrm{T}})}{\partial b_1}\phi(l_4(\overline{b}_{\mathrm{T}}))e^{-2(b_1(c_1-2c_2)+b_{\mathrm{T}}c_2)},
\end{aligned}
$$

*where*

$$\ell_1(\overline{b}_{\mathrm{T}}) := \frac{b_1 + c_1\tau_{\mathrm{T}}}{\sqrt{\tau_{\mathrm{T}}}};\ \ell_2(\overline{b}_{\mathrm{T}}) := \frac{-b_1 + c_1\tau_{\mathrm{T}}}{\sqrt{\tau_{\mathrm{T}}}};$$

$$\ell_3(\overline{b}_{\mathrm{T}}) := \frac{b_1 + (c_1 - 2c_2)\tau_{\mathrm{T}}}{\sqrt{\tau_{\mathrm{T}}}};\ \ell_4(\overline{b}_{\mathrm{T}}) := \frac{-b_1 + (c_1 - 2c_2)\tau_{\mathrm{T}}}{\sqrt{\tau_{\mathrm{T}}}}.$$

*Proof* Use Theorem 3.1, with $b_{\mathrm{I}} = b_1$, $b_{\mathrm{II}} = b_{\mathrm{T}}$, $c_{\mathrm{I}} = c_1$ and $c_{\mathrm{II}} = c_2$, to obtain $q(\overline{b}_{\mathrm{T}})$. Then recall that $q_f(\overline{b}_{\mathrm{T}}) = -\partial q(\overline{b}_{\mathrm{T}})/\partial b_1$. We extensively use the chain rule:

$$\frac{\partial \Psi(f(u))}{\partial u} = -\frac{\partial f(u)}{\partial u}\phi(f(u)).$$

Applying straightforward calculus now gives the desired result. □

Note that

$$
\begin{aligned}
q(\overline{b}) = \mathbb{P}(Q_1 > b_1, Q_{\mathrm{T}} > b_2 + Q_1) &= \int_{b_1}^{\infty} \mathbb{P}(Q_1 = x, Q_{\mathrm{T}} > b_2 + x)\mathrm{d}x\\
&= \int_{b_1}^{\infty} q_f(\overline{x})\mathrm{d}x, \qquad (24)
\end{aligned}
$$

where $\overline{x} \equiv (x, b_2 + x)$. Define

$$m_1(\overline{b}) := \frac{b_1 + c_1\tau}{\sqrt{\tau}};\ m_2(\overline{b}) := \frac{-b_1 + c_1\tau}{\sqrt{\tau}};\ m_4(\overline{b}) := \frac{-b_1 + (c_1 - 2c_2)\tau}{\sqrt{\tau}}.$$

We directly present the main theorem on tandem queues.

**Theorem 4.2** *For each* $b_1, b_2 \geq 0$,

$$q(\overline{b}) = \frac{c_2}{c_1 - c_2}\Psi(m_1(\overline{b})) + \Psi(m_2(\overline{b}))e^{-2b_1c_1} + \frac{c_1 - 2c_2}{c_1 - c_2}(1 - \Psi(m_4(\overline{b})))e^{-2(b_1(c_1-c_2)+b_2c_2)}.$$

*Proof* Use (24) in combination with Lemma 4.1. Note that $q_f(\overline{x})$ consists of 6 terms. Let us start with the first term:

$$\int_{b_1}^{\infty} -\frac{\partial \ell_1(\overline{x})}{\partial x}\phi(\ell_1(\overline{x}))\mathrm{d}x = \Psi(\ell_1(\overline{x}))\Big|_{b_1}^{\infty} = -\Psi(m_1(\overline{b})). \tag{25}$$

Similarly, for the second and third term:

$$\int_{b_1}^{\infty}\left(2c_1\Psi(\ell_2(\overline{x}))e^{-2c_1x} + \frac{\partial \ell_2(\overline{x})}{\partial x}\phi(\ell_2(\overline{x}))e^{-2c_1x}\right)\mathrm{d}x = -\Psi(\ell_2(\overline{x}))e^{-2c_1x}\Big|_{b_1}^{\infty} = \Psi(m_2(\overline{b}))e^{-2b_1c_1}. \tag{26}$$

Proceeding with the fourth term:

$$\int_{b_1}^{\infty}\frac{\partial \ell_3(\overline{x})}{\partial x}\phi(l_3(\overline{x}))e^{-2c_2(b_2+x)}\mathrm{d}x = \int_{b_1}^{\infty}\frac{\partial \ell_3(\overline{x})}{\partial x}\frac{1}{\sqrt{2\pi}}e^{-\frac{\ell_1(\overline{x})^2}{2}}\mathrm{d}x = \int_{b_1}^{\infty}\frac{\partial \ell_1(\overline{x})}{\partial x}\frac{1}{\sqrt{2\pi}}e^{-\frac{\ell_1(\overline{x})^2}{2}}\mathrm{d}x = -\Psi(\ell_1(\overline{x}))\Big|_{b_1}^{\infty} = \Psi(m_1(\overline{b})); \tag{27}$$

here the first equality in (27) follows from the fact that $\exp(-\ell_3(\overline{x})^2/2)\exp(-2c_2(b_2+x)) = \exp(-\ell_1(\overline{x})^2/2)$, whereas the second equality holds due to $\partial\ell_3(\overline{x})/\partial x = \partial\ell_1(\overline{x})/\partial x$. We decompose the fifth term into two parts:

$$2(c_1 - 2c_2)(1 - \Psi(\ell_4(\overline{x})))e^{-2(x(c_1-c_2)+b_2c_2)} = 2(c_1-c_2)(1-\Psi(\ell_4(\overline{x})))e^{-2(x(c_1-c_2)+b_2c_2)} + 2c_2(\Psi(\ell_4(\overline{x}))-1)e^{-2(x(c_1-c_2)+b_2c_2)}.$$

Now, taking the first decomposed fifth term and the sixth term:

$$\int_{b_1}^{\infty}\Big(2(1-\Psi(\ell_4(\overline{x})))(c_1-c_2)e^{-2(x(c_1-c_2)+b_2c_2)} - \frac{\partial \ell_4(\overline{x})}{\partial x}\phi(\ell_4(\overline{x}))e^{-2(x(c_1-c_2)+b_2c_2)}\Big)\mathrm{d}x = -(1 - \Psi(\ell_4(\overline{x})))e^{-2(x(c_1-c_2)+b_2c_2)}\Big|_{b_1}^{\infty} = (1 - \Psi(m_4(\overline{b})))e^{-2(b_1(c_1-c_2)+b_2c_2)}. \tag{28}$$

We are left with the second decomposed fifth term:

$$\int_{b_1}^{\infty} 2c_2(\Psi(\ell_4(\overline{x})) - 1)e^{-2(x(c_1-c_2)+b_2c_2)}\mathrm{d}x = \frac{c_2}{c_1 - c_2}\int_{b_1}^{\infty} 2(c_1 - c_2)(\Psi(\ell_4(\overline{x})) - 1)e^{-2(x(c_1-c_2)+b_2c_2)}\mathrm{d}x = \frac{c_2}{c_1 - c_2}\Psi(m_1(\overline{b})) - \frac{c_2}{c_1 - c_2}(1 - \Psi(m_4(\overline{b})))e^{-2(b_1(c_1-c_2)+b_2c_2)}; \tag{29}$$

here the second equality in (29) is obtained by applying integration by parts, but requires tedious calculus. Adding up (25)–(29) yields the stated. □

*Remark* For $b_1 > 0$ and $b_2 = 0$, we find $q(b_1, 0) = \mathbb{P}(Q_1 > b_1) = \exp(-2b_1c_1)$ in Theorem 4.2, i.e., the well-known exponential distribution with mean $1/(2c_1)$. For $b_1 = 0$ and $b_2 > 0$, Theorem 4.2 yields

$$q(0, b_2) = \mathbb{P}(Q_2 > b_2) = \frac{c_1}{c_1 - c_2}\Psi\left(\frac{c_1}{\sqrt{c_1 - c_2}}\sqrt{b_2}\right) + \frac{c_1 - 2c_2}{c_1 - c_2}e^{-2b_2c_2}\left(1 - \Psi\left(\frac{c_1 - 2c_2}{\sqrt{c_1 - c_2}}\sqrt{b_2}\right)\right),$$

which is in line with Theorem 4.3 in Dębicki et al. (2007a).

## 4.2 Exact large-buffer asymptotics

In this subsection we derive the exact asymptotics of the joint buffer content distribution. Define

$$\alpha_+ := \frac{c_1}{2c_1 - c_2};\ \alpha_- := \frac{c_1 - 2c_2}{2c_1 - 3c_2}.$$

It can be verified that $0 < \alpha_- < \alpha_+ < 1$ if $c_1 > 2c_2$, and $0 < \alpha_+ < 1$ if $c_1 \le 2c_2$. Recall that $\zeta(x) = \left(\sqrt{2\pi}x\right)^{-1}\exp(-x^2/2)$. First we present the counterpart of Lemma 3.2.

**Lemma 4.3** *Let $b_1 = \alpha b$ and $b_2 = (1 - \alpha)b$, with $\alpha \in [0, 1]$. If $b \to \infty$, then*

$$\Psi(m_1(\bar{b})) \sim \zeta(m_1(\bar{b}));$$
$$\Psi(m_2(\bar{b})) \sim \begin{cases} \zeta(m_2(\bar{b})) & if\ \alpha < \alpha_+; \\ 1/2 & if\ \alpha = \alpha_+; \\ 1 & otherwise; \end{cases}$$
$$1 - \Psi(m_4(\bar{b})) \sim \begin{cases} 1 & if\ \alpha < \alpha_-\ and\ c_1 > 2c_2; \\ 1/2 & if\ \alpha = \alpha_-\ and\ c_1 \ge 2c_2; \\ -\zeta(m_4(\bar{b})) & otherwise. \end{cases}$$

*Proof* The proof is as in Lemma 3.2. □

Define

$$\theta(\bar{b}) := \frac{1}{\sqrt{2\pi}}\left(\frac{c_2}{c_1 - c_2}\frac{1}{m_1(\bar{b})} + \frac{1}{m_2(\bar{b})} - \frac{c_1 - 2c_2}{c_1 - c_2}\frac{1}{m_4(\bar{b})}\right) \quad \text{and}$$
$$\delta(\bar{b}) := \frac{(b_1(c_1 - c_2) + b_2c_1)^2}{2b_2(c_1 - c_2)}. \tag{30}$$

The following equalities can shown to hold true:

$$\exp\left(-\frac{m_1(\overline{b})^2}{2}\right) = \exp\left(-\frac{m_2(\overline{b})^2}{2}\right)\exp\left(-2b_1c_1\right)$$
$$= \exp\left(-\frac{m_4(\overline{b})^2}{2}\right)\exp\left(-2(b_1(c_1-c_2)+b_2c_2)\right) = \exp\left(-\delta(\overline{b})\right). \quad (31)$$

The proof of the following two theorems is similar to the proof of Theorem 3.3, but now requires Lemma 4.3 and Eqs. (30) and (31). We omit the proofs.

**Theorem 4.4** *Let* $b_1 = \alpha b$ *and* $b_2 = (1-\alpha)b$*, with* $\alpha \in [0,1]$*. Suppose that* $c_1 > 2c_2$*. For* $b \to \infty$*,*

$$q(\overline{b}) \sim \begin{cases} \frac{c_1-2c_2}{c_1-c_2} e^{-2(b_1(c_1-c_2)+b_2c_2)} & \textit{if } \alpha \in [0,\alpha_-); \\ \frac{1}{2}\frac{c_1-2c_2}{c_1-c_2} e^{-2(b_1(c_1-c_2)+b_2c_2)} & \textit{if } \alpha = \alpha_-; \\ \theta(\overline{b})e^{-\delta(\overline{b})} & \textit{if } \alpha \in (\alpha_-,\alpha_+); \\ \frac{1}{2}e^{-2b_1c_1} & \textit{if } \alpha = \alpha_+; \\ e^{-2b_1c_1} & \textit{if } \alpha \in (\alpha_+,1]. \end{cases}$$

**Theorem 4.5** *Let* $b_1 = \alpha b$ *and* $b_2 = (1-\alpha)b$*, with* $\alpha \in [0,1]$*. Suppose that* $c_1 \le 2c_2$*. For* $b \to \infty$*,*

$$q(\overline{b}) \sim \begin{cases} \theta(\overline{b})e^{-\delta(\overline{b})} & \textit{if } \alpha \in [0,\alpha_+); \\ \frac{1}{2}e^{-2b_1c_1} & \textit{if } \alpha = \alpha_+; \\ e^{-2b_1c_1} & \textit{if } \alpha \in (\alpha_+,1]. \end{cases}$$

*Remark* We note that for $c_1 < 2c_2$ and $b_1 = 0$ ($\alpha = 0$) the asymptotics are not given by $\theta(\overline{b})\exp(-\delta(\overline{b}))$, as it can be verified that $\theta(\overline{b})$ equals 0 in this special case. Therefore we have to rely here on the sharper asymptotic $\left(\sqrt{2\pi}u\right)^{-1}\exp(-u^2/2) - \Psi(u) \sim \left(\sqrt{2\pi}u^3\right)^{-1}\exp(-u^2/2)$. Using this, it can be shown Dębicki et al. (2007a) that

$$q(0,b_2) \sim \frac{1}{\sqrt{2\pi}}\left(\frac{c_1-c_2}{b_2}\right)^{3/2}\frac{4c_2}{c_1^2(c_1-2c_2)^2}e^{-\frac{c_1^2}{2(c_1-c_2)}b_2}.$$

### 4.3 Most probable path

Similar to the parallel system, the large-buffer asymptotics now depend on the model parameters $\alpha$, $c_1$ and $c_2$. Again, we will interpret the corresponding regimes by determining the structure of the MPPs.

We feed $n$ i.i.d. standard Brownian sources into the tandem system, and also scale the link rates and buffer thresholds by $n$: $nc_1$, $nc_2$, $nb_1$ and $nb_2$ respectively. By using (9), we can write

$$q_n(\overline{b}) := \mathbb{P}(Q_{1,n} > nb_1, Q_{2,n} > nb_2) = \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} B_i(\cdot) \in U\right).$$

Clearly, $U \subseteq U^* \subseteq V$, with

$$U^*{:=}\{f\in\Omega|\exists t\geq 0:\exists s\in[0,t]:-f(-s)>b_1+c_1 s,\ f(-s)-f(-t)>b_2+c_2 t-c_1 s\};$$
$$V:=\{f\in\Omega|\exists t\geq 0:\exists s\in[0,t]:-f(-s)>b_1+c_1 s,\ -f(-t)>b_1+b_2+c_2 t\}.$$

Hence, 'Schilder' gives

$$K(\bar{b}) := -\lim_{n\to\infty}\frac{1}{n}\log q_n(\bar{b}) = \inf_{f\in U} I(f) \geq \inf_{f\in V} I(f). \tag{32}$$

Let the MPP in set $V$ be denoted by $f^*$. If $f^* \in U$, then there is clearly equality in (32).

**Theorem 4.6** *Suppose $c_1 > 2c_2$. Then it holds that*

$$K(\bar{b}) = \begin{cases} 2(b_1(c_1-c_2)+b_2c_2) & \textit{if } b_1/(b_1+b_2)\in[0,\alpha_-];\\ \delta(\bar{b}) & \textit{if } b_1/(b_1+b_2)\in(\alpha_-,\alpha_+);\\ 2b_1c_1 & \textit{if } b_1/(b_1+b_2)\in[\alpha_+,1]. \end{cases}$$

*Proof* Consider Theorem 3.10 with $c_{\rm I} = c_1$, $c_{\rm II} = c_2$, $b_{\rm I} = b_1$ and $b_{\rm II} = b_1 + b_2$, i.e., we have $U \subseteq V = S$. The MPPs (in $S = V$) corresponding to each of the regimes of Theorem 3.10 were derived in Sect. 3.3. It can easily be checked that these MPPs are also contained in $U$, and consequently they are the MPPs in $U$. This implies that $K(\bar{b})$ is given by Theorem 3.10. □

**Theorem 4.7** *Suppose $c_1 \leq 2c_2$. Then it holds that*

$$K(\bar{b}) = \begin{cases} \delta(\bar{b}) & \textit{if } b_1/(b_1+b_2)\in[0,\alpha_+);\\ 2b_1c_1 & \textit{if } b_1/(b_1+b_2)\in[\alpha_+,1]. \end{cases}$$

*Proof* Consider Theorem 3.11 with $c_{\rm I} = c_1$, $c_{\rm II} = c_2$, $b_{\rm I} = b_1$ and $b_{\rm II} = b_1 + b_2$. Again, the MPPs corresponding to the second and third regime of Theorem 3.11, are also contained in set $U$, so $K(\bar{b})$ is given by Theorem 3.11 for $b_1/(b_1 + b_2) \in (\alpha_0, 1]$. However, the MPP corresponding to the first regime, i.e., $b_1/(b_1 + b_2) \in [0, \alpha_0]$, is not contained in $U$, so we need a different approach here. In order to obtain a workload in queue 2 at least as large as $b_2$ at time 0, queue 2 needs to start building up at $-\tau = -b_2/(c_1 - c_2)$ at the latest. Set $U$ can now be rewritten as

$$\left\{f\in\Omega \,\middle|\, \exists t\geq\tau:\exists s\in[0,t]:\forall u\in[0,t]: \begin{array}{l} -f(-s)>b_1+c_1 s,\\ f(-u)-f(-t)>b_2+c_2 t-c_1 u \end{array}\right\},$$

which is contained in

$$\{f\in\Omega|\exists t\geq\tau:\exists s\in[0,t]:-f(-s)>b_1+c_1 s,\ -f(-t)>b_1+b_2+c_2 t\} =: W.$$

Using the results of Sect. 3.3, with $b_{\rm I} = b_1$, $b_{\rm II} = b_1 + b_2$, $c_{\rm I} = c_1$ and $c_{\rm II} = c_2$, one can show that if $b_1/(b_1 + b_2) \in [0, \alpha_+)$ and $c_1 \leq 2c_2$, then the MPP in $W$ is given by (23). As (23) is contained in $U$, it is also the MPP in $U$, implying that $K(\bar{b}) = \delta(\bar{b})$. □

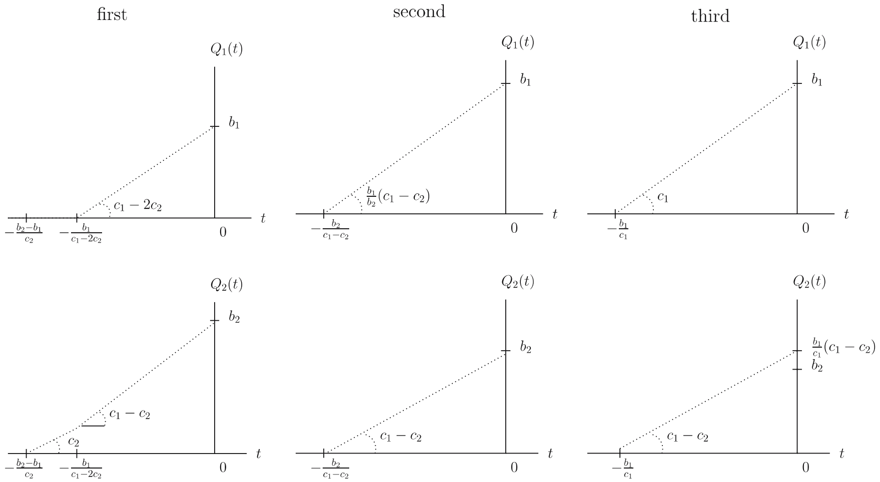


**Fig. 3** The most probable storage path in $\{Q_1 \geq b_1, Q_2 \geq b_2\}$ corresponding to each of the regimes of Theorem 4.6. The most probable storage path corresponding to each of the two regimes of Theorem 4.7, is also given by the most probable storage paths of the last two regimes of Theorem 4.6

Figure 3 depicts for each of the regimes of Theorem 4.6 the most likely way the buffers fill. Clearly, the most likely way the buffers fill for each of the two regimes of Theorem 4.7, coincides with the most probable storage paths of the last two regimes of Theorem 4.6. Interestingly, three types of MPPs are possible. In the first type queue 2 starts to build up earlier than queue 1, but they reach $b_1$ and $b_2$ at the same time. In the second type both queues start to grow at the same time, and reach $b_1$ and $b_2$ at the same time, whereas in the third type both queues start to build up at the same time, but at the time queue 1 reaches $b_1$, queue 2 is strictly larger than $b_2$.

*Remark* If we set $b_1 > 0$ and $b_2 = 0$, then Theorems 4.6 and 4.7 give $K(\bar{b}) = 2b_1c_1$, which indeed is the exponential decay rate of the overflow probability in single queue with standard Brownian input, emptied at rate $c_1$. For $b_1 = 0$ and $b_2 > 0$, Theorems 4.6 and 4.7 yield

$$K(\bar{b}) = \begin{cases} 2b_2c_2 & \text{if } c_1 > 2c_2; \\ \frac{c_1^2}{2(c_1-c_2)}b_2 & \text{otherwise,} \end{cases}$$

which is in line with Sect. 4.1 in Mandjes and van Uitert (2005).

## 4.4 Discussion

As in the two-node parallel queue, we can derive the logarithmic large-buffer asymptotics by using Theorems 4.4 and 4.5. That is,

$$-\lim_{b\to\infty} \frac{1}{b} \log \mathbb{P}(Q_1 > \alpha b, Q_2 > (1-\alpha)b) =: K^*(\bar{b}_\alpha) \quad \text{with} \quad a \in [0,1],\ b \geq 0,$$

where $\overline{b}_\alpha \equiv (\alpha b, (1-\alpha)b)$. With $b_1 = \alpha b$ and $b_2 = (1-\alpha)b$, i.e., $\overline{b}_\alpha = \overline{b}$, it is not hard to see that $K^*(\overline{b}_\alpha)$ and $K(\overline{b})$ coincide; compare Theorems 4.6 and 4.7 with Theorems 4.4 and 4.5, respectively.

Again the results can also be generalized immediately to general Brownian input. Assuming that $c_1 > c_2 > \mu > 0$, this is done by setting $c_i \leftarrow (c_i - \mu)/\sqrt{\lambda}$ and $b_i \leftarrow b_i/\sqrt{\lambda}$, $i = 1, 2$.

## 5 Conclusions

In this paper we analyzed a two-node tandem queue with Brownian input. We obtained the joint distribution function of the workload of the first and second queue, large-buffer asymptotics, and the most probable path leading to overflow. These results were derived by first considering the closely related two-node parallel queue, for which similar results were obtained.

Future research directions include: (1) Analysis of the joint overflow probability in a two-class generalized processor sharing (GPS) system with Brownian inputs. (2) Extending the results obtained in this paper to other input processes. The main approach used in this paper relies on the fact that Brownian motions are characterized by independent increments. Therefore, we expect our approach to be also valid for other input processes that have independent increments (and an LDP), e.g., light-tailed Lévy processes.

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