Heavy tails and upper-tail inequality: The case of Russia

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Abstract Motivated, in part, by the recent surge of interest in robust inequality measurement, cross-country inequality comparisons, applications of heavy-tailed distributions and the study of global and upper-tail inequality, this paper focuses on robust analysis of heavy-tailedness properties and inequality in the upper tails of income distribution in Russia, as measured, mainly, by its tail indices. The study is based on recently developed approaches to robust inference on the degree of heavy-tailedness and their implications for the analysis of upper-tail inequality discussed in the paper. Among other results, the paper provides robust estimates of heavy-tailedness parameters and tail indices for Russian income distribution and their comparisons with the benchmark values in developed economies reported in the previous literature. The estimates point out to important similarity between heavy-tailedness properties of income

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distribution and their implications for the analysis of upper-tail income inequality in Russia and those in developed markets.

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JEL Classification C14 · D31 · P24

1 Introduction and discussion

1.1 Heavy-tailedness and income distribution

Numerous studies in economics and finance have pointed out that many key variables in these fields, including financial returns, foreign exchange rates, income and wealth, have distributions that exhibit heavy tails as in the case of commonly observed Pareto-type power laws (see, for instance, [Embrechts et al. 1997](#), Ch. 7 in [McNeil et al. 2005](#); [Atkinson 2008](#); [Gabaix 2008, 2009](#); [Ibragimov 2009](#); [Milanovic 2005, 2011](#); [Ibragimov et al. 2013, 2015](#); [Ibragimov and Prokhorov 2017](#), and references therein). Heavy-tailed random variables (r.v.'s) $X > 0$ with distribution that has power tails satisfy

$$P(X > x) \sim Cx^{-\zeta}, \quad (1)$$

as $x \rightarrow \infty$, with the tail index $\zeta > 0$ (here and throughout the paper, $f(x) \sim g(x)$ means that $f(x) = g(x)(1 + o(1))$ as $x \rightarrow \infty$).

Important examples of power laws (1) are given by Pareto distributions with parameters (tail indices) ζ where (1) holds exactly for all values x greater than a certain threshold x_m :

$$P(X > x) \sim Cx^{-\zeta}, x \geq x_m, , C = x_m^\zeta, \quad (2)$$

and their modifications such as double Pareto families, and Singh–Maddala distributions that were used in a number of works for income distribution modeling (see [Cowell and Flachaire 2007](#); [Davidson and Flachaire 2007](#); [Toda 2012](#), and references therein).

The tail index ζ characterizes the heaviness (the rate of decay) of the tails of power law distribution (1). An important property of r.v.'s X satisfying a power law with the tail index ζ is that the moments of X are finite if and only if their order is less than ζ : $EX^p < \infty$ if and only if $p < \zeta$. Heavy-tailedness (the tail index ζ) of the variable X (e.g., income, wealth, risk, financial return or foreign exchange rate) governs the likelihood of observing outliers and large observations and fluctuations in the variable. The smaller values of the tail index ζ correspond to a higher degree of heavy-tailedness in X and, thus, to a larger likelihood of observing outliers, extreme observations and large fluctuations in realizations of this variable. In particular, conclusions of tail index parameters being similar for a variable of interest in two or more economies are important because they point out to similarity in likelihoods of occurrence of outliers, extreme observations and large fluctuations in the variable considered. As discussed in the next section, in the case of income distributions, similarity

of tail indices for different economies further implies similarity of their implications concerning inequality in the upper tails of distributions of these variables.

Empirical studies of income and wealth indicate that distributions of these variables in developed economies typically satisfy power laws (1) with the tail index ζ that varies between 1.5 and 3 for income and is rather stable, perhaps around 1.5, for wealth (see, among others, [Atkinson 2004](#); [Atkinson and Piketty 2007](#); [Gabaix 2008, 2009](#); [Atkinson et al. 2011](#), and references therein). This implies, in particular, that the mean is finite for income and wealth distributions (since $\zeta > 1$). However, the variance is infinite for wealth (since $\zeta \approx 1.5 < 2$) and may be infinite for income (if $\zeta \leq 2$). In addition, since their tail indices are smaller than 3, income and wealth distributions have infinite third and higher moments. The problem of infinite variance in income and wealth distributions is important because it may invalidate or make problematic direct applicability of standard inference approaches, including regression analysis, (auto-)correlation based approaches and least squares methods (see the discussion in [Ibragimov et al. 2015](#); [Ibragimov and Prokhorov 2017](#)). In a similar fashion, infinite fourth moments for these variables need to be taken into account in regression and other models involving their volatilities or variances, e.g., in the analysis of permanent and transitory components of income variability and their cross-country comparisons (see [Gorodnichenko et al. 2010](#)) and the study of autocorrelation properties of financial time series and volatility clustering (see the discussion in [Cont 2001](#); [Mikosch and Stariča 2000](#), and references therein). Finiteness of first moments is important because it points out to optimality of diversification and robustness of a number of economic models for the variables considered (see [Ibragimov 2009](#); [Ibragimov et al. 2013, 2015](#); [Ibragimov and Prokhorov 2017](#)). Several recent works in the literature have emphasized robustness as an important aspect in the choice of inequality measures and estimation and inference methods for them (see, among others, [Cowell and Flachaire 2007](#); [Davidson and Flachaire 2007](#); [Ibragimov et al. 2017](#) and references therein). The interest in robust inequality assessment is motivated, in part, by sensitivity of many income inequality measures to changes in different parts of the underlying income distribution, including sensitivity to extremes and outliers.

1.2 Heavy tails and upper-tail inequality

The analysis of heavy-tailedness of income distributions is closely related to the study of inequality in their tails (among the rich and super-rich) and top incomes' dynamics and distribution. The recent years have witnessed a revival of interest in the analysis of the volume and distributions of top incomes and their inequality that are of key importance for a number of reasons, including the enormous gaps between the incomes of the rich and the rest of population, or "the haves and the have-nots," as well as between the rich and very rich, or "the haves and have-mores," e.g., between the top 1 percent and the top 0.1 percent of income and wealth distributions (see, among others, the contributions in [Davies 2008](#), and the discussion in [Milanovic 2011](#)); the huge magnitude of the total income distributed among a few (super-)rich individuals; implications for overall fairness of distribution of economic resources across individuals in a society and important economic impacts such as those on overall growth and resources, taxation and other redistributive policies as well as on overall inequality (see [Atkinson](#)

et al. 2011, and references therein). For instance, as discussed in Milanovic (2011), in 2004, the top 0.1% annual after-tax income per capita in UK was 400,000 GBP, the top 1% was 81,000 GBP, and the average British after-tax per capita income was 11,600 GBP per annum. As indicated in Atkinson et al. (2011), the top 1% of the US income distribution in 2007 captures more than a fifth of total income in the country. Similarly, in the world income distribution, the top 10% receive more than two-thirds of the total world income, while the top 5% receive 45% of the income. As discussed in Milanovic (2011), for the World income distribution, the ratio of the incomes of the top 10% and the bottom 10% incomes (the so-called decile ratio) is about eighty to one, and the ratio for the top 5% and the bottom 5% is almost two hundred to one (see also the World Bank's World Development Indicators database for income shares of different deciles of income distributions in the countries of the World, <http://databank.worldbank.org>). A related fact on the extreme nature of the World income distribution is that the income of the top 1.75% matches the income of the poorest 77%. Using the data on the top 10% and 1% incomes, the (upper) inequality around the world and its changes over time are also emphasized in the recent book by Piketty (2014).¹

Gurieva and Rachinsky (2009) discuss the dynamics of wealth distribution and inequality in the former USSR and Central and Eastern European countries during the transition period. Gurieva and Rachinsky (2009), in particular, point out that the combined wealth of the 26 Russian billionaires in 2004 amounted to 19% of Russian GDP (for comparison, as indicated in the paper, the combined wealth of the 262 US billionaires was only 7% of the US GDP). Using the dataset on incomes of Moscow taxpayers in 2004 that includes the super-rich, Gurieva and Rachinsky (2006) find the Gini coefficient of about 63% for resulting income distribution, which is contrasted with the official value of 41% for the Gini coefficient in Russia in 2004 (see also Table 1 for the official values of the Gini coefficient of income inequality in Russia and the discussion in Gurieva and Rachinsky 2009). Gurieva and Rachinsky (2006) further indicate that the top 10% of individuals get 50% of the total income in the dataset. These and other facts further indicate the extreme nature of income and wealth distributions and inequality in Russia, especially those in the distributions' upper tails (see also the results and discussion in Sects. 4 and 5).

To illustrate the connection between heavy-tailedness and upper-tail inequality, consider the class of power law *Pareto* distributions (2), where (1) holds for *all* values $x \geq x_m$. The Gini coefficient G of inequality over the *whole* income or wealth distribution (2) is equal to

$$G = G(\zeta) = 1/(2\zeta - 1). \quad (3)$$

Power law distribution (1) is approximately Pareto (2) for (*large*) income or wealth levels x in the *upper tails*. Thus, taking into account relation (3), the value of the tail index ζ in (1) may be regarded as a measure of *upper tail* inequality (that is, among the rich, with smaller values of the tail index corresponding to larger inequality in

¹ See also the discussion of inequality at the very top incomes around the world in the inequality issue of the *Economist*, where, in addition to comparisons of Gini coefficients, the top 1% incomes and "tail inequality" measures in the form of a number of (dollar) Billionaires in different countries are considered (<http://www.economist.com/node/21564414>).

the upper tails) that seems to be a useful complement to the Gini coefficient for the whole income or wealth distribution (see also the discussion in [Atkinson 2008](#)).² As expected, a higher degree of heavy-tailedness in the underlying (power law (1)) income distribution and the implied greater likelihood of occurrence of outliers and large fluctuations in it (that is, a smaller value of the tail index ζ) translates into greater inequality in the tails.

1.3 Research objectives and an overview of results

The discussion in the previous sections illustrates the importance of reliable inference on income distributions and their heavy-tailedness properties, in particular, in the context of making conclusions on upper-tail inequality.

Emerging economic markets are likely to be more volatile than their developed counter-parts and subject to more extreme external and internal shocks (see, among others, [Gorodnichenko et al. 2010](#), for the discussion of extreme macroeconomic disturbances accompanying Russia's transition to a market economy following the collapse of the Soviet Union). The higher degree of volatility suffered by emerging countries leads to the expectation that heavy-tailedness properties and distributions of their key economic and financial variables and indicators may be different from those in developed economies. The recent analysis in [Ibragimov et al. \(2013\)](#) (see also [Ibragimov et al. 2015](#)) provides support for the hypothesis that emerging country exchange rates are more pronouncedly heavy-tailed compared to developed country exchange rates. In particular, according to the analysis, in contrast to developed country exchange rates, variances may be infinite for several emerging country exchange rates considered in [Ibragimov et al. \(2013\)](#).

This paper focuses on the robust analysis of heavy-tailedness in income distribution in Russia, with a particular focus on the question whether, similar to the case of emerging country exchange rates, its heavy-tailedness characteristics differ from those of income distributions in developed economies considered in the previous literature. Among other results, using recently proposed robust tail index inference methods, we provide estimates of heavy-tailedness parameters ζ for income distribution in Russia and their comparisons with the benchmark values $\zeta \in (1.5, 3)$ that are well-established for developed economies (see Sect. 4).

Our results point out to important and somewhat surprising similarity between heavy-tailedness properties of income distribution in Russia and those in developed markets. The estimates of the tail index ζ for Russian income distribution obtained in this paper are largely in agreement with the benchmark interval $\zeta \in (1.5, 3)$ for

² This paper focuses on estimation of the degree of heavy-tailedness and tail indices ζ for income and wealth distributions considered, with their above natural interpretation as characteristics of inequality in the tails of these distributions. Recently, there was some interest (see [Jansen and Volpert 2012](#); [Schneider and Tavani 2015](#)) in defining and estimating the Lorenz curves and Gini indices for upper (and lower) tails of power law-type and more general distributions. The quantity $G(\zeta) = 1/(2\zeta - 1)$ appears in [Jansen and Volpert \(2012\)](#), as the limit of the right-sided Gini functions that are based on upper tails of power law distributions considered. The estimates of the tail index in power law distributions (1) developed in this paper, together with the standard errors and confidence intervals can be used to construct semiparametric estimates of upper-tail inequality measures like $G(\zeta)$ considered in the above works and their confidence intervals (see the working paper version [Ibragimov and Ibragimov 2010](#), of this work).

income distributions in developed economies considered in the previous literature. This indicates that the likelihood of observing very large income values and outliers in income distribution in Russia is similar to that in developed countries. It further points out that inequality in the right tail of Russian income distribution (among the rich), as measured by tail indices, appears to be similar to that in developed economies. The above conclusions are in contrast to the case of foreign exchange rates, where, according to the analysis in [Ibragimov et al. \(2013\)](#) discussed above, heavy-tailedness properties in emerging markets, including Russia, are more pronounced compared to developed economies. Furthermore, the estimates obtained in the paper indicate that the tail index of Russian income distribution is greater than 2 and, thus, the distribution has finite variance. The conclusion on the finite variance is especially important since it justifies applicability and validity of the standard OLS regression methods in the analysis of models involving the data on incomes.³

The rest of the paper is organized as follows. Sect. 2 discusses inference methods used in the analysis. Sect. 3 reviews the datasets used in the study. Sect. 4 presents and discusses the estimation results on heavy-tailedness of Russian income distribution obtained in the paper. Sect. 5 makes some concluding remarks and reviews suggestions for the further research. Appendix contains the tables and diagrams on the empirical results obtained or used in the paper.

2 Methodology

Several approaches to inference about the tail index ζ of heavy-tailed distributions are available in the literature (see the review in [Embrechts et al. 1997](#), [Beirlant et al. 2004](#), Ch. 7 in [McNeil et al. 2005](#); [Ibragimov et al. 2015](#), and references therein). The two most commonly used ones are Hill's estimator and the OLS approach using the log–log rank-size regression.

Let $X_1, X_2, \dots, X_N > 0$ be a sample from a population satisfying power law (1) (e.g., a sample of household income or wealth levels). Further, let, for a tail truncation level $n < N$

$$X_{(1)} \geq X_{(2)} \geq \dots \geq X_{(n)} \geq X_{(n+1)} \quad (4)$$

be decreasingly ordered largest (extreme) values of observations in the sample (that is, $n + 1$ upper order statistics for the sample). Typically, in practice, the number n of extreme observations (4) used in estimation of the tail index is taken to be equal to some fraction (e.g., $m\% = 10\%, 5\%$ or 1%) of the total sample size N : $n = mN/100$ (see the discussion in [Gabaix and Ibragimov 2011](#); [Ibragimov et al. 2013, 2015](#), and references therein).⁴

³ As discussed in the working paper version of this work [Ibragimov and Ibragimov \(2010\)](#), empirical results further indicate that the conclusions obtained for income distribution in Russia also hold for several other CIS economies. That paper also provides estimates of heavy-tailedness parameters for the World's wealth distribution that are mostly in agreement with empirical findings of tail indices $\zeta \approx 1.5$ for wealth distributions in developed countries (see the discussion in Sect. 1.1).

⁴ The standard errors of Hill's and log–log rank-size tail index regression estimates decrease with the increase in tail truncation levels and the number n of extreme observations used in tail index estimation. However, as discussed in many works in the literature (see, among others, [Embrechts et al. 1997](#),

Hill's estimator $\hat{\zeta}_{Hill}$ of the tail index ζ is given by (see, among others, Embrechts et al. 1997; Drees et al. 2000; Beirlant et al. 2004; Gabaix 2008; Ibragimov et al. 2013, 2015, and references in those works) $\hat{\zeta}_{Hill} = n / \sum_{t=1}^n [\log(X_{(t)}) - \log(X_{(n+1)})]$. The standard error of the estimator is $s.e._{Hill} = \hat{\zeta}_{Hill} / \sqrt{n}$. The corresponding 95%-confidence interval for the true (unknown) tail index ζ is thus $(\hat{\zeta}_{Hill} - 1.96\hat{\zeta}_{Hill} / \sqrt{n}, \hat{\zeta}_{Hill} + 1.96\hat{\zeta}_{Hill} / \sqrt{n})$.

It was reported in a number of studies that inference on the tail index using Hill's estimator suffers from several problems, including non-robustness with sensitivity to dependence in data and deviations from power laws in the form of additional slowly varying factors in (1), and poor finite sample properties (see, for instance, Embrechts et al. 1997, Ch. 6). Motivated by these problems, several studies have focused on alternative approaches to the tail index estimation. For instance, Huisman et al. (2001) propose a weighted analogue of Hill's estimator that was reported to correct its small sample bias for sample sizes less than 1000. Embrechts et al. (1997), among others, advocate sophisticated nonlinear procedures for tail index estimation.

In addition to Hill's estimates of tail indices of income distributions, we also provide tail index estimates obtained using modifications of log–log rank-size regressions with shifts in ranks recently proposed in Gabaix and Ibragimov (2011) (see also Ibragimov et al. 2015). These estimation procedures use the optimal shifts in ranks and the correct standard errors obtained in Gabaix and Ibragimov (2011).

Despite the availability of more sophisticated methods, a popular way to estimate the tail index ζ is still to run the following OLS log–log rank-size regression with $\lambda = 0$: $\log(t - \lambda) = a - b \cdot \log(X_{(t)})$, $t = 1, \dots, n$, or, in other words, calling t the rank of an observation, and $X(t)$ its size: $\log(Rank - \lambda) = a - b \cdot \log(Size)$ (here and throughout the rest of the paper, $\log(\cdot)$ stands for the natural logarithm). The reason for the popularity of the OLS approach to tail index estimation is arguably the simplicity and robustness of this method, including its robustness to dependence and deviations from power laws (see the review in Gabaix and Ibragimov 2011; Ibragimov et al. 2013, 2015, and the discussion at the end of this section).

Unfortunately, the tail index estimation procedures based on OLS log–log rank-size regressions with $\lambda = 0$ are strongly biased in small samples. The paper by Gabaix and Ibragimov (2011) provides a simple practical remedy for this bias, and argues that, if one wants to use an OLS regression approach to tail index estimation, one should use the $Rank - 1/2$, and run $\log(Rank - 1/2) = a - b \cdot \log(Size)$, that is,

$$\log(t - 1/2) = a - b \cdot \log(X_{(t)}), \quad t = 1, \dots, n. \quad (5)$$

Footnote 4 continued

Beirlant et al. 2004) tail index estimation and truncation level selection for Hill's estimator involves the important problem of bias-variance tradeoff that appears with the increase in tail truncation levels and the number of extreme observations used in tail index estimation. Similar bias-variance tradeoff also holds for log–log rank-size regression tail index estimation. Usually, robustness of tail index estimates to the choice of tail truncation levels is assessed using Hill's plots (see Embrechts et al. 1997; Mikosch and Stariča 2000), and their analogues for the log–log rank-size regression tail index estimates (see Ibragimov et al. 2013, Section 3.2 in Ibragimov et al. 2015, and the discussion at the end of the next section), that is, graphs of tail index estimates and their confidence intervals for different tail truncation levels $m\%$ and the corresponding number $n=mN/100$ of largest observations used in estimation.

In (5), one takes the OLS estimate $\hat{b}$ as the log–log rank-size regression estimate $\hat{\zeta}_{RS}$ of the tail index ζ . The shift of $1/2$ is optimal and reduces the bias to a leading order. The standard error of the estimator $\hat{\zeta}_{RS}$ is $s.e._{RS} = \sqrt{2\hat{\zeta}_{RS}}/\sqrt{n}$ (the standard error is thus different from the OLS standard error given by $\hat{\zeta}_{RS}/\sqrt{n}$). The corresponding correct 95% confidence interval for ζ is thus $(\hat{\zeta}_{RS} - 1.96\sqrt{2\hat{\zeta}_{RS}}/\sqrt{n}, \hat{\zeta}_{RS} + 1.96\sqrt{2\hat{\zeta}_{RS}}/\sqrt{n})$.

Numerical results in Gabaix and Ibragimov (2011) (see also the discussion in Ibragimov et al. 2015) demonstrate the advantage of the proposed approach over the standard OLS estimation procedures with $\lambda = 0$ and indicate that it performs well under auto-correlated processes driven by innovations with deviations from power laws in the form of additional slowly varying factors in (1) and, importantly, for dependent heavy-tailed processes, including GARCH models that are often used as frameworks for modeling the dynamics of dependent heavy-tailed economic and financial time series such as financial returns and foreign exchange rates. The modifications of the OLS log–log rank-size regressions with the optimal shift $\lambda = 1/2$ and the correct standard errors provided by Gabaix and Ibragimov (2011) were subsequently used in a number of works in economics and finance (see the discussion and references in Gabaix and Ibragimov 2011; Ibragimov et al. 2015).

3 Data

The analysis of Russian income distribution in the paper is based on the datasets available from Rosstat (2007) and similar publications by the Rosstat for other years. Rosstat (the Federal State Statistics Service of Russian Federation) conducts sampling surveys of household budgets continuously during a calendar year in all subjects of Russian Federation. The surveys cover about 50 thousands of households. The microdata on the survey results are provided by the Federal State Statistics Service online (<http://www.micro-data.ru>). Table 1 in Appendix referred to in the introduction provides the Rosstat's official values of Gini coefficients for the income distribution in Russia and also their estimates from the Standardized World Income Inequality Database (SWIID, see Solt, 2009).⁵ Table 2 provides some of the basic descriptive statistics for the datasets on income in Russia from the Rosstat.

⁵ According to Rosstat's data in the Table 1, the official values of the Gini coefficient for the whole income distribution in Russia increased from about 0.3 in 1992 to relatively stable values about 0.4 in 1995–2009. While the Gini coefficients in Russia from the SWIID for 1992 (the coefficient of 0.4) and 1995 (the coefficient of 0.45) are greater than the Rosstat's estimates, the estimates of the Gini coefficients in 2000–2009 (the coefficients of about 0.4) from both the sources are rather similar to each other. Gorodnichenko et al. (2010) present a detailed analysis of Russian income inequality dynamics using a panel microdata set from the Russia Longitudinal Monitoring Survey (RLMS) for 1994–2005 (see also Kislitsyna 2003, for a review of the dynamics of income inequality in Russia in the 1990s). The authors argue that, after an initial rise in the early 1990s, the measures of income inequality in Russia stayed relatively high during 1994–1998 and then started falling during 2000–2005. According to the estimates in the paper, the Gini coefficient for earnings in Russia increased from 0.28 in 1985 and 0.32 in 1990 to 0.48 in 1995 and then declined to 0.41 in 2005 (see also Kislitsyna 2003). The latter value of the Gini coefficient in 2005 is just slightly higher than the mean value of Gini coefficients for after-tax household income in upper middle income countries.

4 Empirical results

Table 3 provides the tail index estimates $\hat{\zeta}_{RS}$ for the income distribution in Russia obtained using log–log rank-size regression (5) with the optimal shift $\lambda = 1/2$ and the correct standard errors $s.e._{RS} = \sqrt{2}\hat{\zeta}_{RS}/\sqrt{n}$, as discussed in Sect. 2. The table also provides the correct 95% -confidence intervals for the true tail indices ζ in (1) constructed using these standard errors. The last three columns of Table 3 provide Hill's tail index estimates $\hat{\zeta}_{Hill}$, their standard errors $s.e._{Hill} = \hat{\zeta}_{Hill}/\sqrt{n}$ and the corresponding 95% confidence intervals for the tail indices ζ .

The inference results for Russia in Table 3 are presented for the number n of extreme observations (4) used in estimation equal to $m\% = 10\%$ and 5% of the total sample size N : $n=mN/100$ (see Sect. 2).⁶

The results in Table 3 for the income distribution in Russia are largely in agreement with the empirical results on the tail indices $\zeta \in (1.5, 3)$ for income distribution in developed economies (Sect. 1.1). Namely, all the log–log rank-size regression point estimates $\hat{\zeta}_{RS}$ and Hill's estimates $\hat{\zeta}_{Hill}$ in the table are very close to the value $\zeta = 3$. The most of these point estimates are slightly smaller than 3 and, thus, belong to the benchmark interval $\zeta \in (1.5, 3)$.

Similarly, the most of the confidence intervals constructed using the estimates $\hat{\zeta}_{RS}$ and $\hat{\zeta}_{Hill}$ in the table either lie inside of the interval $(1.5, 3)$ or have their larger parts lying in the interval. Furthermore, this is the case for the tail index estimates and the corresponding confidence intervals constructed using different tail truncation levels ($m\% = 10\%$, 5% and 1% of the total number N of observations). For instance, according to the confidence intervals in the last two rows of Table 3 constructed using the estimates $\hat{\zeta}_{RS}$ and $\hat{\zeta}_{Hill}$ for different truncation levels $m\%$, the tail index ζ of the income distribution in Russia in the 4th quarter of 2007 satisfies $\zeta \in (2.7, 3.2)$ with 95% probability. Similar conclusions also hold for other time periods in the table.

Importantly, the left end points of *all* the confidence intervals in Table 3 are greater than 2. That is, the null hypothesis $H_0 : \zeta = 2$ is rejected in favor of $H_a : \zeta > 2$ at the 2.5% significance level for all the time periods dealt with. These conclusions thus imply that the variance of the income distribution in Russia is finite.

Similar to the point estimates $\hat{\zeta}$, the right-end points of all the confidence intervals in Table 3 are close to the right boundary ($=3$) of the interval $\zeta \in (1.5, 3)$ in developed markets. Thus, similar to the case of developed markets (see the discussion in Sects. 1.1), the third moment is likely to be infinite for the income distribution in Russia.

In addition, the right-end points of all the confidence intervals are smaller than 4. This implies that the null hypothesis $H_0 : \zeta = 4$ is rejected in favor of $H_a : \zeta < 4$ at the 2.5% significance level in all of the time periods in the table. Consequently, similar to the case of developed economies, the income distribution in Russia has infinite fourth moment.

⁶ The conclusions in this section on testing the main hypotheses of interest (that the tail indices are likely to lie in the interval $(1.5, 3)$; are greater than 2 and smaller than 4) obtained using the 10% and 5% tail truncation levels are also supported by the empirical results obtained with 1% of largest observations. However, in a number of cases, the point estimates of the tail indices and the corresponding confidence intervals obtained using the 1% tail truncation levels differ quite significantly from those for $m\%=5\%$ and 10% . It suggests that the results with $m\%=1\%$ are downward biased (see also the discussion at the end of this section).

The qualitative agreement of the results in Table 3 with those for developed economies in the literature indicates that the likelihood of observing very large income values and outliers in income distribution in Russia is similar to that in developed countries. Importantly, the above similarity conclusions are in contrast to the case of foreign exchange rates, where, according to the analysis in Ibragimov et al. (2013) discussed in Sect. 1.3, heavy-tailedness properties in emerging markets are more pronounced compared to developed economies. Furthermore, the estimates obtained in this section indicate that the tail index of Russian income distribution is greater than 2 and, thus, the distribution has finite variance. As discussed in the introduction, this conclusion is especially important since it justifies applicability and validity of the standard OLS regression methods in the analysis of models involving the income data.

Similar to Figures 3.1–3.3 and the discussion in Section 3 in Ibragimov et al. (2013) and Section 3.2 in Ibragimov et al. (2015), in order to illustrate the appropriateness of the tail truncation levels ($m\% = 5\%$ and 10% of the total number N of observations) used in this section for income distribution in Russia (see Table 3), we follow the analysis and suggestions in Embrechts et al. (1997) and Mikosch and Stariča (2000) and present the analogues of Hill's plots for the log–log rank-size regression tail index estimates for the Russian income distribution in the 4th quarter of 2007 (Figure 1). These are graphs of the log–log rank-size regression point estimates $\hat{\zeta}_{RS}$ of the tail index of income distribution in Russia in that period for different truncation levels ($n=mN/100$ largest observations, $m\%$ of the total sample size N , used in tail index estimation). Figure 1 also presents the corresponding 95% confidence intervals for the true tail index of the income distribution computed using the estimated log–log rank-size regressions. The diagram points out to relative stability of the log–log rank-size regression tail index estimates across different truncation levels. In particular, similar to Figures 3.1–3.3 in Ibragimov et al. (2013), we note that the 95% confidence intervals for the true tail index in the diagram constructed using log–log regressions with different truncation levels intersect. This means that the tail indices in corresponding power law approximations (1) for the tails of income distribution in Russia are statistically indistinguishable. In particular, the above conclusions on statistical tests for the tail index of the Russian income distribution (e.g., whether $\zeta \in (1.5, 3)$, $\zeta > 2$ and $\zeta < 4$) remain the same regardless of truncation levels for the log–log rank-size regression tail index estimates and the corresponding confidence intervals the tests are based on.⁷

5 Conclusion and further research

Emerging and developing economies are likely to be more volatile than their developed counter-parts and subject to more extreme external and internal shocks. The higher

⁷ Similar Hill's plots for the tail index of income distribution in Russia indicate less stability of Hill's tail index estimates compared to log–log rank-size regression estimates. Although confidence intervals constructed using Hill's estimates with many truncation levels intersect, it is not the case for truncation levels that are too far apart. Nevertheless, similar to statistical tests on the true tail index (e.g., $\zeta \in (1.5, 3)$, $\zeta > 2$ and $\zeta < 4$) in this section conducted using log–log rank-size regression estimates and the corresponding confidence intervals, the conclusions of the tests based on Hill's estimates with different truncation levels and confidence intervals corresponding to them remain the same.

degree of volatility leads to the expectation that heavy-tailedness properties and distributions of key variables in these markets, including income and wealth, may differ from those in developed economies. However, the results obtained in this paper point out to important and somewhat surprising similarities between the heavy-tailedness properties of (and corresponding conclusions on upper-tail inequality in) income distribution in Russia and those in developed markets. This is in contrast to the case of foreign exchange rates, where, according to the analysis in [Ibragimov et al. \(2013\)](#), heavy-tailedness properties in emerging markets are more pronounced compared to developed economies. According to the empirical results in the working paper version [Ibragimov and Ibragimov \(2010\)](#) of this work, similar conclusions also hold for income distributions in several CIS economies. Among other implications, the analysis in the paper points out to the necessity of the use of inference methods that are robust to heavy tails, infinite moments, heterogeneity and outliers in the analysis of income and wealth data.

Of key importance is the long-standing problem of quantifying and further explaining the relationship between key macroeconomic variables, such as economic development and mobility in labor markets, and inequality and income distribution, including its behavior in the upper tails. Concerning this research direction, as discussed at the beginning of the paper, the dynamics of inequality is strongly correlated with such important macroeconomic factors as mobility in labor markets and changes in labor market conditions (see [Kopczuk et al. 2010](#), who focus on the analysis of earnings inequality and mobility in the United States in the 20th century). In addition, income distribution and its heavy-tailedness, and the dynamics of inequality, upper-tail inequality and their structure are affected by political and macroeconomic structural breaks such as wars and economic crises (e.g., see [Piketty and Saez 2003](#), for the discussion of effects of the Great Depression and World War II on top capital incomes and top wage shares in the United States in the last century), changes in government policies and labor market regulations and other institutional factors. Further research may focus on the analysis of the important problem of the effects of economic and financial crises and other structural shocks on (upper tail) income and wealth inequality in different countries, including Russia and emerging economies, and international and global inequality. Similar to the discussion in this paper, the analysis of the crisis impact on upper tail inequality is related to the study of its effects on heavy-tailedness properties of income and wealth distributions. Such a study may be conducted using robust confidence intervals for tail indices and inequality measures for the pre-crisis and post-crisis periods constructed similar to this paper (see [Ibragimov et al. 2013, 2015](#), for the analysis of the effects of the on-going global crisis on heavy-tailedness characteristics of emerging and developed country exchange rates). Further research topics of key interest include the analysis of economic explanations for the obtained empirical results on heavy-tailedness of income distribution in Russia and other CIS countries. This also concerns observed similarities in heavy-tailedness properties and upper-tail inequality in emerging economies like Russia and those in developed markets, as measured by tail indices of their income distributions.

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6 Appendix: Tables, figures and estimation results

See Tables 1, 2 and 3 and Fig. 1

Table 1 The Gini coefficient of income inequality in Russia Source: Rosstat, the Federal State Statistics Service of Russian Federation

Years	1992	1995	2000	2004	2005	2006	2007	2008	2009
Gini coefficient	0.289	0.387	0.395	0.409	0.409	0.416	0.423	0.422	0.422

Table 2 Descriptive statistics for distribution of household incomes in Russia, 2005–2007 (quarterly data, average rubles per month)

Year, quarter	Sample size	Mean	Median	Moda	Std. dev.	Max.	Min.	Skewness	Kurtosis	Gini coef.
2005, 1	46974	22820	17075	10000	23940	1429393	0.02	14.5	570.1	0.414
2005, 2	53132	25086	18975	10000	25290	1269834	0.25	10.3	294.4	0.412
2005, 3	53129	27000	20302	9500	28032	1849601	0.00	13.2	539.3	0.415
2005, 4	53135	29668	22430	13500	28740	1119837	0.04	8.6	210.6	0.412
2006, 1	53093	28619	21793	9500	30548	2152383	0.07	20.6	1133.6	0.409
2006, 2	53094	30445	23139	9500	33214	2110964	0.03	19.6	954.5	0.411
2006, 3	53089	32435	24635	13500	31426	1981988	0.05	10.5	415.3	0.412
2006, 4	53072	35784	27422	16000	32746	2104030	0.18	9.5	409.1	0.407
2007, 1	50589	34561	26523	17000	34235	1718131	0.07	12.8	478.3	0.406
2007, 2	49884	37562	28636	13000	34819	1869102	0.05	7.8	240.5	0.410
2007, 3	53104	41355	31363	15000	47198	3246727	0.17	23.6	1285.2	0.418
2007, 4	53096	46789	35585	18000	46460	2446446	0.06	10.0	305.6	0.413

Table 3 Tail index estimates for income distribution in Russia

1	2	3	4	5	6	7	8	9	10
Year, quarter	Sample size, N	% of largest observations, m	n	$\hat{\xi}_{RS}$	$s.e._{RS} = \sqrt{\frac{2}{n} \hat{\xi}_{RS}}$	95% CI	$\hat{\xi}_{Hill}$	$s.e._{Hill} = \frac{\hat{\xi}_{Hill}}{\sqrt{n}}$	95% CI
2005, 1	46974	10	4697	2.942	0.061	(2.823, 3.060)	2.827	0.041	(2.746, 2.908)
		5	2349	2.950	0.086	(2.781, 3.118)	3.012	0.062	(2.890, 3.134)
2005, 2	53132	10	5313	2.887	0.056	(2.777, 2.996)	2.786	0.038	(2.711, 2.861)
		5	2656	2.916	0.080	(2.760, 3.073)	2.997	0.058	(2.883, 3.111)

Table 3 continued

1	2	3	4	5	6	7	8	9	10
Year, quarter	Sample size, N	% of largest observa- tions, m	n	$\hat{\zeta}_{RS}$	$s.e._{RS} =$ $\sqrt{\frac{2}{n} \hat{\zeta}_{RS}}$	95% CI	$\hat{\zeta}_{Hill}$	$s.e._{Hill} =$ $\frac{\hat{\zeta}_{Hill}}{\sqrt{n}}$	95% CI
2005, 3	53129	10	5313	2.871	0.056	(2.762, 2.980)	2.755	0.038	(2.681, 2.829)
		5	2656	2.877	0.079	(2.722, 3.031)	2.972	0.058	(2.859, 3.085)
2005, 4	53135	10	5313	3.045	0.059	(2.929, 3.161)	2.879	0.040	(2.801, 2.956)
		5	2656	3.075	0.084	(2.909, 3.240)	3.202	0.062	(3.080, 3.324)
2006, 1	53093	10	5309	2.999	0.058	(2.885, 3.113)	2.873	0.039	(2.796, 2.950)
		5	2655	2.971	0.082	(2.811, 3.131)	3.199	0.062	(3.077, 3.321)
2006, 2	53094	10	5309	2.880	0.056	(2.771, 2.990)	2.853	0.039	(2.777, 2.930)
		5	2655	2.828	0.078	(2.676, 2.980)	2.992	0.058	(2.878, 3.106)
2006, 3	53089	10	5309	2.994	0.058	(2.880, 3.108)	2.792	0.038	(2.717, 2.867)
		5	2655	3.076	0.084	(2.910, 3.241)	3.048	0.059	(2.932, 3.164)
2006, 4	53072	10	5309	3.238	0.063	(3.115, 3.361)	2.886	0.040	(2.808, 2.963)
		5	2655	3.489	0.096	(3.301, 3.677)	3.237	0.063	(3.114, 3.360)
2007, 1	50589	10	5059	3.019	0.060	(2.902, 3.137)	2.891	0.041	(2.811, 2.970)
		5	2530	3.023	0.085	(2.857, 3.190)	3.109	0.062	(2.988, 3.231)
2007, 2	49884	10	4988	3.088	0.062	(2.967, 3.210)	2.857	0.041	(2.778, 2.937)
		5	2500	3.257	0.092	(3.076, 3.437)	3.094	0.062	(2.973, 3.215)
2007, 3	53104	10	5310	2.912	0.057	(2.801, 3.023)	2.835	0.039	(2.759, 2.911)
		5	2655	2.904	0.080	(2.748, 3.060)	3.009	0.058	(2.894, 3.123)
2007, 4	53096	10	5310	2.941	0.057	(2.829, 3.053)	2.811	0.039	(2.735, 2.886)
		5	2655	2.972	0.082	(2.812, 3.131)	3.077	0.060	(2.960, 3.194)

Fig. 1 Log-log rank-size estimates and 95% confidence intervals the tail index of income distribution in Russia in the 4th quarter of 2007

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