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Self-similarity in fully developed homogeneous isotropic turbulence using the lyapunov analysis

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Abstract In this work, we calculate the self-similar longitudinal velocity correlation function and the statistics of the velocity difference, using the results of the Lyapunov analysis of the fully developed isotropic homogeneous turbulence just presented by the author in a previous work (de Divitiis, Theor Comput Fluid Dyn, doi:10.1007/s00162-010-0211-9). There, a closure of the von Kármán-Howarth equation is proposed and the statistics of velocity difference is determined through a specific statistical analysis of the Fourier-transformed Navier-Stokes equations. The longitudinal correlation functions correspond to steady-state solutions of the von Kármán-Howarth equation under the self-similarity hypothesis introduced by von Kármán. These solutions and the corresponding statistics of the velocity difference are numerically determined for different Taylor-scale Reynolds numbers. The obtained results adequately describe the several properties of the fully developed isotropic turbulence.

Keywords Self-similarity · Lyapunov analysis · Von Kármán-Howarth equation · Velocity difference statistics

1 Introduction

A recent work of the author dealing with the homogeneous isotropic turbulence [1], suggests a novel method to analyze the fully developed turbulence through a specific Lyapunov analysis of the relative motion between two fluid particles. The analysis expresses the velocity fluctuation as the combined effect of the exponential growth rate of the fluid velocity in the Lyapunov basis, and of the rotation of the same basis with respect to the fixed frame of reference. The results of this analysis lead to the closure of the von Kármán-Howarth equation and give an explanation of the mechanism of the energy cascade. A constant skewness of the velocity derivative $\partial u_r / \partial r$ is calculated whose value is in agreement with the various source of the literature. Moreover, the statistics of the velocity difference can be inferred looking at the Fourier series of the fluid velocity. This is a non-Gaussian statistics, where the constancy of this skewness implies that the other higher absolute moments increase with the Taylor-scale Reynolds number.

The present work represents a further contribution of Ref. [1]. Here, the self-similar solutions of the von Kármán-Howarth equation are numerically calculated using the closure obtained in the previous work and the statistics of the velocity difference is determined.

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2 Analysis

For sake of convenience, this section reports the main results of the Lyapunov analysis obtained in the Ref. [1].

As well known, the pair correlation function f of the longitudinal velocity u_r for fully developed isotropic and homogeneous turbulence, satisfies the von Kármán-Howarth equation [2]

$$\frac{\partial f}{\partial t} = \frac{K(r)}{u^2} + 2\nu \left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r} \frac{\partial f}{\partial r} \right) - 10\nu f \frac{\partial^2 f}{\partial r^2}(0) \quad (1)$$

the boundary conditions of which are

$$\begin{aligned} f(0) &= 1, \\ \lim_{r \rightarrow \infty} f(r) &= 0 \end{aligned} \quad (2)$$

where r is the separation distance and u is the standard deviation of u_r , which satisfies [2,3]

$$\frac{du^2}{dt} = 10\nu u^2 \frac{\partial^2 f}{\partial r^2}(0) \quad (3)$$

This equation gives the rate of the kinetic energy and is determined putting $r = 0$ into Eq. (1) [2,3]. The function $K(r)$, related to the triple velocity correlation function, represents the effect of the inertia forces and expresses the mechanism of energy cascade. Thus, the von Kármán-Howarth equation provides the relationship between the statistical moments $\langle (\Delta u_r)^2 \rangle$ and $\langle (\Delta u_r)^3 \rangle$ in function of r , where Δu_r is the longitudinal velocity difference.

Following the Lyapunov analysis presented in Ref [1], $K(r)$ is in terms of f and its space derivative

$$K(r) = u^3 \sqrt{\frac{1-f}{2}} \frac{\partial f}{\partial r} \quad (4)$$

This expression satisfies the two conditions $\partial K / \partial r(0) = 0$, $K(0) = 0$, which represent, respectively, the homogeneity and the condition that K does not modify the fluid kinetic energy [2,3]. The skewness of Δu_r is calculated as [3]

$$H_3(r) = \frac{\langle (\Delta u_r)^3 \rangle}{\langle (\Delta u_r)^2 \rangle^{3/2}} = \frac{6k(r)}{(2(1-f(r)))^{3/2}} \quad (5)$$

where $k(r)$ is the longitudinal triple velocity correlation function, related to $K(r)$ through [3]

$$K(r) = u^3 \left(\frac{\partial}{\partial r} + \frac{4}{r} \right) k(r) \quad (6)$$

As a result, $H_3(0) = -3/7$ does not depend upon the Reynolds number [1]. In line with the analysis of Ref.[1], the higher moments are consequently determined, taking into account that the analytical structure of Δu_r is

$$\frac{\Delta u_r}{\sqrt{\langle (\Delta u_r)^2 \rangle}} = \frac{\xi + \psi (\chi(\eta^2 - 1) - (\zeta^2 - 1))}{\sqrt{1 + 2\psi^2 (1 + \chi^2)}} \quad (7)$$

Equation (7), arising from statistical considerations about the Fourier-transformed Navier-Stokes equations, expresses the internal structure of the fully developed isotropic turbulence, where ξ , η and ζ are independent centered random variables which exhibit the gaussian distribution functions $p(\xi)$, $p(\eta)$ and $p(\zeta)$ whose standard deviation is equal to the unity. Thus, the moments of Δu_r are easily calculated from Eq. (7) [1]

$$H_n \equiv \frac{\langle (\Delta u_r)^n \rangle}{\langle (\Delta u_r)^2 \rangle^{n/2}} = \frac{1}{(1 + 2\psi^2 (1 + \chi^2))^{n/2}} \sum_{k=0}^n \binom{n}{k} \psi^k \langle \xi^{n-k} \rangle \langle (\chi(\eta^2 - 1) - (\zeta^2 - 1))^k \rangle \quad (8)$$

where

$$\begin{aligned} \langle (\chi(\eta^2 - 1) - (\zeta^2 - 1))^k \rangle &= \sum_{i=0}^k \binom{k}{i} (-\chi)^i \langle (\zeta^2 - 1)^i \rangle \langle (\eta^2 - 1)^{k-i} \rangle \\ \langle (\eta^2 - 1)^i \rangle &= \sum_{l=0}^i \binom{i}{l} (-1)^l \langle \eta^{2(i-l)} \rangle \end{aligned} \quad (9)$$

In particular, H_3 , related to the energy cascade, is

$$H_3 = \frac{8\psi^3 (\chi^3 - 1)}{(1 + 2\psi^2 (1 + \chi^2))^{3/2}} \quad (10)$$

where ψ is a function of r and of the Reynolds number [1]

$$\psi(\mathbf{r}, R) = \sqrt{\frac{R}{15\sqrt{15}}} \hat{\psi}(r) \quad (11)$$

being $\lambda_T = u/\sqrt{\langle (\partial u_r / \partial r)^2 \rangle}$ and $R = u\lambda_T/\nu$, respectively, the Taylor scale and the Taylor-scale Reynolds number, whereas the function $\hat{\psi}(r)$ is determined through Eq. (10) as soon as $H_3(r)$ is known. The parameter χ is also a function of R and is implicitly calculated putting $r = 0$ into Eq. (10) [1]

$$\frac{8\psi_0^3 (1 - \chi^3)}{(1 + 2\psi_0^2 (1 + \chi^2))^{3/2}} = \frac{3}{7} \quad (12)$$

where $\psi_0 = \psi(R, 0)$ and $\hat{\psi}_0 = 1.075$ [1].

From Eqs. (7) and (11), all the quantities $|H_p(r)|$, for $p > 3$, rise with R , indicating that the intermittency of Δu_r increases with the Reynolds number.

The PDF of Δu_r can be formally expressed through the gaussian PDFs $p(\xi)$, $p(\eta)$ and $p(\zeta)$, using the Frobenius-Perron equation

$$F(\Delta u'_r) = \int_{\xi} \int_{\eta} \int_{\zeta} p(\xi) p(\eta) p(\zeta) \delta(\Delta u_r - \Delta u'_r) d\xi d\eta d\zeta \quad (13)$$

where δ is the Dirac delta.

The spectra $E(\kappa)$ and $T(\kappa)$ are the Fourier Transforms of f and K [3]

$$\begin{bmatrix} E(\kappa) \\ T(\kappa) \end{bmatrix} = \frac{1}{\pi} \int_0^\infty \begin{bmatrix} u^2 f(r) \\ K(r) \end{bmatrix} \kappa^2 r^2 \left(\frac{\sin \kappa r}{\kappa r} - \cos \kappa r \right) dr \quad (14)$$

3 Self-similarity

An ordinary differential equation for describing the spatial evolution of f is now derived from the von Kármán-Howarth equation, under the hypothesis of self-similarity [4] and using the closure (4). Equation (1), representing a boundary problem with initial conditions, is here transformed into an initial condition problem in the variable r .

Far from the initial condition, it is reasonable that the simultaneous effects of the cascade of energy and of the viscosity act keeping f and $E(\kappa)$ similar in the time. This is the idea of self-preserving correlation function which was originally introduced by von Kármán (see ref. [4] and reference therein). In order to analyse this self-similarity, it is convenient to express f in terms of the dimensionless variables $\hat{r} = r/\lambda_T$ and $\hat{t} = tu/\lambda_T$, i.e., $f = f(\hat{t}, \hat{r})$. As a result, Eq. (1) reads as follows

$$\frac{\partial f}{\partial \hat{t}} \frac{\lambda_T}{u} \frac{d}{dt} \left(\frac{tu}{\lambda_T} \right) - \frac{\partial f}{\partial \hat{r}} \frac{\hat{r}}{u} \frac{d\lambda_T}{dt} = \sqrt{\frac{1-f}{2}} \frac{\partial f}{\partial \hat{r}} + \frac{2}{R} \left(\frac{\partial^2 f}{\partial \hat{r}^2} + \frac{4}{\hat{r}} \frac{\partial f}{\partial \hat{r}} \right) + \frac{10}{R} f \quad (15)$$

where $\partial^2 f / \partial \hat{r}^2(0) \equiv -1$. This is a non-linear partial differential equation whose coefficients vary in the time according to the rate of kinetic energy

$$\frac{du^2}{dt} = -\frac{10\nu u^2}{\lambda_T^2} \quad (16)$$

and to $d\lambda_T/dt$. This latter is calculated considering the terms of the order $O(r^2)$ into Eqs. (1) [2,3] and (4) [1]

$$\frac{d\lambda_T}{dt} = -\frac{u}{2} + \nu \left(\frac{7}{3} \frac{\partial^4 f}{\partial r^4}(0) \lambda_T^3 - \frac{5}{\lambda_T} \right) \quad (17)$$

If the self-similarity is assumed, all the coefficients of Eq. (15) must not vary with the time [2,4], thus

$$R = \text{const} \quad (18)$$

$$a_1 \equiv \frac{\lambda_T}{u} \frac{d}{dt} \left(\frac{tu}{\lambda_T} \right) = \text{const} \quad (19)$$

$$a_2 \equiv \frac{1}{u} \frac{d\lambda_T}{dt} = \text{const}, \quad (20)$$

From Eqs. (16) and (18), λ_T and u will depend on t according to

$$\lambda_T(t) = \lambda_T(0) \sqrt{1 + 10\nu t / \lambda_T^2(0)}, \quad u(t) = \frac{u(0)}{\sqrt{1 + 10\nu t / \lambda_T^2(0)}}. \quad (21)$$

Therefore, a_1 and a_2 are determined substituting Eqs. (21) into Eqs. (19) and (20)

$$\begin{aligned} a_1 &\equiv \frac{\lambda_T}{u} \frac{d}{dt} \left(\frac{tu}{\lambda_T} \right) = \frac{1}{1 + 10\nu t / \lambda_T^2(0)} \\ a_2 &\equiv \frac{1}{u} \frac{d\lambda_T}{dt} = \frac{5}{R} \end{aligned} \quad (22)$$

For $t \rightarrow \infty$, $a_1 = 0$ and a_2 tends to a constant, therefore the self-similar correlation function $f(\hat{r})$ does not depend on the initial condition and obeys to the following non-linear ordinary differential equation

$$\frac{5}{R} \frac{df}{d\hat{r}} \hat{r} + \sqrt{\frac{1-f}{2}} \frac{df}{d\hat{r}} + \frac{2}{R} \left(\frac{d^2 f}{d\hat{r}^2} + \frac{4}{\hat{r}} \frac{df}{d\hat{r}} \right) + \frac{10}{R} f = 0 \quad (23)$$

In line with von Kármán [2,4], we search the self-similar solutions over the whole range of $\hat{r}$, with the exception of the dimensionless distances whose order magnitude exceed R . This corresponds to assume the self-similarity for all the frequencies of the energy spectrum, but for the lowest ones [2,4]. Accordingly, the first term of Eq. (23), representing the time derivative of f , can be neglected with respect to the other ones

$$\sqrt{\frac{1-f}{2}} \frac{df}{d\hat{r}} + \frac{2}{R} \left(\frac{d^2 f}{d\hat{r}^2} + \frac{4}{\hat{r}} \frac{df}{d\hat{r}} \right) + \frac{10}{R} f = 0 \quad (24)$$

The boundary conditions of Eq. (24) are from Eqs. (2) [2]

$$f(0) = 1 \quad (25)$$

$$\lim_{\hat{r} \rightarrow \infty} f(\hat{r}) = 0 \quad (26)$$

Since the solutions $f \in C^2[0, \infty)$ exponentially tend to zero when $r \rightarrow \infty$, and λ_T is considered to be an assigned quantity, the boundary condition (26) can be substituted by the homogeneity condition

$$\frac{df(0)}{d\hat{r}} = 0 \quad (27)$$

Therefore, the boundary problem represented by Eqs. (24), (25) and (26), is replaced by the following initial condition problem written in the Cauchy normal form

$$\begin{aligned}\frac{df}{d\hat{r}} &= F \\ \frac{dF}{d\hat{r}} &= -5f - \left(\frac{1}{2} \sqrt{\frac{1-f}{2}} R + \frac{4}{\hat{r}} \right) F\end{aligned}\quad (28)$$

the initial condition of which is

$$f(0) = 1, \quad F(0) = 0 \quad (29)$$

where

$$\lim_{\hat{r} \rightarrow 0} \frac{F(\hat{r})}{\hat{r}} = \lim_{\hat{r} \rightarrow 0} \frac{dF(\hat{r})}{d\hat{r}} = -1 \quad (30)$$

4 Results and discussion

In this section, the solutions Eqs. (28) are qualitatively studied and numerically calculated for several values of the Taylor-scale Reynolds numbers. The statistics of the longitudinal velocity difference is also investigated through the analysis seen in the Sect. 2 (Eqs. (7)-(13)).

The analysis of Eqs. (28) shows that, for large $\hat{r}$, f exponentially decreases, thus all the integral scales of f are finite quantities and the energy spectrum is a definite quantity whose integral over the Fourier space gives the turbulent kinetic energy. The relation between the integral moments of f and the derivatives of $E(\kappa)$ calculated at $\kappa = 0$, is [3]

$$\frac{\partial^{2m} E(0)}{\partial \kappa^{2m}} = \frac{4m}{\pi} (m-1)(-1)^m u^2 \int_0^\infty r^{2m} f(r) dr \quad (31)$$

Thus, $\partial^2 E(0)/\partial \kappa^2 = 0$, $\partial^4 E(0)/\partial \kappa^4 \neq 0$, that is $E(\kappa)$ satisfies the incompressibility condition.

Next, observe that, where $K(r)$ is about constant, f behaves like

$$f - 1 \approx \hat{r}^{2/3} \quad (32)$$

as a result, there exists an interval of the Fourier space which identifies the inertial subrange of Kolmogorov, where $E(\kappa) \propto \kappa^{-5/3}$.

Furthermore, the solution of the system (28, 29), in the vicinity of the origin is

$$f(\hat{r}) = 1 - \frac{\hat{r}^2}{2} + \frac{10+R}{112} \hat{r}^4 + O(\hat{r}^6) \quad (33)$$

Since the variations of f near the origin are responsible for the behavior of $E(\kappa)$ at high wavenumbers [3], we assume that the minimum length scale of f depends upon the second and the third term of Eq. (33). Under this scale the effects of the inertia and pressure forces are negligible with respect to the viscous forces. This scale is determined as two times the separation distance $\hat{r} > 0$ where the first derivative of $-\hat{r}^2/2 + (10+R)/112 \hat{r}^4$ vanishes. This is $\Delta \hat{r}_{min} = \sqrt{28/(10+R)}$, and the corresponding wave-number is also related to R through the relationship

$$\kappa_{max} \approx \frac{2\pi}{\lambda_T} \sqrt{\frac{10+R}{112}} \quad (34)$$

This value of κ_{max} should indicate the order of magnitude of the separation wave-number between the Kolmogorov inertial subrange and the dissipative interval. Hence, for $\kappa > \kappa_{max}$, the viscosity acts on f in such a way that $E(\kappa)$ decreases much more rapidly than in the inertial subrange. In fact, in the dissipative interval, $E(\kappa)$ roughly coincides with the Fourier transformation of $f \simeq 1 - \hat{r}^2/2 \simeq 1/(1 + \hat{r}^2/2)$, therefore one could expect that the energy spectrum behaves like [5]

$$E(\kappa) \approx \exp(-a\lambda_T \kappa), \quad \kappa > \kappa_{max} \quad (35)$$

where $a = O(1) > 0$ is a proper parameter which depends upon the Reynolds number.

Fig. 1 Longitudinal correlation function for different Taylor-Scale Reynolds numbers

Fig. 2 Longitudinal triple correlation function for different Taylor-Scale Reynolds numbers

Now, several numerical solutions of Eqs. (28) are calculated for different values of the Taylor-scale Reynolds numbers by means of the fourth-order Runge-Kutta method. The fixed step of the integrator scheme is selected on the basis of the behavior of Eqs. (28) when $\hat{r} \rightarrow \infty$. There, Eq. (24) tends to the following linear differential equation

$$\frac{d^2 f}{d\hat{r}^2} + \frac{R}{2\sqrt{2}} \frac{df}{d\hat{r}} + 5f = 0 \quad (36)$$

whose characteristic exponents $\alpha = -R/4\sqrt{2} \pm \sqrt{(R/4\sqrt{2})^2 - 5}$ suggest that $\Delta\hat{r} = \sqrt{2}/R$ is an adequate step of integration [6] for the accuracy of the numerical solutions of Eqs. (28). Since $\Delta\hat{r} < \Delta\hat{r}_{\min}$, this step provides a fairly accurate description of correlation function and energy spectrum at the dissipation scales.

The cases here studied correspond to $R = 100, 200, 300, 400, 500$ and 600 . Figures 1 and 2 show, respectively, double and triple longitudinal correlation functions in terms of $\hat{r}$. Due to the mechanism of energy cascade expressed by Eq. (4), the tail of f rises with R in agreement with Eq. (33) and this determines that, for an assigned value of λ_T , all the integral scales of f are rising functions of R . Since large Reynolds numbers require also small Δr , these calculations correspond to a great number of steps of integration which diverges with R . Therefore, the present analysis only considers the cases with $R \leq 600$.

Fig. 3 Turbulent energy spectrum for different Taylor-Scale Reynolds numbers

Fig. 4 “Transfer function $T(\kappa)$ ” for several Taylor-Scale Reynolds numbers

According to Eq. (4), k decays more slowly than f and its characteristic scales increase with R as prescribed by Eqs. (28). The maximum of $|k|$, which gives the entity of the mechanism of energy cascade, is slightly less than 0.05. These results are in very good agreement with the numerous data of the literature (see [3] and Refs. therein).

Figures 3 and 4 show the corresponding spectra $E(\kappa)$ and $T(\kappa)$ calculated with Eq. (14). As a consequence of Eq. (31), $E(\kappa) \approx \kappa^4$ near the origin, and after a maximum is about parallel to the dashed line $\kappa^{-5/3}$ in a given interval of the wave-numbers. The size of this latter, which defines the inertial range of Kolmogorov, increases with R and this is due to the fact that $f - 1 \approx r^{2/3}$ in a certain spatial interval where K is about constant.

For $\kappa > \kappa_{\max}$, $E(\kappa)$ decreases more rapidly than in the Kolmogorov subrange and its variations follow Eq. (35) with a which varies about from 0.53 to 0.26 when R is 100 and 600, respectively. The wavenumber of separation between the two regions varies with R according to Eq. (34).

Since K does not modify the fluid kinetic energy, $\int_0^\infty T(\kappa) d\kappa = 0$, for all the Reynolds numbers.

Figures 5 and 6 illustrate the variations of skewness and flatness of Δu_r in function of $\hat{r}$ for the same values of the Reynolds numbers. The skewness H_3 is first calculated with Eq. (5) and thereafter the flatness H_4 has been determined using Eq. (8). Although $H_3(0) = -3/7$ does not depend upon the Reynolds number, $H_3(\hat{r})$ rises with R and goes to zero for $r \rightarrow \infty$. Equation (8) states that also H_4 is a rising function of R . This is significantly greater than 3 for $r = 0$ and tends to 3 as $r \rightarrow \infty$. Again thanks to Eq. (8), $H_4 - 3$ goes to zero more rapidly than H_3 . The constancy of $H_3(0)$ and the quadratic terms of Eq. (7) cause an intermittency of Δu_r

Fig. 5 Skewness of Δu_r at different Taylor-Scale Reynolds numbers

Fig. 6 Flatness of Δu_r at different Taylor-Scale Reynolds numbers

(i.e. H_4 and higher moments) which increases with R . These results, which are the consequence of Eqs. (4) and (7), are in agreement with the several experimental evidence and theoretical topics of the literature [3].

Next, the Kolmogorov function $Q(r)$ and the Kolmogorov constant C , are determined using the previous results. According to the theory, the Kolmogorov function, defined as

$$Q(r) = -\frac{\langle (\Delta u_r)^3 \rangle}{r\varepsilon} \quad (37)$$

is constant with respect to r , and is equal to $4/5$ as long as $r/\lambda_T = O(1)$, where $\varepsilon = -3/2 du^2/dt$ is the rate of energy dissipation. In Fig. 7, $Q(r)$, calculated through $H_3(\hat{r})$, is shown in terms of $\hat{r}$. This exhibits a maximum $Q_{\max}$ for $\hat{r} = O(1)$ and quite small variations for $\hat{r} > 1$, as R increases. $Q_{\max}$ rises with R and seems to tend toward the limit $4/5$ prescribed by the Kolmogorov theory. Figure 8 shows $Q_{\max}$ in terms of R , where the present results (“x” symbols) are compared with those of Ref. [7] (circular filled symbols and continuous line). There, the Kolmogorov equation with a forcing term, is compared with experimental measurements, for helium gas at low temperature. It is apparent that the values here calculated with Eq. (37) are less than those given by Ref. [7], and such difference changes with R with an average value of about 12%. This disagreement could be due to the self-similarity hypothesis here assumed or to possible differences in the estimation of the Taylor-scale. Specifically, λ_T is here analytically calculated or assumed, whereas in Ref. [7], it is determined through measurements of ε and u . Nevertheless, the two set of data can be considered comparable, since the two diagrams almost exhibit the same trend.

Fig. 7 Kolmogorov function for several Taylor-Scale Reynolds numbers

Fig. 8 Maximum of the Kolmogorov Function in terms of the Taylor-Scale Reynolds number. These data are from Ref. [7]. The symbols “X” are for the present result

Table 1 Kolmogorov constant for different Taylor-Scale Reynolds number

R	C
100	1.8860
200	1.9451
300	1.9704
400	1.9847
500	1.9940
600	2.0005

The Kolmogorov constant C , defined by $E(\kappa) \approx C\varepsilon^{2/3}/\kappa^{5/3}$ in the inertial subrange, is here calculated as

$$C = \max_{\kappa \in (0, \infty)} \frac{E(\kappa)\kappa^{5/3}}{\varepsilon^{2/3}} \quad (38)$$

The obtained values of C , shown in Table 1, increase with the Reynolds number and are in good agreement with the numerical and experimental values known from the literature [8–10].

The spatial structure of Δu_r , expressed by Eq. (7), is also studied with the previous results. According to the various works [11–13], Δu_r behaves quite similarly to a multifractal system, where Δu_r obeys to a law of the kind $\Delta u_r(r) \approx r^q$, being q a fluctuating exponent. This implies that the statistical moments of $\Delta u_r(r)$ are expressed through different scaling exponents $\zeta(n)$, i.e.

$$\langle (\Delta u_r)^n \rangle = A_n r^{\zeta(n)} \quad (39)$$

Fig. 9 Statistical moments of Δu_r in terms of the separation distance, for $R = 600$

Fig. 10 Scaling exponents of Δu_r for several R . Solid symbols are for the present data. Dashed line is for Kolmogorov K41 data [14]. Dotted line is for Kolmogorov K62 data [11]. Continuous line is for She-Leveque data [13]

In order to calculate $\zeta(n)$, the statistical moments of Δu_r are first calculated using Eqs. (8), in function of $\hat{r}$ (see for instance Fig. 9). These scaling exponents are identified through a best fitting procedure, in the intervals $(\hat{r}_1, \hat{r}_2)$, where the endpoints $\hat{r}_1$ and $\hat{r}_2$ have to be determined. The calculation of $\zeta(n)$ and A_n is carried out through a minimum square method which, for each statistical moment, is applied to the following optimization problem

$$J_n(\zeta(n), A_n) \equiv \int_{\hat{r}_1}^{\hat{r}_2} \left(\langle (\Delta u_r)^n \rangle - A_n r^{\zeta(n)} \right)^2 dr = \min, \quad n = 1, 2, \dots \quad (40)$$

where $\langle (\Delta u_r)^n \rangle$ are calculated with Eqs. (8), $\hat{r}_1$ is assumed to be equal to 0.1, whereas $\hat{r}_2$ is taken in such a way that $\zeta(3) = 1$ for all the Reynolds numbers. The so obtained scaling exponents, shown in Table (2), are compared in Fig. 10 (solid symbols) with those of the Kolmogorov theories K41 [14] (dashed line) and K62 [11] (dotted line), and with the exponents calculated by She-Leveque [13] (continuous curve). It is found that, near the origin $\zeta(n) \simeq n/3$, and in general the values of $\zeta(n)$ are in good agreement with the She-Leveque results. In particular, the scaling exponents here calculated are lightly greater than those by She-Leveque for $n > 8$.

Table 2 Scaling exponents of the longitudinal velocity difference for several Taylor-Scale Reynolds number

R	100	200	300	400	500	600
$\zeta(1)$	0.35	0.35	0.35	0.35	0.35	0.35
$\zeta(2)$	0.70	0.71	0.71	0.71	0.71	0.71
$\zeta(3)$	1.00	1.00	1.00	1.00	1.00	1.00
$\zeta(4)$	1.30	1.29	1.29	1.29	1.29	1.29
$\zeta(5)$	1.56	1.55	1.55	1.55	1.55	1.55
$\zeta(6)$	1.82	1.81	1.81	1.81	1.81	1.81
$\zeta(7)$	2.06	2.05	2.05	2.04	2.04	2.05
$\zeta(8)$	2.31	2.28	2.28	2.28	2.28	2.28
$\zeta(9)$	2.53	2.50	2.50	2.50	2.50	2.51
$\zeta(10)$	2.76	2.72	2.73	2.72	2.72	2.73
$\zeta(11)$	2.97	2.93	2.93	2.93	2.93	2.94
$\zeta(12)$	3.18	3.14	3.14	3.13	3.13	3.15
$\zeta(13)$	3.39	3.33	3.34	3.33	3.33	3.35
$\zeta(14)$	3.59	3.53	3.54	3.53	3.53	3.55
$\zeta(15)$	3.79	3.73	3.73	3.72	3.73	3.75

Fig. 11 Probability distribution functions of the longitudinal velocity derivative for the different Taylor-Scale Reynolds numbers

Following the present analysis, these characteristic laws $\zeta = \zeta(n)$, which make Δu_r multifractal, are the consequence of the combined effect of the quadratic terms of Eq. (7) and of the peculiar form of K calculated through Eqs. (28).

The distribution functions of $\partial u_r / \partial \hat{r}$ can be formally determined with Eqs. (13) and (7). These are really calculated with sequences of the variables ξ , η and ζ generated by gaussian random numbers generators. The distribution functions are then calculated through the statistical elaboration of these data and Eq. (7). The results are shown in Fig. 11a and 11b in terms of the dimensionless abscissa

$$s = \frac{\partial u_r / \partial \hat{r}}{\left\langle (\partial u_r / \partial \hat{r})^2 \right\rangle^{1/2}}$$

where the dashed curve represents the gaussian PDF. These distribution functions are normalized, in order that their standard deviations are equal to the unity. In particular, Fig. 11b shows the enlarged region of Fig. 11a, where $5 < s < 8$. The tails of the PDFs change with R according to Eq. (7) in such a way that the intermittency

risks with R . The values of the PDF, especially for $5 < s < 8$, can be compared with those obtained by Tabeling et al. in Ref. [15], where in an experiment using low temperature helium gas between two counter-rotating cylinders (closed cell), the authors measure the PDF of $\partial u_r / \partial r$ and its moments. Although these experiments pertain wall-bounded flows, where the flow could be quite far from the isotropy condition, the PDFs and the corresponding slopes here calculated for $5 < s < 8$, exhibit the same order of magnitude of those obtained by Tabeling et al. [15] for $100 < R < 600$.

5 Conclusions

The self-similar solutions of the von Kármán-Howarth equation calculated with the proposed closure, and the corresponding characteristic statistics of the velocity difference are shown to be in very good agreement with the various properties of the isotropic turbulence from several points of view.

In particular:

- The energy spectrum satisfies the continuity equation and follows the Kolmogorov law in a certain range of wave-numbers whose size increases with the Reynolds number. Thereafter, the spectrum decays according to an exponential law.
- The Kolmogorov function exhibits a maximum and relatively small variations in proximity of $r = O(\lambda_T)$. This maximum value rises with the Reynolds number and seems to tend toward the limit $4/5$, prescribed by the Kolmogorov theory.
- The Kolmogorov constant moderately rises with the Reynolds number with an average value around to 1.95 when R varies from 100 to 600.
- The scaling exponents of the moments of velocity difference are calculated through a best fitting procedure in an opportune range of the separation distance. The values of these exponents are in good agreement with the results known from the literature.

These results represent a further test of the analysis presented in Ref. [1] which adequately describes many of the properties of the isotropic turbulence.

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