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Maehara-style modal nested calculi

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Abstract

We develop multi-conclusion nested sequent calculi for the fifteen logics of the intuitionistic modal cube between IK and IS5. The proof of cut-free completeness for all logics is provided both syntactically via a Maehara-style translation and semantically by constructing an infinite birelational countermodel from a failed proof search. Interestingly, the Maehara-style translation for proving soundness syntactically fails due to the hierarchical structure of nested sequents. Consequently, we only provide the semantic proof of soundness. The countermodel construction used to prove completeness required a completely novel approach to deal with two independent sources of nontermination in the proof search present in the case of transitive and Euclidean logics.

Keywords Proof theory · Sequent calculus · Nested sequents · Modal logic · Intuitionistic logic · Cut elimination · Multiple conclusion · Intuitionistic modal logic

Mathematics Subject Classification 03B45 · 03B60 · 03B62 · 03B70 · 03F03 · 03F05 · 03F07 · 03F55

1 Introduction

Ever since Gentzen's LK and LI[1], it is almost considered common knowledge that sequent systems for intuitionistic logic are single conclusion, in other words, one must

[1] This calculus has been erroneously called LJ almost since the beginning. The mistake is due to the peculiarities of the Sütterlin handwriting used by Gentzen. LI with 'I' standing for the German word *intuitionistische* the same way as 'K' stands for the German word *klassische* in LK certainly makes much more sense than LJ (for more details, see [1, p. 83]).

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restrict the succedent to (no more than) one formula. Gentzen [2] himself obtained this as a natural consequence of the natural deduction presentation, which has only one conclusion. In effect, the ability to have several formulas in the succedent was an additional feature introduced by Gentzen to incorporate the principle of the excluded middle. Despite this near-consensus, a multi-conclusion sequent calculus for intuitionistic propositional (and predicate) logic is almost as old as Gentzen's LI. It was proposed by Maehara [3] as the auxiliary calculus L′J used to translate intuitionistic reasoning into classical one. It is hard to divine Maehara's thinking: the terse language of results, the whole results, and nothing but results was much in vogue at the time. But on the face of it, his system amounts to an observation that most classical sequent rules remain valid intuitionistically, with only a couple of propositional rules requiring the singleton-succedent restriction in the intuitionistic case. Thus, the blanket restriction of succedent to (at most) singleton sets can be seen as an overreaction. Even the interpretation of the succedent as the disjunction of its formulas is retained in Maehara's calculus.

One possible criticism of this calculus could be that it was introduced as an auxiliary, artificial construct bridging the gap between the natural(-deduction) inspired LI and the fully symmetric LK. This criticism is, however, unfounded. It has been noted (see, for instance, the excellent in-depth survey of various intuitionistic calculi by Dyckhoff [4]) that Maehara's calculus is essentially a notational variant of tableaux from Fitting's Ph.D. thesis [5] (which Fitting himself attributes to Beth [6]). In fact, the same system can be found in [7] and, according to von Plato [1], a similar system was considered by Gentzen himself.

In other words, this calculus is quite natural, has been discovered by several researchers independently, and has a distinction of correlating with the semantic presentation of intuitionistic reasoning much better than LI. Indeed, tableau rules are typically read from the semantics, and Beth–Fitting's destructive tableaux match intuitionistic Kripke models perfectly. It should also be noted that Egly and Schmitt [8] demonstrated that LI cannot polynomially simulate Maehara's calculus, meaning that the latter is more efficient with respect to proof search.

The idea of extending intuitionistic reasoning with modalities is equally natural but less straightforward. There have been multiple approaches over the years with each classical modal logic receiving several alternative intuitionizations. We refer the reader to Simpson's Ph.D. [9] for the discussion of these approaches and concentrate on what eventually became officially known as *intuitionistic modal logics*. Similar to their classical counterparts, one can talk about the intuitionistic modal cube consisting of 15 logics. And similar to their classical brethren, ordinary sequent systems seem inadequate to describe these logics, but *nested sequent* systems [10–12] exist for all of them [13]. A nested sequent is a tree of ordinary sequents, referred to as *sequent nodes*. The tableau analog of nested sequents is prefixed tableaux [14], the underlying idea being that the tree structure of a sequent is homomorphically mapped into the accessibility structure of a Kripke model.

To our knowledge, all nested sequent calculi for these intuitionistic modal logics that have been published so far [13,15,16] are globally single-conclusion: exactly one of the sequent nodes is allowed to have a non-empty succedent, and it contains exactly one formula.[2] One can say that they are the modalizations of LI in the sense that the propositional rules are local and identical to those in LI if the rest of the nested structure is ignored. The goal of this paper is to construct a modalization of the Maehara-style calculus with propositional rules conforming to the birelational semantics of intuitionistic modal logics. We formulate the calculi for the 15 intuitionistic modal logics and prove their completeness in two alternative ways:

1. by a syntactic translation from the single-conclusion calculi of [13] and
2. by a direct semantic proof of cut admissibility.

The syntactic-translation method originally employed by Maehara in [3] for translating from multi-conclusion systems to single-conclusion systems is not applicable in the setting of nested sequents because formulas in the succedent of the sequent can occur in various places in the nested sequent tree, and there is no immediate equivalence to their disjunction as it is the case with ordinary sequents. Also the method by [8] that is based on rule permutations does not work in our case, again, because the formulas in the succedent do not necessarily occur in the same sequent node. For this reason, we prove soundness for our multi-conclusion systems by a direct semantic argument.

This paper is organized as follows. First, in Sect. 2 we recall the syntax and semantics of intuitionistic modal logics. In Sect. 3, we present our nested sequent systems and show completeness of the multi-conclusion systems by using completeness of the single-conclusion systems demonstrated in [13]. Then we show semantically in Sect. 4 the soundness of the multi-conclusion systems and, finally, in Sect. 5, we give a semantic argument for the completeness of the multi-conclusion systems with respect to the birelational Kripke models.

2 Syntax and semantics of intuitionistic modal logics

Definition 2.1 (Language of intuitionistic modal logic) *We start from a countable set $\mathcal{A}$ of* propositional variables *(or* atoms*). Then the set $\mathcal{M}$ of* formulas *of intuitionistic modal logic (IML) is generated by the grammar*

$$\mathcal{M} ::= \mathcal{A} \mid \bot \mid (\mathcal{M} \wedge \mathcal{M}) \mid (\mathcal{M} \vee \mathcal{M}) \mid (\mathcal{M} \supset \mathcal{M}) \mid \Box\mathcal{M} \mid \Diamond\mathcal{M}$$

We denote atoms by lowercase Latin letters, like a, b, c, and formulas by capital Latin letters, like A, B, C. Negation can be defined as $\neg A := A \supset \bot$, and the constant $\top$ is defined as $\top := \neg\bot$.

[2] For the *intuitionistic modal logics* that we study in this paper, there are also natural deduction systems [17] based on the data structure of nested sequents. For other "intuitionistic" variants of modal logics, multiple-conclusion sequent style systems do exist in the literature, e.g., for *constructive modal logics* in [18], and for $\Box$-only fragments in [19].

d : $\Box A \supset \Diamond A$	$\forall w.\ \exists v.\ wRv$	(serial)
t : $(A \supset \Diamond A) \wedge (\Box A \supset A)$	$\forall w.\ wRw$	(reflexive)
b : $(A \supset \Box\Diamond A) \wedge (\Diamond\Box A \supset A)$	$\forall w.\ \forall v.\ wRv \supset vRw$	(symmetric)
4 : $(\Diamond\Diamond A \supset \Diamond A) \wedge (\Box A \supset \Box\Box A)$	$\forall w.\ \forall v.\ \forall u.\ wRv \wedge vRu \supset wRu$	(transitive)
5 : $(\Diamond A \supset \Box\Diamond A) \wedge (\Diamond\Box A \supset \Box A)$	$\forall w.\ \forall v.\ \forall u.\ wRv \wedge wRu \supset vRu$	(Euclidean)

Fig. 1 Intuitionistic modal axioms d, t, b, 4, and 5 with corresponding frame conditions

Definition 2.2 (Logic IK) *A formula is a theorem of* IK*, an intuitionistic variant of the modal logic* K*, if it can be derived from the axioms of intuitionistic propositional logic (IPL) extended with the* k-axioms

$$\begin{aligned} &\mathsf{k}_1 : \Box(A \supset B) \supset (\Box A \supset \Box B)\ , \\ &\mathsf{k}_2 : \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)\ , \\ &\mathsf{k}_3 : \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)\ , \\ &\mathsf{k}_4 : (\Diamond A \supset \Box B) \supset \Box(A \supset B)\ , \quad \textit{and} \\ &\mathsf{k}_5 : \Diamond\bot \supset \bot\ , \end{aligned} \tag{1}$$

using the rules

$$\mathsf{nec}\ \frac{A}{\Box A} \quad \textit{and} \quad \mathsf{mp}\ \frac{A \qquad A \supset B}{B} \tag{2}$$

called necessitation, modus ponens respectively.

Note that in the classical case the axioms k_2–k_5 from (1) would follow from k_1, but due to the lack of De Morgan duality, this is not the case in intuitionistic logic.

This variant of IK that contains all 5 axioms k_1–k_5 has first been studied in [20,21] and investigated in detail in [9]. There exist other intuitionistic variants of K, e.g., [22–25], the most prominent being the one which has only the axioms k_1 and k_2 from (1). There is now consensus in the literature to call this variant *constructive modal logic*, e.g., [18,26,27].

Besides the axioms (1) we also consider the axioms d, t, b, 4, and 5 shown in the left column of Fig. 1. By adding a subset of these five axioms, we can *a priori* define 32 different logics. But some of them coincide, and we get (as in the classical case) only 15 different logics, which can be organized in the intuitionistic version of the "modal cube" [28] shown in Fig. 2.

Definition 2.3 (Logics IK+X) *For any* X $\subseteq$ {d, t, b, 4, 5}*, the logic* IK+X *is obtained from* IK *by adding all axioms in* X*. We typically simplify the name of the logic by dropping the plus and capitalizing the names of axioms that are letters. For example, the logic* ID45 *in Fig. 2 is* IK + {d, 4, 5}*. Additionally,* IS4 := IK + {t, 4} *and* IS5 := IK + {t, 4, 5}*. We write* IK + X $\vdash$ *A to state that A is a theorem of* IK + X*.*

Let us now recall the *birelational models* [21,29] for intuitionistic modal logics, which are a combination of the Kripke semantics for propositional intuitionistic logic and for classical modal logic.

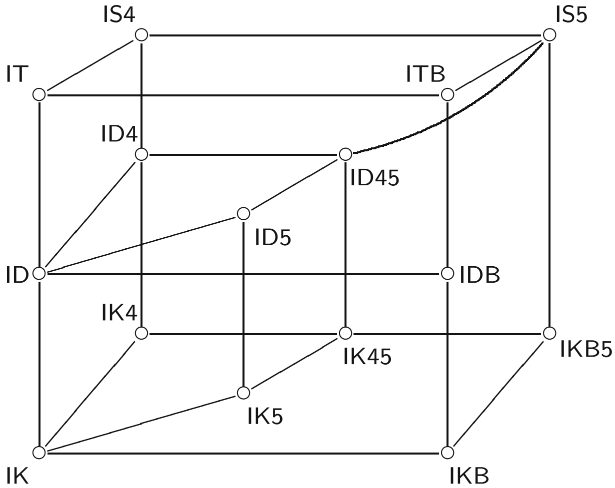


Fig. 2 The intuitionistic "modal cube"

Definition 2.4 (Birelational semantics) *A* frame $\langle W, \leq, R\rangle$ *is a non-empty set* W *of* worlds *together with two binary relations* $\leq, R \subseteq W \times W$, *where* $\leq$ *is a preorder (i.e., reflexive and transitive), such that the following two conditions hold:*

(F1) *For all* w, v, v', *if* wRv *and* $v \leq v'$, *then there is a* w' *such that* $w \leq w'$ *and* $w'Rv'$.

(F2) *For all* w', w, v, *if* $w \leq w'$ *and* wRv, *then there is a* v' *such that* $w'Rv'$ *and* $v \leq v'$.

A (birelational) model $\mathfrak{M}$ *is a quadruple* $\langle W, \leq, R, V\rangle$, *where* $\langle W, \leq, R\rangle$ *is a frame, and* V, *called* valuation, *is a monotone function from the* $\leq$*-ordered set* $\langle W, \leq\rangle$ *of worlds into the set* $\langle 2^{\mathcal{A}}, \subseteq\rangle$ *of subsets of propositional variables ordered by inclusion. The valuation* V *maps each world* w *to the set of propositional variables that are true in* w. *We write* $\mathfrak{M}, w \Vdash a$ *if* $a \in V(w)$. *The relation* $\Vdash$ *is extended to all formulas as follows:*

- $\mathfrak{M}, w \Vdash A \wedge B$ *iff* $\mathfrak{M}, w \Vdash A$ *and* $\mathfrak{M}, w \Vdash B$;
- $\mathfrak{M}, w \Vdash A \vee B$ *iff* $\mathfrak{M}, w \Vdash A$ *or* $\mathfrak{M}, w \Vdash B$;
- $\mathfrak{M}, w \Vdash A \supset B$ *iff for all* $w' \geq w$ *we have that* $\mathfrak{M}, w' \Vdash A$ *implies* $\mathfrak{M}, w' \Vdash B$;
- $\mathfrak{M}, w \Vdash \Box A$ *iff for all* w' *and* v' *such that* $w' \geq w$ *and* $w'Rv'$ *we have* $\mathfrak{M}, v' \Vdash A$;
- $\mathfrak{M}, w \Vdash \Diamond A$ *iff there is a* v *such that* wRv *and* $\mathfrak{M}, v \Vdash A$.

When $\mathfrak{M}, w \Vdash A$ *we say that* w forces A *in* $\mathfrak{M}$. *We write* $\mathfrak{M}, w \nVdash A$ *(*w does not force A in $\mathfrak{M}$*) if* $\mathfrak{M}, w \Vdash A$ *does not hold. We omit the name of the model* $\mathfrak{M}$ *when it does not create confusion.*

In particular, note that $w \nVdash \bot$ and $w \Vdash \top$ for all worlds w in all models. It is easy to show that:

Proposition 2.5 (Monotonicity) *Let* $\mathfrak{M} = \langle W, \leq, R, V\rangle$ *be a model. If* $w \leq w'$ *and* $\mathfrak{M}, w \Vdash A$, *then* $\mathfrak{M}, w' \Vdash A$.

Definition 2.6 (Validity for formulas) *We say that a formula A* is valid in a model $\mathfrak{M}$ *and write $\mathfrak{M} \Vdash A$ if every world in $\mathfrak{M}$ forces A.*

Definition 2.7 (X-model and X-validity for formulas) *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$ *and* $\mathfrak{M} = \langle W, \leq, R, V\rangle$ *be a birelational model. If the relation R obeys all frame conditions in the second column of Fig. 1 that correspond to the axioms in* X*, then we call $\mathfrak{M}$ an* X-model. *We say that a formula A is* X-valid *and write* $\mathsf{X} \Vdash A$ *if A is valid in every* X*-model.*

The *raison d'être* of the birelational models is the following theorem, for which a proof can be found in [9].

Theorem 2.8 (Soundness and completeness) *For any* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$*, a formula A is a theorem of* $\mathsf{IK} + \mathsf{X}$ *if and only if it is valid in all* X*-models, i.e.,*

$$\mathsf{IK} + \mathsf{X} \vdash A \quad \iff \quad \mathsf{X} \Vdash A \ .$$

If we collapse the relation $\leq$ by letting $w \leq v$ iff $w = v$ we obtain the standard Kripke models for classical modal logics.

3 Nested sequents for modal logics

Ordinary one-sided sequents are usually multisets of formulas separated by commas:

$$A_1, \ldots, A_n \ . \tag{3}$$

The intended meaning of such a sequent is given by its *corresponding formula*

$$A_1 \vee \cdots \vee A_n \ .$$

Ordinary two-sided sequents are pairs of such multisets of formulas usually separated by the sequent arrow $\Rightarrow$. The corresponding formula of a two-sided sequent

$$B_1, B_2, \ldots, B_h \Rightarrow C_1, C_2, \ldots, C_l \tag{4}$$

is the formula

$$B_1 \wedge B_2 \wedge \cdots \wedge B_h \supset C_1 \vee C_2 \vee \cdots \vee C_l \ .$$

In their original formulation for classical modal logics [10,11], *nested sequents* are a generalization of ordinary one-sided sequents: a nested sequent is a tree whose nodes are multisets of formulas. More precisely, it is of the form

$$A_1, \ldots, A_n, [\Gamma_1], \ldots, [\Gamma_m] \tag{5}$$

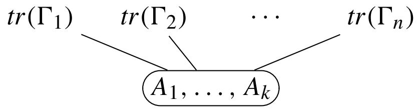
where $A_1, \ldots, A_n$ are formulas and $\Gamma_1, \ldots, \Gamma_m$ are nested sequents. The *corresponding formula* for the sequent in (5) in the classical case is

$$A_1 \vee \cdots \vee A_n \vee \Box fm(\Gamma_1) \vee \cdots \vee \Box fm(\Gamma_m)$$

where $fm(\Gamma_i)$ is the corresponding formula of Γ_i for $i \in \{1, \ldots, m\}$. In the following, we just write *sequent* for nested sequent.

Definition 3.1 (Sequent tree) *For a sequent* Γ *we write* $tr(\Gamma)$ *to denote its* sequent tree *whose nodes (called* sequent nodes *from now on and denoted by lowercase Greek letters, like* γ, δ, σ*) are multisets of formulas. We slightly abuse the notation and write* $\gamma \in \Gamma$ *instead of* $\gamma \in tr(\Gamma)$.

$tr(\Gamma) :=$



(6)

The depth *of a sequent is defined to be the depth of its tree.*

For capturing intuitionistic logic, we need "two-sided" nested sequents. For this, we follow [13] and assign each formula in the nested sequent a unique *polarity* that can be either $\bullet$ for *input/left polarity* (representing "being in the antecedent of the sequent" or "on the left of the sequent arrow, if there were a sequent arrow"), and $\circ$ for *output/right polarity* (representing "being in the succedent of the sequent" or "on the right of the sequent arrow, if there were a sequent arrow").

Definition 3.2 (Two-sided nested sequent) *A two-sided nested sequent is of the shape*

$$B_1^\bullet, \ldots, B_h^\bullet, C_1^\circ \ldots, C_l^\circ, [\Gamma_1], \ldots, [\Gamma_m] \tag{7}$$

where $B_1, \ldots, B_h, C_1, \ldots, C_l$ *are formulas and* $\Gamma_1, \ldots, \Gamma_m$ *are two-sided nested sequents.*

In a classical setting, the corresponding formula of (7) is simply

$$B_1 \wedge \cdots \wedge B_h \supset C_1 \vee \cdots \vee C_l \vee \Box fm(\Gamma_1) \vee \cdots \vee \Box fm(\Gamma_m)\ . \tag{8}$$

However, in the intuitionistic setting, the situation is not as simple. The systems presented in [13,15,16] follow Gentzen's idea of having exactly one formula of output polarity in the sequent. Such a sequent is generated by the grammar

Definition 3.3 (Single-conclusion two-sided nested sequent)

$$\Lambda ::= \varnothing \mid \Lambda, B^\bullet \mid \Lambda, [\Lambda] \qquad \Gamma ::= \Lambda, C^\circ \mid \Lambda, [\Gamma] \tag{9}$$

In (9)*,* Λ *stands for a sequent that contains only formulas with input polarity, and* Γ *for a sequent that contains exactly one formula with output polarity. The* corresponding formula *of a sequent in* (9) *is defined as follows:*

$$\bot^{\bullet}\,\frac{}{\Gamma\{\bot^{\bullet}\}} \qquad \mathsf{id}\,\frac{}{\Gamma\{a^{\bullet},a^{\circ}\}} \qquad \mathsf{c}^{\bullet}\,\frac{\Gamma\{A^{\bullet},A^{\bullet}\}}{\Gamma\{A^{\bullet}\}} \qquad \mathsf{c}^{\circ}\,\frac{\Gamma\{A^{\circ},A^{\circ}\}}{\Gamma\{A^{\circ}\}}$$

$$\wedge^{\bullet}\,\frac{\Gamma\{A^{\bullet}\}}{\Gamma\{A\wedge B^{\bullet}\}} \qquad \wedge^{\circ}\,\frac{\Gamma\{A^{\circ}\}\quad\Gamma\{B^{\circ}\}}{\Gamma\{A\wedge B^{\circ}\}} \qquad \supset^{\bullet}\,\frac{\Gamma\{A^{\circ}\}\quad\Gamma\{B^{\bullet}\}}{\Gamma\{A\supset B^{\bullet}\}} \qquad \supset^{\circ}\,\frac{\Gamma\{A^{\bullet},B^{\circ}\}}{\Gamma\{A\supset B^{\circ}\}}$$

$$\wedge^{\bullet}\,\frac{\Gamma\{B^{\bullet}\}}{\Gamma\{A\wedge B^{\bullet}\}} \qquad \vee^{\circ}\,\frac{\Gamma\{A^{\circ}\}}{\Gamma\{A\vee B^{\circ}\}} \qquad \Box^{\bullet}\,\frac{\Gamma\{[A^{\bullet},\Delta]\}}{\Gamma\{\Box A^{\bullet},[\Delta]\}} \qquad \Box^{\circ}\,\frac{\Gamma\{[A^{\circ}]\}}{\Gamma\{\Box A^{\circ}\}}$$

$$\vee^{\bullet}\,\frac{\Gamma\{A^{\bullet}\}\quad\Gamma\{B^{\bullet}\}}{\Gamma\{A\vee B^{\bullet}\}} \qquad \vee^{\circ}\,\frac{\Gamma\{B^{\circ}\}}{\Gamma\{A\vee B^{\circ}\}} \qquad \Diamond^{\bullet}\,\frac{\Gamma\{[A^{\bullet}]\}}{\Gamma\{\Diamond A^{\bullet}\}} \qquad \Diamond^{\circ}\,\frac{\Gamma\{[A^{\circ},\Delta]\}}{\Gamma\{\Diamond A^{\circ},[\Delta]\}}$$

Fig. 3 System NKK

$$\mathsf{d}^{\circ}\,\frac{\Gamma\{[A^{\circ}]\}}{\Gamma\{\Diamond A^{\circ}\}} \qquad \mathsf{t}^{\circ}\,\frac{\Gamma\{A^{\circ}\}}{\Gamma\{\Diamond A^{\circ}\}} \qquad \mathsf{b}^{\circ}\,\frac{\Gamma\{[\Delta],A^{\circ}\}}{\Gamma\{[\Delta,\Diamond A^{\circ}]\}} \qquad 4^{\circ}\,\frac{\Gamma\{[\Diamond A^{\circ},\Delta]\}}{\Gamma\{\Diamond A^{\circ},[\Delta]\}} \qquad 5^{\circ}\,\frac{\Gamma\{\varnothing\}\{\Diamond A^{\circ}\}}{\Gamma\{\Diamond A^{\circ}\}\{\varnothing\}}$$

$$\mathsf{d}^{\bullet}\,\frac{\Gamma\{[A^{\bullet}]\}}{\Gamma\{\Box A^{\bullet}\}} \qquad \mathsf{t}^{\bullet}\,\frac{\Gamma\{A^{\bullet}\}}{\Gamma\{\Box A^{\bullet}\}} \qquad \mathsf{b}^{\bullet}\,\frac{\Gamma\{[\Delta],A^{\bullet}\}}{\Gamma\{[\Delta,\Box A^{\bullet}]\}} \qquad 4^{\bullet}\,\frac{\Gamma\{[\Box A^{\bullet},\Delta]\}}{\Gamma\{\Box A^{\bullet},[\Delta]\}} \qquad 5^{\bullet}\,\frac{\Gamma\{\varnothing\}\{\Box A^{\bullet}\}}{\Gamma\{\Box A^{\bullet}\}\{\varnothing\}}$$

Fig. 4 Rules for the axioms in Fig. 1; 5° and $5^{\bullet}$ have proviso $depth(\Gamma\{\ \}\{\emptyset\}) \geq 1$

$$\supset^{\bullet}_{\mathsf{s}}\,\frac{\Gamma^{\downarrow}\{A^{\circ}\}\quad\Gamma\{B^{\bullet}\}}{\Gamma\{A\supset B^{\bullet}\}} \qquad \supset^{\circ}_{\mathsf{m}}\,\frac{\Gamma^{\downarrow}\{A^{\bullet},B^{\circ}\}}{\Gamma\{A\supset B^{\circ}\}} \qquad \Box^{\circ}_{\mathsf{m}}\,\frac{\Gamma^{\downarrow}\{[A^{\circ}]\}}{\Gamma\{\Box A^{\circ}\}} \qquad \mathsf{d}^{[]}\,\frac{\Gamma\{[\ ]\}}{\Gamma\{\varnothing\}}$$

Fig. 5 Rules needed for defining the intuitionistic systems used in this paper

$$\begin{aligned} fm(\varnothing) &:= \bot\ , \\ fm(\Lambda, B^{\bullet}) &:= fm(\Lambda) \wedge B\ , \\ fm(\Lambda, [\Lambda']) &:= fm(\Lambda) \wedge \Diamond fm(\Lambda')\ , \\ fm(\Lambda, C^{\circ}) &:= fm(\Lambda) \supset C\ , \\ fm(\Lambda, [\Gamma]) &:= fm(\Lambda) \supset \Box fm(\Gamma)\ . \end{aligned} \tag{10}$$

Note that the corresponding formula for a single-conclusion sequent significantly depends on the position of the output formula. Moving this formula to a different node drastically changes the corresponding formula. Thus, there is no natural way to generalize this to the multi-conclusion case, and unfortunately, it seems that no such formula exists. For this reason, in the next section we provide an alternative definition of validity for multi-conclusion sequents.

The next step is to show the inference rules. But before we can do so, we need to introduce an additional notation.

Definition 3.4 (Sequent context) A (sequent) context *is a nested sequent with a hole* $\{\ \}$, *taking the place of a formula. Contexts are denoted by* $\Gamma\{\ \}$, *and* $\Gamma\{\Delta\}$ *is the sequent obtained from* $\Gamma\{\ \}$ *by replacing the occurrence of* $\{\ \}$ *with* Δ. *We write* $\Gamma\{\emptyset\}$ *for the sequent obtained from* $\Gamma\{\ \}$ *by removing the* $\{\ \}$ *(i.e., the hole is filled with nothing). We also allow* binary contexts *which are sequents with two occurrences of* $\{\ \}$, *that can be* *filled independently, i.e., if* $\Gamma\{\ \}\{\ \}$ *is a binary context, then* $\Gamma\{\Delta\}\{\ \}$ *and* $\Gamma\{\ \}\{\Delta\}$ *are two different (unary) contexts. The* depth *of a context* $\Gamma\{\ \}$*, denoted by* $depth(\Gamma\{\ \})$*, is defined inductively as follows:* $depth(\{\ \}) = 0$ *and* $depth(\Gamma', \Gamma\{\ \}) = depth(\Gamma\{\ \})$ *and* $depth([\Gamma\{\ \}]) = depth(\Gamma\{\ \}) + 1$*, i.e., it is the length of the path from the sequent node with the hole to the root of the tree.*

Definition 3.5 (Multi-conclusion nested sequent calculi $\mathsf{NKK} + \mathsf{X}$ for classical modal logics) *Figure 3 shows the system for the classical modal logic* K*, which is just the two-sided version of Brünnler's system* [11] *(see also* [15]*), extended with the rules for* $\bot$ *and* $\supset$.[3] *Then, Fig. 4 shows the rules for the axioms* d, t, b, $\mathsf{4}$, *and* $\mathsf{5}$ *from Fig. 1. For* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *we write* $\mathsf{X}^\bullet$ *and* X° *to be the corresponding subsets of* $\{\mathsf{d}^\bullet, \mathsf{t}^\bullet, \mathsf{b}^\bullet, \mathsf{4}^\bullet, \mathsf{5}^\bullet\}$ *and* $\{\mathsf{d}^\circ, \mathsf{t}^\circ, \mathsf{b}^\circ, \mathsf{4}^\circ, \mathsf{5}^\circ\}$ *respectively. Then we write*

$$\mathsf{NKK} + \mathsf{X} := \mathsf{NKK} \cup \mathsf{X}^\bullet \cup \mathsf{X}^\circ\ .$$

As usual, we denote derivability in these and other nested sequent calculi by using $\vdash$. Before we can state soundness and completeness for the classical system, we need the notion of *45-closure* due to Brünnler [11], which is needed for completeness.

Definition 3.6 (45-closure) *A set* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *is called* 45-closed *if the following two conditions are fulfilled:*

1. *if* $\mathsf{IK} + \mathsf{X} \vdash \mathsf{4}$*, then* $\mathsf{4} \in \mathsf{X}$*;*
2. *if* $\mathsf{IK} + \mathsf{X} \vdash \mathsf{5}$*, then* $\mathsf{5} \in \mathsf{X}$.

The 45-closure *of* X*, denoted by* $\hat{\mathsf{X}}$*, is the smallest 45-closed set that contains* X.

Theorem 3.7 (Brünnler [11]) *For a 45-closed set* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*, the system* $\mathsf{NKK}+\mathsf{X}$ *is sound and complete w.r.t. the classical modal logic* K *extended with the axioms* X.

We can now straightforwardly obtain an intuitionistic variant of the system NKK by demanding that each sequent occurring in a proof contains exactly one output formula. Note that almost all rules in Figs. 1 and 3 preserve this property when going from conclusion to premise, and can therefore remain unchanged. There are only two rules that violate this condition: c° and $\supset^\bullet$. We therefore forbid the use of c° and change $\supset^\bullet$ in that we delete the old output formula in the left premise.

Definition 3.8 (Single-conclusion nested sequent calculi $\mathsf{NIKs}+\mathsf{X}$ and $\mathsf{NIKs}+\mathsf{X}'$ for intuitionistic modal logics) *We define*

$$\mathsf{NIKs} := \mathsf{NKK} \setminus \{\mathsf{c}^\circ, \supset^\bullet\} \cup \{\supset^\bullet_{\mathsf{s}}\}\ ,$$

where the rule $\supset^\bullet_{\mathsf{s}}$ *is shown on the left in Fig. 5, where* $\Gamma^{\downarrow}\{\ \}$ *stands for the context* $\Gamma\{\ \}$ *with all output formulas removed. For each* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*, we define*

$$\mathsf{NIKs} + \mathsf{X} := \mathsf{NIKs} \cup \mathsf{X}^\bullet \cup \mathsf{X}^\circ\ .$$

[3] We use here a system with explicit contraction and the additive versions for $\wedge$ and $\vee$ because in this way the two intuitionistic systems we show later are just restrictions of the classical system.

We write $\mathsf{NIKs{+}X'}$ *for the system obtained from* $\mathsf{NIKs{+}X}$ *by replacing the two rules* $\mathsf{d}^\bullet$ *and* d°*, if they are present, with the rule* $\mathsf{d}^{[]}$ *(shown on the right in Fig. 5):*

$$\mathsf{NIKs{+}X'} := \begin{cases} \mathsf{NIKs{+}X} \setminus \{\mathsf{d}^\bullet, \mathsf{d}^\circ\} \cup \{\mathsf{d}^{[]}\} & \textit{if } \mathsf{d} \in \mathsf{X}\ , \\ \mathsf{NIKs{+}X} & \textit{otherwise.} \end{cases}$$

In a similar way we can define $\mathsf{NKK{+}X'}$*.*

Theorem 3.9 (Straßburger, [13]) *For a 45-closed set* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$*,* $\mathsf{NIKs{+}X'}$ *is sound and complete w.r.t. the intuitionistic modal logic* $\mathsf{IK} + \mathsf{X}$*.*[4]

The proof in [13] is done via cut elimination where the cut rule is shown on the left below:

$$\mathsf{cut_s}\ \frac{\Gamma^{\downarrow}\{A^\circ\} \quad \Gamma\{A^\bullet\}}{\Gamma\{\emptyset\}}\ , \qquad\qquad \mathsf{cut_m}\ \frac{\Gamma\{A^\circ\} \quad \Gamma\{A^\bullet\}}{\Gamma\{\emptyset\}}\ . \tag{11}$$

The variant of the cut rule on the right above is the version for the systems without the restriction of having only one output formula in a sequent.

This brings us to the actual purpose of this paper: multiple-conclusion systems for the logics $\mathsf{IK} + \mathsf{X}$, in the style of Maehara [3].

Definition 3.10 (Multi-conclusion nested sequent calculi $\mathsf{NIKm} + \mathsf{X}$ and $\mathsf{NIKm{+}X'}$ for intuitionistic modal logics) *As before, we start from the classical system and define*

$$\mathsf{NIKm} = \mathsf{NKK} \setminus \{\supset^\circ, \Box^\circ\} \cup \{\supset^\circ_{\mathsf{m}}, \Box^\circ_{\mathsf{m}}\}$$

where the rules $\supset^\circ_{\mathsf{m}}$ *and* $\Box^\circ_{\mathsf{m}}$ *are given in the center of Fig. 5 and* $\Gamma^{\downarrow}\{\ \}$ *is defined as in Definition 3.8. Then, the systems* $\mathsf{NIKm{+}X}$ *and* $\mathsf{NIKm{+}X'}$ *are defined analogously to* $\mathsf{NIKs{+}X}$ *and* $\mathsf{NIKs{+}X'}$*.*

In all these systems, the weakening rule

$$\mathsf{w}\ \frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}}$$

is depth-preserving admissible:

Lemma 3.11 (dp-admissibility of weakening) *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$*. Then the weakening rule* w *is depth-preserving admissible in* $\mathsf{NKK{+}X}$*, in* $\mathsf{NIKs{+}X}$*, and in* $\mathsf{NIKm{+}X}$*, i.e., if* $\Gamma\{\emptyset\}$ *has a proof, then* $\Gamma\{\Delta\}$ *has a proof of at most the same depth.*

Proof The proof is a straightforward induction on the depth of the derivation (see [11] for details). □

The following lemma clarifies the relationship between the rule $\mathsf{d}^{[]}$ and the rules $\mathsf{d}^\bullet$ and d°.

[4] In [13], the theorem is incorrectly stated for $\mathsf{NIKs{+}X}$. However, as observed in [30], in the absence of c° the rule $\mathsf{d}^{[]}$ is not admissible in the general case (see Lemma 3.12 and Remark 3.13 below).

Lemma 3.12 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$. *If* $\mathsf{d} \in \mathsf{X}$ *then* $\mathsf{d}^{[]}$ *is admissible in* $\mathsf{NKK} + \mathsf{X}$ *and in* $\mathsf{NIKm} + \mathsf{X}$. *Furthermore,* $\mathsf{d}^{\bullet}$ *and* d° *are derivable in* $\{\mathsf{d}^{[]}, \Box^{\bullet}, \Diamond^{\circ}\}$.

Proof The proof of the first statement is by induction on the derivation depth with case distinction based on the last rule used in this derivation. It is obvious that the empty bracket can be removed from any initial sequent. For most rules, the statement for the conclusion easily follows from the IH for the premises. If the bracket to be removed became empty because the last rule was $\Box^{\bullet}$ with $\Delta = \varnothing$, then these $\Box^{\bullet}$ followed by $\mathsf{d}^{[]}$ can be replaced with $\mathsf{d}^{\bullet}$. This also proves the derivability of $\mathsf{d}^{\bullet}$ from $\Box^{\bullet}$ and $\mathsf{d}^{[]}$. Similarly, $\Diamond^{\circ}$ with $\Delta = \varnothing$ followed by $\mathsf{d}^{[]}$ can be replaced with d°, making the latter derivable from $\Diamond^{\circ}$ and $\mathsf{d}^{[]}$. The cases for the rules $\mathsf{4}^{\circ}$, $\mathsf{4}^{\bullet}$, $\mathsf{5}^{\circ}$, and $\mathsf{5}^{\bullet}$ are similar. We only show the transformation for $\mathsf{5}^{\circ}$:

$$\mathsf{d}^{[]}\,\dfrac{\mathsf{5}^{\circ}\,\dfrac{\Gamma\{\varnothing\}\{[\Diamond A^{\circ}]\}}{\Gamma\{\Diamond A^{\circ}\}\{[\ ]\}}}{\Gamma\{\Diamond A^{\circ}\}\{\varnothing\}} \quad\rightsquigarrow\quad \mathsf{c}^{\circ}\,\dfrac{\mathsf{5}^{\circ}\,\dfrac{\mathsf{d}^{\circ}\,\dfrac{\mathsf{5}^{\circ}\,\dfrac{w\,\dfrac{\Gamma\{\varnothing\}\{[\Diamond A^{\circ}]\}}{\Gamma\{\varnothing\}\{[\Diamond A^{\circ}, A^{\circ}]\}}}{\Gamma\{\Diamond A^{\circ}\}\{[A^{\circ}]\}}}{\Gamma\{\Diamond A^{\circ}\}\{\Diamond A^{\circ}\}}}{\Gamma\{\Diamond A^{\circ}, \Diamond A^{\circ}\}\{\varnothing\}}}{\Gamma\{\Diamond A^{\circ}\}\{\varnothing\}}$$

Note that the first transition is by weakening, which is admissible in all our systems by Lemma 3.11, and that the proviso for both applications of $\mathsf{5}^{\circ}$ in the transformed derivation is satisfied whenever it is satisfied in the original derivation. □

Remark 3.13 As observed by Marin [30], Lemma 3.12 fails to hold for $\mathsf{NIKs}+\mathsf{X}$ because of the absence of c°. Below is an example of a derivation from which $\mathsf{d}^{[]}$ cannot be eliminated:

$$\supset^{\circ}\,\dfrac{\mathsf{d}^{[]}\,\dfrac{\mathsf{4}^{\circ}\,\dfrac{\mathsf{d}^{\circ}\,\dfrac{\supset^{\circ}\,\dfrac{\supset^{\bullet}_{\mathsf{s}}\,\dfrac{\Diamond^{\circ}\,\dfrac{\Diamond^{\circ}\,\dfrac{\mathsf{id}\,\dfrac{}{[[a^{\circ}, a^{\bullet}]]}}{[\Diamond a^{\circ}, [a^{\bullet}]]}}{\Diamond\Diamond a^{\circ}, [[a^{\bullet}]]} \qquad \bot^{\bullet}\,\dfrac{}{\bot^{\bullet}, [[a^{\bullet}, b^{\circ}]]}}{\Diamond\Diamond a \supset \bot^{\bullet}, [[a^{\bullet}, b^{\circ}]]}}{\Diamond\Diamond a \supset \bot^{\bullet}, [[a \supset b^{\circ}]]}}{\Diamond\Diamond a \supset \bot^{\bullet}, [\Diamond(a \supset b)^{\circ}]}}{\Diamond\Diamond a \supset \bot^{\bullet}, \Diamond(a \supset b)^{\circ}, [\ ]}}{\Diamond\Diamond a \supset \bot^{\bullet}, \Diamond(a \supset b)^{\circ}}}{(\Diamond\Diamond a \supset \bot) \supset \Diamond(a \supset b)^{\circ}}$$

In all systems presented so far, the identity rule id is restricted to atomic formulas, but the general form is derivable.

Proposition 3.14 (Non-atomic initial sequents) *For every formula A and every appropriate context* $\Gamma\{\ \}$*, the sequent*

$$\Gamma\{A^\bullet, A^\circ\}$$

is derivable in NKK*, in* NIKs*, and in* NIKm.

Proof By a straightforward induction on A. □

Remark 3.15 The appropriateness of the context only plays a role for NIKs, where $\Gamma\{\ \}$ is not allowed to contain output formulas.

Maehara shows [3] the equivalence of his multiple conclusion system to Gentzen's single conclusion system from [2] by translating a multiple conclusion sequent into a single conclusion sequent whereby the multiple formulas on the right are replaced by one, their disjunction. This is not possible in the nested sequent setting because "the formulas on the right" are generally scattered all over the sequent tree.

However, one direction is straightforward:

Theorem 3.16 (Translation from single- to multi-conclusion) *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *and* Γ *be a single-conclusion sequent.*

$$\mathsf{NIKs{+}X} \vdash \Gamma \quad\Longrightarrow\quad \mathsf{NIKm{+}X} \vdash \Gamma\ ,$$
$$\mathsf{NIKs{+}X'} \vdash \Gamma \quad\Longrightarrow\quad \mathsf{NIKm{+}X'} \vdash \Gamma\ .$$

Proof The only rule in NIKs + X (resp. NIKs+X′) that is not an instance of a rule in NIKm + X (resp. NIKm+X′) is $\supset^\bullet_{\mathsf{s}}$. But it can be derived using $\supset^\bullet$ and weakening. Thus, the theorem follows from Lemma 3.11. □

Corollary 3.17 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*. If a sequent* Γ *is provable in* NIKs+X *or in* NIKs+X′*, then* Γ *is also provable both in* NIKm+X *and in* NIKm+X′.

Proof This follows immediately from Theorem 3.16 using Lemma 3.12. □

Note that in Corollary 3.17, it is implicitly assumed that the sequent Γ has exactly one output formula because otherwise it could not be the endsequent of a correct derivation in NIKs+X or NIKs+X′.

Corollary 3.18 (Formula-level completeness of NIKm+X and NIKm+X′) *For a 45-closed set* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*,*

$$\mathsf{X} \Vdash B \quad\Longrightarrow\quad \mathsf{NIKm{+}X} \vdash B^\circ\ ,$$
$$\mathsf{X} \Vdash B \quad\Longrightarrow\quad \mathsf{NIKm{+}X'} \vdash B^\circ\ .$$

Proof If B is X-valid, then B° is derivable in NIKs+X′ by Theorem 3.9. Thus, B° is derivable both in NIKm+X and in NIKm+X′ by Corollary 3.17. □

4 Semantic proof of soundness

In this section we show that every rule in NIKm+X is sound with respect to X-models. For this, we first have to extend the notion of validity from formulas to sequents.

Definition 4.1 ($\mathfrak{M}$-map) *For a sequent* Γ *and a birelational model* $\mathfrak{M}=\langle W, \leq, R, V\rangle$*, an* $\mathfrak{M}$-map for Γ *is a map* $f : tr(\Gamma) \to W$ *from nodes of the sequent tree to worlds in the model such that, whenever* δ *is a child of* γ *in* $tr(\Gamma)$*, then* $f(\gamma)Rf(\delta)$.

Definition 4.2 (Forcing for sequents) *A sequent* Γ *is* satisfied *by an* $\mathfrak{M}$*-map* f *for* Γ*, written* $f \Vdash \Gamma$*, iff*

$$\mathfrak{M}, f(\gamma) \Vdash A \quad \textit{for all } A^\bullet \in \gamma \in \Gamma \qquad \Longrightarrow \qquad \mathfrak{M}, f(\delta) \Vdash B \quad \textit{for some } B^\circ \in \delta \in \Gamma \ .$$

If Γ *is not satisfied by* f*, it is* refuted *by it.*

Remark 4.3 This definition works for both single- and multi-conclusion sequents.

Definition 4.4 (X-validity for sequents) *For every* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$*, a sequent* Γ *is* X-valid*, written* $\mathsf{X} \Vdash \Gamma$*, iff it is satisfied by all* $\mathfrak{M}$*-maps for* Γ *for all* X*-models* $\mathfrak{M}$*. A sequent is* X-refutable *iff there is an* $\mathfrak{M}$*-map for an* X*-model* $\mathfrak{M}$ *that refutes it.*

Lemma 4.5 (Sequent validity extends formula validity) *A formula* B *is* X*-valid in all* X*-models if and only if the sequent* B° *is:*

$$\mathsf{X} \Vdash B \qquad \Longleftrightarrow \qquad \mathsf{X} \Vdash B^\circ \ .$$

Proof This follows immediately from the definition of validity. □

Theorem 4.6 (Soundness) *For any* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$,

$$\mathsf{NIKm{+}X} \vdash \Sigma \qquad \Longrightarrow \qquad \mathsf{X} \Vdash \Sigma \ .$$

Proof We prove the contrapositive: if Σ is X-refutable, then $\mathsf{NIKm} + \mathsf{X}$ does not prove Σ. To demonstrate this, it is sufficient to show that, whenever the conclusion of a rule from $\mathsf{NIKm} + \mathsf{X}$ is X-refutable, then so is at least one of the premises of this rule.

Let $\mathfrak{M} = \langle W, \leq, R, V\rangle$ be an arbitrary X-model and f be an arbitrary $\mathfrak{M}$-map for the conclusion Γ of a given rule. Let $\gamma \in \Gamma$ be the node with the hole of this rule. Since the model $\mathfrak{M}$ is never modified, we omit its mentions in this proof. Note that an $\mathfrak{M}$-map refutes a sequent iff it maps its nodes into worlds of $\mathfrak{M}$ in a way that makes all input formulas forced and all output formulas not forced.

Initial sequents. The statement is vacuously true for $\bot^\bullet$ and id because neither $\Gamma\{\bot^\bullet\}$ nor $\Gamma\{a^\bullet, a^\circ\}$ can be refuted in any birelational model.

Local propositional rules $\vee^\bullet$**,** $\wedge^\bullet$**,** $\supset^\bullet$**,** $\vee^\circ$**,** $\wedge^\circ$**.** Since propositional rules (including contraction rules) are local in that they act within one node of the sequent tree, the node we called γ, the proof for them is analogous to the case of propositional intuitionistic logic. Namely, for all propositional rules except $\supset^{\circ}_{\mathsf{m}}$, for any birelational model $\mathfrak{M}$, any $\mathfrak{M}$-map refuting the conclusion must refute one of the premises. Consider, for instance, an instance of the rule $\vee^{\bullet}$ and an $\mathfrak{M}$-map f that refutes its conclusion $\Gamma\{A \vee B^{\bullet}\}$. In particular, it forces all input formulas from $\Gamma\{\ \}$, forces none of output formulas from $\Gamma\{\ \}$ (each formula at the world assigned by f), and satisfies $f(\gamma) \Vdash A \vee B$. For the latter to happen, either $f(\gamma) \Vdash A$, making f refute the left premise $\Gamma\{A^{\bullet}\}$, or $f(\gamma) \Vdash B$, in which case it is the right premise $\Gamma\{B^{\bullet}\}$ that is refuted by f.

Rule $\supset^{\circ}_{\mathsf{m}}$. Assume that all input formulas in the conclusion of the rule

$$\supset^{\circ}_{\mathsf{m}} \frac{\Gamma^{\downarrow}\{B^{\bullet}, C^{\circ}\}}{\Gamma\{B \supset C^{\circ}\}}$$

are forced and all output formulas are not forced by an $\mathfrak{M}$-map f in their respective worlds, in particular, $f(\gamma) \nVdash B \supset C$. Then there exists a world $w \geq f(\gamma)$ where $w \Vdash B$ and $w \nVdash C$. It is easy to show using (F1) and (F2) that there exists another $\mathfrak{M}$-map g for the conclusion such that $g(\gamma) = w$ and $g(\delta) \geq f(\delta)$ for each node $\delta \in \Gamma\{B \supset C^{\circ}\}$. By monotonicity (Proposition 2.5), all input formulas in the conclusion are also forced by g in their respective worlds. Since additionally $g(\gamma) \Vdash B$ and $g(\gamma) \nVdash C$, it follows that in the premise all input formulas are forced and the only output formula, C, is not forced by g in their respective worlds. Thus, the constructed g refutes the premise in the same model.

Rules $\Box^{\bullet}$, $\Diamond^{\circ}$, $\mathsf{t}^{\bullet}$, t°, $\mathsf{b}^{\bullet}$, b°, 4°, and 5°. Although these rules are not local in that they affect two nodes of the sequent tree, their treatment is much the same as that of propositional rules: any $\mathfrak{M}$-map refuting the conclusion must also refute the premise. Consider, for instance,

$$4^{\circ} \frac{\Gamma\{[\Diamond A^{\circ}, \Delta]\}}{\Gamma\{\Diamond A^{\circ}, [\Delta]\}} .$$

Assume that all input formulas in the conclusion are forced and all output formulas are not forced by an $\mathfrak{M}$-map f in their respective worlds of a transitive model $\mathfrak{M}$, in particular, $f(\gamma) \nVdash \Diamond A$. Let δ be the node corresponding to the displayed bracket. Note that $f(\gamma) R f(\delta)$. Consider any world w such that $f(\delta) R w$. Then, by transitivity, $f(\gamma) R w$ and $w \nVdash A$. We have shown that $w \nVdash A$ whenever $f(\delta) R w$. Thus, $f(\delta) \nVdash \Diamond A$, which is sufficient to demonstrate that f refutes the premise of the rule. We also show the argument for

$$5^{\circ} \frac{\Gamma\{\emptyset\}\{\Diamond A^{\circ}\}}{\Gamma\{\Diamond A^{\circ}\}\{\emptyset\}} .$$

Assume that all input formulas in the conclusion are forced and all output formulas are not forced by an $\mathfrak{M}$-map f in their respective worlds of a Euclidean model $\mathfrak{M}$, in particular, $f(\gamma) \nVdash \Diamond A$ for the displayed $\Diamond A$ in the conclusion (here γ is node with the hole containing the principle formula). Let δ be the node containing the other hole and ρ be the root of the sequent tree. Then $f(\rho)R^k f(\gamma)$ and $f(\rho)R^l f(\delta)$ for some $k, l \geq 0$. Moreover, the proviso for the rule demands that $k > 0$. Consider any world w such that $f(\delta)Rw$. Then both $f(\gamma)$ and w are accessible from $f(\rho)$ in one or more R steps. It is an easy corollary of Euclideanity that $f(\gamma)Rw$, meaning that $w \nVdash A$. We have shown that $w \nVdash A$ whenever $f(\delta)Rw$. Thus, $f(\delta) \nVdash \Diamond A$, which is sufficient to demonstrate that f refutes the premise of the rule.

Rules $\mathsf{4}^\bullet$ and $\mathsf{5}^\bullet$ are similar in nature but require an additional consideration in the proof. We explain it on the example of

$$\mathsf{4}^\bullet \frac{\Gamma\{[\Box A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [\Delta]\}} .$$

As in the case of $\mathsf{4}^\circ$, we deal with two nodes: parent γ and its child δ, the latter corresponding to the displayed bracket. We assume that $f(\gamma) \Vdash \Box A$ and need to show that $f(\delta) \Vdash \Box A$. The difference lies in the fact that apart from worlds accessible from $f(\delta)$ itself, as in the case of $\mathsf{4}^\circ$, we have to consider also worlds accessible from futures of $f(\delta)$. However, the condition (F1) and transitivity ensure that any world accessible from a future of $f(\delta)$ is also accessible from some future of $f(\gamma)$ making it possible to apply the assumption.

Rules $\Diamond^\bullet$, $\mathsf{d}^\bullet$, and d°. All these rules are similar to the majority of modal rules, except for the fact that one needs to choose a new world for the premise. For rules $\mathsf{d}^\bullet$ and d°, this world is chosen as any world accessible from $f(\gamma)$ by seriality. For the rule $\Diamond^\bullet$, the assumption is that $\Diamond A$ is forced at $f(\gamma)$, which implies that there exists an accessible world forcing A, and it is this world that is chosen for the extra node in the sequent tree of the premise. Consider, e.g., an instance of $\Diamond^\bullet$ and assume that f refutes its conclusion $\Gamma\{\Diamond A^\bullet\}$. In particular, $f(\gamma) \Vdash \Diamond A$. Thus, there exists a world $w \in W$ such that $f(\gamma)Rw$ and $w \Vdash A$. We define an $\mathfrak{M}$-map g for the premise $\Gamma\{[A^\bullet]\}$ to act like f on all nodes that are present in the conclusion and to map the node δ corresponding to the displayed bracket to w. Then, just like f, the map g forces all input formulas in $\Gamma\{\ \}$ and none of output formulas in $\Gamma\{\ \}$ and, in addition, $g(\delta) \Vdash A$, meaning that g refutes the premise.

Rule $\Box^\circ_{\mathsf{m}}$. Assume that all input formulas in the conclusion of the rule

$$\Box^\circ_{\mathsf{m}} \frac{\Gamma^{\downarrow}\{[A^\circ]\}}{\Gamma\{\Box A^\circ\}}$$

are forced and all output formulas are not forced by an $\mathfrak{M}$-map f in their respective worlds, in particular, $f(\gamma) \nVdash \Box A$. Then there exist worlds u and w such that $u \geq f(\gamma)$, and uRw, and $w \nVdash A$. It is easy to show using (F1) and (F2) that there exists an $\mathfrak{M}$-map g for the premise such that $g(\gamma) = u$, $g(\delta) = w$ for the node δ present in the premise but not in the conclusion, and $g(\vartheta) \geq f(\vartheta)$ for each node $\vartheta \in \Gamma\{\Box A^\circ\}$. By monotonicity (Proposition 2.5), all input formulas in the conclusion are also forced by g in their respective worlds. Since additionally $g(\delta) \nVdash A$, it follows that in the premise all input formulas are forced and the only output formula, A, is not forced by g in their respective worlds. Thus, the constructed g refutes the premise in the same model.

This completes the proof of soundness. □

5 Semantic proof of completeness

In this section we show the completeness of our multiple conclusion systems semantically. To simplify the argument, we work with a modified system cNIKm+X′, that is defined as follows. For every inference rule in NIKm+X (and NIKm+X′), except for $\supset^\circ_{\mathsf{m}}$ and $\Box^\circ_{\mathsf{m}}$, we can define its *contraction variant*, denoted by the subscript c, that keeps the principal formula of the conclusion in all premises. Below are three examples:

$$\Box^\bullet_{\mathsf{c}} \frac{\Gamma\{\Box A^\bullet, [A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [\Delta]\}} , \qquad \Diamond^\bullet_{\mathsf{c}} \frac{\Gamma\{\Diamond A^\bullet, [A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}} ,$$

$$\supset^\bullet_{\mathsf{c}} \frac{\Gamma\{A \supset B^\bullet, A^\circ\} \quad \Gamma\{A \supset B^\bullet, B^\bullet\}}{\Gamma\{A \supset B^\bullet\}} .$$

Note that $\bot^\bullet$ and $\bot^\bullet_{\mathsf{c}}$ are identical (as are $\mathsf{d}^{[]}$ and $\mathsf{d}^{[]}_{\mathsf{c}}$). We denote by cNIKm+X′ the system obtained from NIKm+X′ by removing $\mathsf{c}^\bullet$ and c°, and by replacing every rule, except for $\supset^\circ_{\mathsf{m}}$ and $\Box^\circ_{\mathsf{m}}$, with its contraction variant.

Definition 5.1 (Equivalent systems) *Two systems* S_1 *and* S_2 *are* equivalent *if for every derivation in* S_1*, there is a derivation in* S_2 *of the same endsequent, and vice versa.*

Lemma 5.2 (Equivalence of Kleene'ing) *For every* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$*, the systems*

$$\mathsf{NIKm{+}X}, \qquad \mathsf{NIKm{+}X'}, \qquad \textit{and} \qquad \mathsf{cNIKm{+}X'}$$

are pairwise equivalent.

Proof Every rule $\mathsf{r_c}$ is derivable via r and $\mathsf{c}^\bullet$ or c°, and conversely, every rule r is derivable from $\mathsf{r_c}$ and w. Hence, the statement follows from Lemmas 3.11 and 3.12. □

We can now state the completeness theorem:

Theorem 5.3 (Completeness) *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$ *be a 45-closed set, and let* Υ *be a sequent.*

$$\mathsf{X} \Vdash \Upsilon \qquad \Longrightarrow \qquad \mathsf{NIKm} + \mathsf{X} \vdash \Upsilon .$$

Remark 5.4 Note that this is stronger than the completeness result proved syntactically in Corollary 3.18 which was formulated for single formulas rather than arbitrary sequents. While the argument used to prove Corollary 3.18 extends as is to single-conclusion sequents, the result in this section shows completeness for all multi-conclusion sequents.

The rest of this section is dedicated to the proof of Theorem 5.3, and we let X and Υ be fixed. We prove the contrapositive: if $\mathsf{NIKm} + \mathsf{X} \nvdash \Upsilon$, then Υ is X-refutable. By Lemma 5.2 we can work with the system cNIKm+X′, which is equivalent to $\mathsf{NIKm} + \mathsf{X}$. We work with the (almost) complete proof search tree $\mathfrak{T}$ in cNIKm+X′ that is constructed as follows: the nodes of $\mathfrak{T}$ are sequents, and the root of $\mathfrak{T}$ is the endsequent Υ. For each possible unary rule application r to a sequent Γ in $\mathfrak{T}$ the premise of r is a child of Γ in $\mathfrak{T}$, and for each possible binary rule application to Γ, both premises of r are children of Γ in $\mathfrak{T}$. (Recall that we mean here upward rule applications.) There are only two exceptions to prevent the creation of infinitely many redundant children: along each branch of the proof search tree

1. the rule $\Diamond^{\bullet}_{\mathsf{c}}$ is not applied to discharged occurrences of $\Diamond A^{\bullet}$; whenever the rule $\Diamond^{\bullet}_{\mathsf{c}}$ is applied to an occurrence of $\Diamond A^{\bullet}$, this occurrence is deemed discharged along this branch starting from the premise of the rule;
2. the rule $\mathsf{d}^{[]}$ is not applied to create children of discharged sequent nodes; whenever the rule $\mathsf{d}^{[]}$ is applied to create a child of a sequent node, this sequent node is deemed discharged along this branch starting from the premise of the rule (thus, the rule $\mathsf{d}^{[]}$ is applied to every node in a sequent Γ exactly once).

The countermodel that we are going to construct will be based on the tree $\underline{\mathfrak{T}}$ that is obtained from $\mathfrak{T}$ by removing all subtrees that have derivable sequents as roots. In the following, we use Γ, Δ, etc. to denote sequent occurrences in $\underline{\mathfrak{T}}$ rather than sequents.

The method we use is a modification of the one used for classical modal logics to the case of infinite proof-search tree. Ordinarily, a proof-search tree used to construct a countermodel is obtained by postponing non-invertible rules as long as possible. In other words, a non-invertible rule is only to be used after the sequent is completely saturated by all the applicable invertible rules. Saturated sequents can then be used to model individual worlds of the countermodel because they contain enough information to guarantee Boolean values required from the countermodel. For intuitionistic modal logics, however, it can happen that saturation is never achieved, that invertible rules can be applied indefinitely. Thus, instead of only using a non-invertible rule once the saturation is reached, the same non-invertible rule has to be used at every stage of partial saturation. Intuitively, the saturated state is achieved after at most countably many applications of invertible rules and the non-invertible rule is applied to the resulting "(possibly infinite) nested sequent." Formally, we consider infinitely many saturation stages of the sequent, where saturation stages of the same sequent node are connected by the correspondence relation $\approx$. For each equivalence class with respect to $\approx$, the saturated state is simply the limit over all individual instances from the class, and the non-invertible rule operates on these saturated equivalent classes. In order to synchronize different individual applications of non-invertible rules, we use the notion of *level*.

We distinguish three types of unary rules:

1. the *leveling* rules $\supset^{\circ}_{\mathsf{m}}$ and $\Box^{\circ}_{\mathsf{m}}$, which are non-invertible,
2. the *node creating* rules $\Diamond^{\bullet}_{\mathsf{c}}$ and $\mathsf{d}^{[]}$, and
3. all other unary rules (which are invertible), that we call *simple*.

Definition 5.5 (Level) *The* level *of* Γ *(and of every* $\gamma \in \Gamma$*) is the total number of leveling-rule instances on the path from* Υ *to* Γ *in* $\underline{\Sigma}$*. Sequents* Γ *and* Δ *with the same level are* equilevel.

In contrast to the soundness proof, we now distinguish between nodes in the premise and conclusion of the rule, which necessitates the following

Definition 5.6 (Corresponding nodes) *Let* Ω *be the set of all sequent nodes of all sequent occurrences in* $\underline{\Sigma}$*. We define the* correspondence relation $\approx$ *on* Ω *recursively:*

- *if* γ *and* δ *can be traced to the same sequent node in the endsequent* Υ*, then* $\gamma \approx \delta$*;*
- *if* γ *and* δ *are created by instances of* $\Diamond^{\bullet}_{\mathsf{c}}$ *with the same principal formula* $\Diamond A^{\bullet}$ *in nodes* $\gamma' \approx \delta'$ *respectively, then* $\gamma \approx \delta$*;*
- *if* γ *and* δ *are created by instances of* $\mathsf{d}^{[]}$ *from nodes* $\gamma' \approx \delta'$ *respectively, then* $\gamma \approx \delta$*;*
- *if* γ *and* δ *are created by applications of* $\Box^{\circ}_{\mathsf{m}}$ *with the same principal formula* $\Box A^{\circ}$ *in equilevel nodes* $\gamma' \approx \delta'$ *respectively, then* $\gamma \approx \delta$.

If $\gamma \approx \delta$*, we also say that* γ *and* δ *are* corresponding.

Clearly, $\approx$ is an equivalence relation. It is easy to see that distinct nodes of the same sequent occurrence cannot be corresponding. For a sequent node $\gamma \in \Omega$ and a sequent occurrence Δ we denote by γ_Δ the unique sequent node of Δ corresponding to γ (if it exists). If γ is the parent of δ in $tr(\Gamma)$ and both γ_Σ and δ_Σ exist for some sequent occurrence Σ, then γ_Σ is the parent of δ_Σ in $tr(\Sigma)$.

Definition 5.7 (Superior sequent) *We call* Δ *a* superior *of* Γ*, written* $\Gamma \sqsubseteq \Delta$*, if* γ_Δ *exists for all* $\gamma \in \Gamma$ *and satisfies* $\gamma \subseteq \gamma_\Delta$ *as multisets of formulas.*

It is clear that $\sqsubseteq$ is reflexive and transitive.

Definition 5.8 (Corresponding rules) *Two instances of the same rule* r *are called* corresponding *if they are applied to nodes* $\gamma \approx \delta$*, to the same principal formula in* γ *and* δ *(this requirement is dropped for* $\mathsf{d}^{[]}$*, which has no principal formulas), to corresponding children of* γ *and* δ *(for* $\Box^{\bullet}_{\mathsf{c}}$, $\Diamond^{\circ}_{\mathsf{c}}$, $4^{\bullet}_{\mathsf{c}}$*, and* 4°_{c}*), and to corresponding second nodes (for* $5^{\bullet}_{\mathsf{c}}$ *and* 5°_{c}*).*

Clearly, rule correspondence is also an equivalence relation.

Definition 5.9 (Rule transfer) *Let* r *be a rule instance and* Δ *be the conclusion of a corresponding rule instance. We denote the first premise of this corresponding rule instance by* $\mathsf{prem}_{\mathsf{r}}(\Delta)$ *and, in case of binary rules, the second premise by* $\mathsf{prem}'_{\mathsf{r}}(\Delta)$.

The following lemma is a direct consequence of the definition:

Lemma 5.10 (Corresponding rules for superior sequents) *Let* Γ *be the conclusion of a rule instance* r *and* Δ *be a superior of* Γ.

- *If* r *is not node-creating, then* Δ *is the conclusion of a corresponding rule instance and* $\mathsf{prem}_{\mathsf{r}}(\Gamma) \sqsubseteq \mathsf{prem}_{\mathsf{r}}(\Delta)$ *(also* $\mathsf{prem}'_{\mathsf{r}}(\Gamma) \sqsubseteq \mathsf{prem}'_{\mathsf{r}}(\Delta)$ *for binary rules).*
- *If* r *is node-creating, then either*

1. *a corresponding rule instance has already been used on the path from* Υ *to* Δ *and* $\mathsf{prem}_\mathsf{r}(\Gamma) \sqsubseteq \Delta$, *or*
2. Δ *is the conclusion of a corresponding rule instance and* $\mathsf{prem}_\mathsf{r}(\Gamma) \sqsubseteq \mathsf{prem}_\mathsf{r}(\Delta)$.

In the former case, we define $\mathsf{prem}_\mathsf{r}(\Delta) := \Delta$ *to unify the notation.*

In the following, we use $\mathfrak{G}$ to denote an arbitrary subset of the set of sequent occurrences in $\mathfrak{T}$. We write $\mathfrak{G} \subseteq \underline{\mathfrak{T}}$ if all occurrences are taken from $\underline{\mathfrak{T}}$.

Definition 5.11 (Confluent sets) *A set* $\mathfrak{G} \subseteq \underline{\mathfrak{T}}$ *is called* confluent *iff the following condition is satisfied: for any* $\Gamma, \Delta \in \mathfrak{G}$, *the sequent occurrences* Γ *and* Δ *are equilevel and there is a sequent occurrence* $\Sigma \in \mathfrak{G}$ *that is a superior of both* Γ *and* Δ. *The set* $\mathfrak{G}$ *is* maximal confluent *if it is confluent and has no proper confluent supersets in* $\underline{\mathfrak{T}}$.

It is an immediate corollary of Zorn's Lemma that

Lemma 5.12 ("Lindenbaum") *Each confluent set can be extended to a maximal confluent set.*

Definition 5.13 *Let* $\mathfrak{G} \subseteq \underline{\mathfrak{T}}$ *be a set of sequent occurrences, and let* r *be a rule instance with conclusion in* $\mathfrak{G}$. *Then we define*

$$\mathsf{prem}_\mathsf{r}(\mathfrak{G}) := \{\mathsf{prem}_\mathsf{r}(\Delta) \mid \Delta \in \mathfrak{G} \textit{ and } \mathsf{prem}_\mathsf{r}(\Delta) \textit{ is defined}\} \ .$$

For binary rules, we use $\mathsf{prem}'_\mathsf{r}(\mathfrak{G})$ *for the second premises.*

Lemma 5.14 (Properties of confluent sets) *Let the set* $\mathfrak{G} \subseteq \underline{\mathfrak{T}}$ *be confluent, and let* prem_r *be a rule instance with conclusion in* $\mathfrak{G}$.

1. *If* r *is unary, then* $\mathsf{prem}_\mathsf{r}(\mathfrak{G})$ *is confluent, and* $\mathsf{prem}_\mathsf{r}(\mathfrak{G}) \subseteq \underline{\mathfrak{T}}$. *If* r *is not a leveling rule, then* $\mathfrak{G} \cup \mathsf{prem}_\mathsf{r}(\mathfrak{G})$ *is also confluent, and* $\mathfrak{G} \cup \mathsf{prem}_\mathsf{r}(\mathfrak{G}) \subseteq \underline{\mathfrak{T}}$.
2. *If* r *is simple and* $\mathfrak{G}$ *is maximal confluent, then* $\mathsf{prem}_\mathsf{r}(\mathfrak{G}) \subseteq \mathfrak{G}$, *in other words, maximal confluent sets are closed with respect to applications of simple rules.*
3. *If* r *is a binary rule, then at least one of* $\mathsf{prem}_\mathsf{r}(\mathfrak{G})$ *and* $\mathsf{prem}'_\mathsf{r}(\mathfrak{G})$ *is a confluent set and contained in* $\underline{\mathfrak{T}}$, *and additionally must be a subset of* $\mathfrak{G}$ *if the latter is maximal confluent.*

Proof 1. Let $\mathsf{prem}_\mathsf{r}(\Pi), \mathsf{prem}_\mathsf{r}(\Delta) \in \mathsf{prem}_\mathsf{r}(\mathfrak{G})$, where $\Pi, \Delta \in \mathfrak{G}$. By the confluence of $\mathfrak{G}$, there is $\Sigma \in \mathfrak{G}$ such that $\Pi, \Delta \sqsubseteq \Sigma$. By Lemma 5.10,

$$\mathsf{prem}_\mathsf{r}(\Pi), \mathsf{prem}_\mathsf{r}(\Delta) \sqsubseteq \mathsf{prem}_\mathsf{r}(\Sigma) \in \mathsf{prem}_\mathsf{r}(\mathfrak{G}) \ . \tag{12}$$

This demonstrates that $\mathsf{prem}_\mathsf{r}(\mathfrak{G})$ is confluent.
For a non-leveling r, take two sequents from $\mathfrak{G} \cup \mathsf{prem}_\mathsf{r}(\mathfrak{G})$. If both belong to $\mathfrak{G}$ or both belong to $\mathsf{prem}_\mathsf{r}(\mathfrak{G})$, the two sequents have a superior in the same set by its confluence. If $\Pi \in \mathfrak{G}$ and $\mathsf{prem}_\mathsf{r}(\Delta) \in \mathsf{prem}_\mathsf{r}(\mathfrak{G})$, then there is a superior $\Sigma \sqsupseteq \Pi, \Delta$ in $\mathfrak{G}$ by its confluence. By Lemma 5.10, (12) holds again. Given that $\mathsf{prem}_\mathsf{r}(\Pi) \sqsupseteq \Pi$ because r is not a leveling rule, $\mathsf{prem}_\mathsf{r}(\Sigma) \in \mathsf{prem}_\mathsf{r}(\mathfrak{G})$ is a superior of both Π and $\mathsf{prem}_\mathsf{r}(\Delta)$.

2. Follows from Clause 1 and the maximality of $\mathfrak{G}$.
3. We prove that either $\mathsf{prem}_\mathsf{r}(\mathfrak{G}) \subseteq \underline{\mathfrak{T}}$ or $\mathsf{prem}'_\mathsf{r}(\mathfrak{G}) \subseteq \underline{\mathfrak{T}}$ by contradiction. Otherwise, there would have been $\Pi, \Delta \in \mathfrak{G}$ such that $\mathsf{prem}_\mathsf{r}(\Pi)$ and $\mathsf{prem}'_\mathsf{r}(\Delta)$ are both derivable. For a superior $\Sigma \sqsupseteq \Pi, \Delta$, which would have existed by the confluence of $\mathfrak{G}$, both $\mathsf{prem}_\mathsf{r}(\Sigma)$ and $\mathsf{prem}'_\mathsf{r}(\Sigma)$ would have been derivable by admissibility of weakening (Lemma 3.11), making $\Sigma \in \mathfrak{G} \subseteq \underline{\mathfrak{T}}$ derivable by r, in contradiction to our assumptions. Whichever of $\mathsf{prem}_\mathsf{r}(\mathfrak{G})$ or $\mathsf{prem}'_\mathsf{r}(\mathfrak{G})$ is within $\underline{\mathfrak{T}}$ must be confluent (and contained in $\mathfrak{G}$ for maximal confluent sets) as in Clause 1.

□

Definition 5.15 (Limit) *For a confluent set* $\mathfrak{G}$ *we define its* limit $\hat{\mathfrak{G}} = (V_{\hat{\mathfrak{G}}}, E_{\hat{\mathfrak{G}}}, L_{\hat{\mathfrak{G}}})$ *as a (possibly infinite) graph whose nodes are labelled with sets of formulas as follows:*[5]

$$V_{\hat{\mathfrak{G}}} := \{[\gamma]_{\mathfrak{G}} \mid \gamma \in \Gamma \textit{ for some } \Gamma \in \mathfrak{G}\}$$

where the quotients $[\gamma]_{\mathfrak{G}}$ *are taken with respect to the equivalence relation* $\approx$. *We define* $([\gamma]_{\mathfrak{G}}, [\delta]_{\mathfrak{G}}) \in E_{\hat{\mathfrak{G}}}$ *iff there are* $\gamma' \in [\gamma]_{\mathfrak{G}}$ *and* $\delta' \in [\delta]_{\mathfrak{G}}$ *such that* γ' *is the parent of* δ' *in* $tr(\Gamma)$ *for some nested sequent* $\Gamma \in \mathfrak{G}$. *The labelling function* $L_{\hat{\mathfrak{G}}}$ *from nodes of* $\hat{\mathfrak{G}}$ *to sets of polarized formulas is defined by*

$$\begin{array}{lll} A^\bullet \in L_{\hat{\mathfrak{G}}}([\gamma]_{\mathfrak{G}}) & \textit{iff} & A^\bullet \in \delta \textit{ for some } \delta \in [\gamma]_{\mathfrak{G}} \\ A^\circ \in L_{\hat{\mathfrak{G}}}([\gamma]_{\mathfrak{G}}) & \textit{iff} & A^\circ \in \delta \textit{ for some } \delta \in [\gamma]_{\mathfrak{G}} \end{array}$$

Lemma 5.16 $\hat{\mathfrak{G}}$ *is a tree. Furthermore, if* ρ *is the root of some* $\Gamma \in \mathfrak{G}$, *then* $[\rho]_{\mathfrak{G}}$ *is the root of* $\hat{\mathfrak{G}}$.

Proof It immediately follows from Definition 5.6 that $\rho_1 \approx \rho_2$ for the roots ρ_1 and ρ_2 of any two sequents $\Sigma_1, \Sigma_2 \in \mathfrak{G}$ because ρ_1 and ρ_2 can be traced down to the root ρ_Υ of the endsequent Υ. Thus, $[\rho]_{\mathfrak{G}}$ includes all roots of all sequents in $\mathfrak{G}$. To show that there is a path from $[\rho]_{\mathfrak{G}}$ to $[\gamma]_{\mathfrak{G}} \in V_{\hat{\mathfrak{G}}}$ for any node γ from $\Sigma \in \mathfrak{G}$, it remains to note that there is a path from ρ_Σ to γ in $tr(\Sigma)$ and that $\rho_\Sigma \approx \rho$.

Further, it is easy to show that for $\gamma \approx \delta$ with γ from a sequent $\Sigma \in \mathfrak{G}$ and δ from a sequent $\Pi \in \mathfrak{G}$, the path from ρ_Σ to γ has the same length as the path from ρ_Π to δ. In other words, each edge in $\hat{\mathfrak{G}}$ increases the minimal distance from the root (in each member sequent), which prevents directed cycles.

Finally, we show that each node in $\hat{\mathfrak{G}}$ has at most one parent. Indeed, assume $([\gamma]_{\mathfrak{G}}, [\delta]_{\mathfrak{G}}) \in E_{\hat{\mathfrak{G}}}$ and $([\gamma']_{\mathfrak{G}}, [\delta]_{\mathfrak{G}}) \in E_{\hat{\mathfrak{G}}}$. This means that γ_Σ is the parent of δ_Σ and γ'_Π is the parent of δ_Π for some sequent occurrences Σ and Π from the confluent set $\mathfrak{G}$. They must have a superior $\Lambda \in \mathfrak{G}$. Since all nodes $\delta_\Lambda \approx \delta_\Sigma \approx \delta_\Pi$, and $\gamma_\Lambda \approx \gamma_\Sigma$, and $\gamma'_\Lambda \approx \gamma'_\Pi$ exist by superiority of Λ and since both γ_Λ and γ'_Λ must coincide with the unique parent of δ_Λ in $tr(\Lambda)$, it follows that $\gamma_\Lambda = \gamma'_\Lambda$ and, consequently, $\gamma_\Sigma \approx \gamma'_\Pi$. In other words, $[\gamma]_{\mathfrak{G}} = [\gamma']_{\mathfrak{G}}$. □

[5] We will shortly show that this graph is a tree, which could be seen as an infinite nested sequent tree, but we did not define nested sequents to be infinite. Hence, we will just call it a tree.

Definition 5.17 (Countermodel) *We define* $\mathfrak{M} := \langle W, \leq, R, V\rangle$*, where*

- $W := \{[\gamma]_{\mathfrak{G}} \in V_{\hat{\mathfrak{G}}} \mid \gamma \in \Gamma \in \mathfrak{G}$*, and* $\mathfrak{G}$ *is maximal confluent, and* $\mathfrak{G} \subseteq \underline{\mathfrak{T}}\}$.
- *The binary relation* $R_0 := \bigsqcup\{E_{\hat{\mathfrak{G}}} \mid \mathfrak{G}$ *is maximal confluent, and* $\mathfrak{G} \subseteq \underline{\mathfrak{T}}\}$*. The binary relation* R *on* W *is defined as the closure of* R_0 *with respect to the frame properties corresponding to the axioms from the 45-closed* X*, except for seriality.*
- *For* $[\gamma]_{\mathfrak{G}}, [\delta]_{\mathfrak{H}} \in W$*, where* $\mathfrak{G}$ *and* $\mathfrak{H}$ *are maximal confluent sets of sequents, we define* $[\gamma]_{\mathfrak{G}} \leq_0 [\delta]_{\mathfrak{H}}$ *iff for some conclusion* $\Gamma \in \mathfrak{G}$ *and premise* $\Delta \in \mathfrak{H}$ *of a leveling-rule instance* $\mathsf{prem_r}$ *with* $\mathsf{prem_r}(\mathfrak{G}) \subseteq \mathfrak{H}$ *we have* $\gamma_\Gamma \approx \delta_\Delta$*. The binary relation* $\leq$ *on* W *is the reflexive and transitive closure of* $\leq_0$.
- *For* $[\gamma]_{\mathfrak{G}} \in W$*, we define* $V([\gamma]_{\mathfrak{G}}) := \{a \mid a^\bullet \in [\gamma]_{\mathfrak{G}}\}$.

Note that this model construction is a distant relative of the canonical models. Indeed, the structure of the proof-search tree is almost completely ignored: only levels are used to prevent maximal confluent sets from reaching over leveling rules. We use the completeness of the (infinite) proof search to demonstrate the properties of maximal confluent trees, there is no direct translation of rule applications in the proof search to the accessibility relation in the model.

Lemma 5.18 (Input formula preservation) *If* $A^\bullet \in [\gamma]_{\mathfrak{G}}$ *and* $[\gamma]_{\mathfrak{G}} \leq [\delta]_{\mathfrak{H}}$*, then* $A^\bullet \in [\delta]_{\mathfrak{H}}$.

Proof The statement for $\leq$ follows from that for $\leq_0$. Assume $A^\bullet \in [\gamma]_{\mathfrak{G}}$ and $[\gamma]_{\mathfrak{G}} \leq_0 [\delta]_{\mathfrak{H}}$. Then

1. $A^\bullet \in \gamma_\Pi$ for some $\Pi \in \mathfrak{G}$ and
2. for some conclusion $\Gamma \in \mathfrak{G}$ and premise $\Delta \in \mathfrak{H}$ of a leveling-rule instance $\mathsf{prem_r}$ with $\mathsf{prem_r}(\mathfrak{G}) \subseteq \mathfrak{H}$ we have $\gamma_\Gamma \approx \delta_\Delta$.

By confluence of $\mathfrak{G}$, there is a superior Σ to both Π and Γ. We have $A^\bullet \in \gamma_\Pi \subseteq \gamma_\Sigma$ and $\mathsf{prem_r}(\Sigma) \in \mathsf{prem_r}(\mathfrak{G}) \subseteq \mathfrak{H}$. Thus, $A^\bullet \in \gamma_{\mathsf{prem_r}(\Sigma)}$. Since

$$\gamma_{\mathsf{prem_r}(\Sigma)} \approx \gamma_\Gamma \approx \delta_\Delta \approx \delta \ ,$$

it follows that $\gamma_{\mathsf{prem_r}(\Sigma)} \in [\delta]_{\mathfrak{H}}$ and $A^\bullet \in [\delta]_{\mathfrak{H}}$. □

Lemma 5.19 (Correctness) $\mathfrak{M}$ *is an* X*-model.*

Proof Clearly, $W \neq \varnothing$. By construction, R satisfies all requisite frame conditions other than seriality, and $\leq$ is a preorder.
Seriality. If seriality is explicitly required, then $\mathsf{d}^{[]}$ is a rule of $\mathsf{cNIKm+X'}$. By Lemma 5.14, for every $[\gamma]_{\mathfrak{G}} \in W$, where $\gamma \in \Gamma \in \mathfrak{G}$, we have that $\mathsf{prem_r}(\Gamma) \in \mathfrak{G}$ for the instance $\mathsf{prem_r}$ of $\mathsf{d}^{[]}$ applied to γ. Since $\gamma_{\mathsf{prem_r}(\Gamma)}$ must have a child δ in $tr(\mathsf{prem_r}(\Gamma))$, we have $[\gamma]_{\mathfrak{G}} R [\delta]_{\mathfrak{G}}$.
Monotonicity. To show monotonicity of V along $\leq$, assume $a \in V([\gamma]_{\mathfrak{G}})$ and $[\gamma]_{\mathfrak{G}} \leq [\delta]_{\mathfrak{H}}$. Then $a^\bullet \in [\gamma]_{\mathfrak{G}}$ and $a^\bullet \in [\delta]_{\mathfrak{H}}$ by Lemma 5.18. Hence, $a \in V([\delta]_{\mathfrak{H}})$.
(F1)–(F2). Since the proofs of these two properties are similar, we only show (F2). We first show (F2) for $\leq_0$ and R_0. Assume that

$$[\gamma]_{\mathfrak{G}} \leq_0 [\delta]_{\mathfrak{H}} \qquad \text{and} \qquad [\gamma]_{\mathfrak{G}} R_0 [\sigma]_{\mathfrak{G}}$$

for some $[\gamma]_{\mathfrak{G}}, [\sigma]_{\mathfrak{G}}, [\delta]_{\mathfrak{H}} \in W$. This means that:

1. for some conclusion $\Gamma \in \mathfrak{G}$ and premise $\Delta \in \mathfrak{H}$ of a leveling-rule instance $\mathsf{prem_r}$ with $\mathsf{prem_r}(\mathfrak{G}) \subseteq \mathfrak{H}$ we have $\gamma_\Gamma \approx \delta_\Delta$ and
2. for some $\Pi \in \mathfrak{G}$, the node γ_Π is the parent of σ_Π in $tr(\Pi)$.

By confluence of $\mathfrak{G}$, there is a superior Σ to both Π and Γ. In $tr(\Sigma)$, the node γ_Σ is the parent of σ_Σ. Further $\mathsf{prem_r}(\Sigma) \in \mathsf{prem_r}(\mathfrak{G}) \subseteq \mathfrak{H}$ and $\gamma_{\mathsf{prem_r}(\Sigma)}$ is the parent of $\sigma_{\mathsf{prem_r}(\Sigma)}$ in $tr(\mathsf{prem_r}(\Sigma))$. Since $\sigma_{\mathsf{prem_r}(\Sigma)} \approx \sigma_\Sigma \approx \sigma$,

$$[\sigma]_{\mathfrak{G}} \leq_0 [\sigma_{\mathsf{prem_r}(\Sigma)}]_{\mathfrak{H}} \ .$$

Since $\gamma_{\mathsf{prem_r}(\Sigma)} \approx \gamma_\Gamma \approx \delta_\Delta \approx \delta$, it follows that

$$[\delta]_{\mathfrak{H}} R_0 [\sigma_{\mathsf{prem_r}(\Sigma)}]_{\mathfrak{H}} \ .$$

Extending (F2) to $\leq$ and R_0 is straightforward.

It remains to note that, if (F1)–(F2) hold for $\leq$ and R_0, then they hold for $\leq$ and R, which is the closure of R_0 with respect to the frame properties of X . This is proved by induction on the length of derivation of an R-link from the R_0-links. (Recall that no seriality closure is performed.) We show only (F1) for the Euclidean closure: assuming by IH that the R-links

$$[\gamma]_{\mathfrak{G}} R [\delta]_{\mathfrak{G}} \qquad \text{and} \qquad [\gamma]_{\mathfrak{G}} R [\sigma]_{\mathfrak{G}}$$

satisfy both (F1)–(F2), we show that

$$[\delta]_{\mathfrak{G}} R [\sigma]_{\mathfrak{G}}$$

satisfies (F1). Let $[\delta]_{\mathfrak{G}} \leq [\delta']_{\mathfrak{H}}$. By (F1) for $[\gamma]_{\mathfrak{G}} R [\delta]_{\mathfrak{G}}$, there is $[\gamma']_{\mathfrak{H}}$ such that

$$[\gamma']_{\mathfrak{H}} R [\delta']_{\mathfrak{H}} \qquad \text{and} \qquad [\gamma]_{\mathfrak{G}} \leq [\gamma']_{\mathfrak{H}} \ .$$

By (F2) for $[\gamma]_{\mathfrak{G}} R [\sigma]_{\mathfrak{G}}$, there is $[\sigma']_{\mathfrak{H}}$ such that

$$[\gamma']_{\mathfrak{H}} R [\sigma']_{\mathfrak{H}} \qquad \text{and} \qquad [\sigma]_{\mathfrak{G}} \leq [\sigma']_{\mathfrak{H}} \ .$$

Finally, we have $[\delta']_{\mathfrak{H}} R [\sigma']_{\mathfrak{H}}$ by Euclideanity. The cases for transitivity, reflexivity, and symmetry are similar. □

Lemma 5.20 (Modal saturation) *Assume that $[\gamma]_{\mathfrak{G}} R [\delta]_{\mathfrak{G}}$ in the constructed model $\mathfrak{M}$.*

1. *If $\Box A^\bullet \in [\gamma]_{\mathfrak{G}}$, then $A^\bullet \in [\delta]_{\mathfrak{G}}$. In addition, if R was obtained by applying (among others) transitive closure to R_0, then $\Box A^\bullet \in [\delta]_{\mathfrak{G}}$.*
2. *If $\Diamond A^\circ \in [\gamma]_{\mathfrak{G}}$, then $A^\circ \in [\delta]_{\mathfrak{G}}$. In addition, if R was obtained by applying (among others) transitive closure to R_0, then $\Diamond A^\circ \in [\delta]_{\mathfrak{G}}$. (transitive case)*

Both additional statements also hold when the closure includes Euclideanity and $[\gamma]_{\mathfrak{G}}$ is not the root of $\hat{\mathfrak{G}}$. (Euclidean case)

Proof The last claim is a direct consequence of the closure of $\mathfrak{G}$ w.r.t. $5_c^\bullet$ and 5_c°. The other claims are proved by induction on the length of a *minimal* derivation of an R-link from the R_0-links. For the remaining statements, consider $\Box A^\bullet$ because $\Diamond A^\circ$ is completely analogous.

R_0-links. For R_0-links, $A^\bullet \in [\delta]_{\mathfrak{G}}$ follows from the closure of maximal confluent sets w.r.t. $\Box_c^\bullet$. For transitive logics, additionally $\Box A^\bullet \in [\delta]_{\mathfrak{G}}$ because of $4_c^\bullet$.

Reflexive closure. If the link $[\gamma]_{\mathfrak{G}} R [\gamma]_{\mathfrak{G}}$ is obtained by reflexivity, then $\Box A^\bullet \in [\gamma]_{\mathfrak{G}}$ trivially and $A^\bullet \in [\gamma]_{\mathfrak{G}}$ because of $t_c^\bullet$.
Symmetric closure. If the link $[\gamma]_{\mathfrak{G}} R [\delta]_{\mathfrak{G}}$ is obtained by symmetry from $[\delta]_{\mathfrak{G}} R [\gamma]_{\mathfrak{G}}$, then by minimality it is neither an R_0-link nor a reflexive loop.

- In the absence of transitivity closure, $[\delta]_{\mathfrak{G}} R_0 [\gamma]_{\mathfrak{G}}$ by minimality: if $[\delta]_{\mathfrak{G}} R [\gamma]_{\mathfrak{G}}$ were added by Euclideanity from $[\sigma]_{\mathfrak{G}} R [\delta]_{\mathfrak{G}}$ and $[\sigma]_{\mathfrak{G}} R [\gamma]_{\mathfrak{G}}$, then adding $[\gamma]_{\mathfrak{G}} R [\delta]_{\mathfrak{G}}$ instead would have been shorter. Thus, $A^\bullet \in [\delta]_{\mathfrak{G}}$ because of $b_c^\bullet$.
- In the transitive case, $5 \in X$ by 45-closure. If neither $[\gamma]_{\mathfrak{G}}$ nor $[\delta]_{\mathfrak{G}}$ is the root of $\hat{\mathfrak{G}}$, then $\Box A^\bullet$ is in $[\delta]_{\mathfrak{G}}$ and its parent by $5_c^\bullet$ and $A^\bullet \in [\delta]_{\mathfrak{G}}$ by $\Box_c^\bullet$. If $[\gamma]_{\mathfrak{G}}$ is the root, both $A^\bullet$ and $\Box A^\bullet$ belong to all nodes of $\hat{\mathfrak{G}}$ by $\Box_c^\bullet$, $4_c^\bullet$, and $b_c^\bullet$ (for $A^\bullet \in [\gamma]_{\mathfrak{G}}$). If $[\delta]_{\mathfrak{G}}$ but not $[\gamma]_{\mathfrak{G}}$ is the root, then $\Box A^\bullet$ is in $[\delta]_{\mathfrak{G}}$ and all its children, which exist, by $5_c^\bullet$ and $A^\bullet$ is in $[\delta]_{\mathfrak{G}}$ by $b_c^\bullet$.

Transitive closure. If the link $[\gamma]_{\mathfrak{G}} R [\delta]_{\mathfrak{G}}$ is obtained by transitivity from $[\gamma]_{\mathfrak{G}} R [\sigma]_{\mathfrak{G}}$ and $[\sigma]_{\mathfrak{G}} R [\delta]_{\mathfrak{G}}$, then both of them have shorter derivations and, by IH, $A^\bullet, \Box A^\bullet \in [\sigma]_{\mathfrak{G}}$. Hence, by IH, both $A^\bullet, \Box A^\bullet \in [\delta]_{\mathfrak{G}}$.

Euclidean closure. Assume the link $[\gamma]_{\mathfrak{G}} R [\delta]_{\mathfrak{G}}$ is obtained by Euclideanity from $[\sigma]_{\mathfrak{G}} R [\gamma]_{\mathfrak{G}}$ and $[\sigma]_{\mathfrak{G}} R [\delta]_{\mathfrak{G}}$. It is sufficient to show that $\Box A^\bullet$ is present in all nodes of $\hat{\mathfrak{G}}$, including $[\sigma]_{\mathfrak{G}}$ and $[\delta]_{\mathfrak{G}}$, from which the main statement follows by IH from $[\sigma]_{\mathfrak{G}} R [\delta]_{\mathfrak{G}}$:

- If $[\gamma]_{\mathfrak{G}}$ is not the root of $\hat{\mathfrak{G}}$, then $\Box A^\bullet$ is in all nodes by $5_c^\bullet$.
- If $[\gamma]_{\mathfrak{G}}$ is the root, then we claim that transitivity must also hold. Otherwise, by 45-closure of X, none of reflexive, symmetric or transitive closure would apply to R and Euclidean closure alone would not have added any incoming links into the root $[\gamma]_{\mathfrak{G}}$. This means that transitive closure has also been applied. Therefore, from $\Box A^\bullet \in [\gamma]_{\mathfrak{G}}$ it immediately follows by $4_c^\bullet$ that $\Box A^\bullet$ is present in all nodes.

□

Lemma 5.21 (Truth Lemma) *If $C^\bullet \in [\gamma]_{\mathfrak{G}}$, then $[\gamma]_{\mathfrak{G}} \models C$; if $C^\circ \in [\gamma]_{\mathfrak{G}}$, then $[\gamma]_{\mathfrak{G}} \not\models C$.*

Proof The proof is reasonably standard and relies on Lemma 5.20 for input □'s and output ◊'s, as well as on Lemma 5.18 for input ⊃'s and □'s. The cases for $C^\bullet = a^\bullet$, $C^\bullet = \bot^\bullet$, and $C^\circ = \bot^\circ$ are trivial.
Case $C^\circ = a^\circ$. If $a^\circ \in [\gamma]_{\mathfrak{G}}$, then $a^\bullet \notin [\gamma]_{\mathfrak{G}}$. Indeed, if $a^\bullet \in \gamma_\Pi$ and $a^\circ \in \gamma_\Delta$ for some $\Pi, \Delta \in \mathfrak{G}$, then any superior Σ of Π and Δ, would be derivable due to both $a^\bullet, a^\circ \in \gamma_\Sigma$, whereas the confluent $\mathfrak{G}$ must contain a non-derivable superior of Π and Δ.

Case $C^\bullet = A \wedge B^\bullet$. If $A \wedge B^\bullet \in [\gamma]_{\mathfrak{G}}$, then both $A^\bullet, B^\bullet \in [\gamma]_{\mathfrak{G}}$ by Lemma 5.14. Thus, $[\gamma]_{\mathfrak{G}} \vDash A$ and $[\gamma]_{\mathfrak{G}} \vDash B$ by IH, making $[\gamma]_{\mathfrak{G}} \vDash A \wedge B$.
Case $C^\circ = A \vee B^\circ$ is similar.

Case $C^\circ = A \wedge B^\circ$. If $A \wedge B^\circ \in [\gamma]_{\mathfrak{G}}$, then either $A^\circ \in [\gamma]_{\mathfrak{G}}$ or $B^\circ \in [\gamma]_{\mathfrak{G}}$ by Lemma 5.14. Thus, either $[\gamma]_{\mathfrak{G}} \nvDash A$ or $[\gamma]_{\mathfrak{G}} \nvDash B$ by IH, making $[\gamma]_{\mathfrak{G}} \nvDash A \wedge B$.
Case $C^\bullet = A \vee B^\bullet$ is similar.
Case $C^\circ = A \supset B^\circ$. If $A \supset B^\circ \in [\gamma]_{\mathfrak{G}}$, then there is a maximal confluent set $\mathfrak{H} \supseteq \mathsf{prem}_{\mathsf{r}}(\mathfrak{G})$ with $[\gamma_{\mathsf{prem}_{\mathsf{r}}(\Gamma)}]_{\mathfrak{H}} \geq_0 [\gamma]_{\mathfrak{G}}$ for the application r of $\supset^\circ$ to $A \supset B^\circ$ in some γ_Γ for some $\Gamma \in \mathfrak{G}$. In that case, $A^\bullet, B^\circ \in [\gamma_{\mathsf{prem}_{\mathsf{r}}(\Gamma)}]_{\mathfrak{H}}$. Thus, by IH $[\gamma_{\mathsf{prem}_{\mathsf{r}}(\Gamma)}]_{\mathfrak{H}} \vDash A$ and $[\gamma_{\mathsf{prem}_{\mathsf{r}}(\Gamma)}]_{\mathfrak{H}} \nvDash B$ making $[\gamma]_{\mathfrak{G}} \nvDash A \supset B$.
Case $C^\bullet = A \supset B^\bullet$. Let $A \supset B^\bullet \in [\gamma]_{\mathfrak{G}}$ and $[\delta]_{\mathfrak{H}} \geq [\gamma]_{\mathfrak{G}}$. By monotonicity of input formulas (Lemma 5.18), $A \supset B^\bullet \in [\delta]_{\mathfrak{H}}$. By Lemma 5.14, either $A^\circ \in [\delta]_{\mathfrak{H}}$ or $B^\bullet \in [\delta]_{\mathfrak{H}}$. Thus, for any $[\delta]_{\mathfrak{H}} \geq [\gamma]_{\mathfrak{G}}$, we have by IH that either $[\delta]_{\mathfrak{H}} \nvDash A$ or $[\delta]_{\mathfrak{H}} \vDash B$. Thus, $[\gamma]_{\mathfrak{G}} \vDash A \supset B$.
Case $C^\bullet = \Diamond A^\bullet$. If $\Diamond A^\bullet \in [\gamma]_{\mathfrak{G}}$, then by Lemma 5.14, there is another sequent $\Delta \in \mathfrak{G}$ with $A^\bullet \in \delta$ for the child δ of γ_Δ in $tr(\Delta)$. Thus, $[\delta]_{\mathfrak{G}} \vDash A$ by IH. Since $[\gamma]_{\mathfrak{G}} R_0 [\delta]_{\mathfrak{G}}$, we have $[\gamma]_{\mathfrak{G}} \vDash \Diamond A$.
Case $C^\circ = \Diamond A^\circ$. Let $\Diamond A^\circ \in [\gamma]_{\mathfrak{G}}$ and $[\gamma]_{\mathfrak{G}} R [\delta]_{\mathfrak{G}}$. Then, by Lemma 5.20, we have $A^\circ \in [\delta]_{\mathfrak{G}}$. Thus, by IH $[\delta]_{\mathfrak{G}} \nvDash A$ whenever $[\gamma]_{\mathfrak{G}} R [\delta]_{\mathfrak{G}}$. We have shown that $[\gamma]_{\mathfrak{G}} \nvDash \Diamond A$.
Case $C^\circ = \Box A^\circ$. If $\Box A^\circ \in [\gamma]_{\mathfrak{G}}$, then there is a maximal confluent set $\mathfrak{H} \supseteq \mathsf{prem}_{\mathsf{r}}(\mathfrak{G})$ with $[\gamma_{\mathsf{prem}_{\mathsf{r}}(\Gamma)}]_{\mathfrak{H}} \geq [\gamma]_{\mathfrak{G}}$ for the application r of $\Box^\circ$ to $\Box A^\circ$ in some γ_Γ for some $\Gamma \in \mathfrak{G}$. In that case, $A^\circ \in [\delta]_{\mathfrak{H}}$ for the child δ of $\gamma_{\mathsf{prem}_{\mathsf{r}}(\Gamma)}$ in $tr(\mathsf{prem}_{\mathsf{r}}(\Gamma))$. Thus, by IH $[\delta]_{\mathfrak{H}} \nvDash A$. Since $[\gamma]_{\mathfrak{G}} \leq_0 [\gamma_{\mathsf{prem}_{\mathsf{r}}(\Gamma)}]_{\mathfrak{H}} R_0 [\delta]_{\mathfrak{H}}$, we have $[\gamma]_{\mathfrak{G}} \nvDash \Box A$.
Case $C^\bullet = \Box A^\bullet$ combines the monotonicity argument for input implications with the use of Lemma 5.20 for output diamonds. □

We can now complete the proof of Theorem 5.3.

Proof of Theorem 5.3 By Lemma 5.12, the endsequent Υ belongs to some maximal confluent set $\mathfrak{G}$. The map $f\colon \gamma \mapsto [\gamma]_{\mathfrak{G}}$ embeds $tr(\Upsilon)$ into $\mathfrak{M}$. By the Truth Lemma 5.21, this map refutes the endsequent. □

Corollary 5.22 (Cut admissibility) *For a 45-closed* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*, the rule* $\mathsf{cut_m}$ *is admissible in* NIKm+X.

Proof If NIKm+X + $\mathsf{cut_m} \vdash \Gamma$, then Γ is X-valid by Soundness Theorem 4.6 and the obvious fact that $\mathsf{cut_m}$ preserves validity. Thus, NIKm+X $\vdash \Gamma$ by Completeness Theorem 5.3.

6 Conclusion

In this paper we have presented a multiple-conclusion calculus for all intuitionistic modal logics in the intuitionistic S5-cube, using nested sequents. The observation made by Egly and Schmitt [8], that multiple conclusion calculi for intuitionistic logic can provide exponentially shorter proofs than single-conclusion calculi, does also apply to our case, which makes our calculi interesting for possible applications in proof search.

This raises the question whether we can obtain a focused variant for the multiple-conclusion calculus, in the same way as for the single-conclusion calculus in [16]. The answer is not as easy as one might expect: due to the non-invertibility of the rules $\supset^{\circ}_{\mathsf{m}}$ and $\Box^{\circ}_{\mathsf{m}}$, we have to make the connectives $\supset$ and $\Box$ positive. But in a focused system also $\Diamond$ has to be positive. On the one hand, due to the absence of De Morgan duality, we certainly can make both modalities positive. But on the other hand, this is against the "spirit of focusing" which demands to make as much as possible negative—the more connectives are negative, the less choices we have to make and the less backtracking is needed. Having a "focused system" in which every connective is positive is trivial and not interesting.

However, there is something more to say about $\Diamond$. It can be seen as "morally negative" because when the $\Diamond^{\circ}$ and X°-rules are applicable, they can be applied such that no backtracking is needed (using contraction and the multiple conclusion setting). But this is not "negative" in the sense of focusing: we cannot dispose of $\Diamond$ after the rule application because we might have to wait for an instance of $\Box$ to unfold first. This is a topic of ongoing research.

It remains an open problem whether the multi-conclusion calculi have a formula interpretation.

It would also be useful to remove the condition of 45-closure from the completeness proof, in the style of [15].

The semantic proof of completeness required a novel method of model construction. However, both the original proof-search tree and the constructed model are generally infinite. It is an interesting task to attempt to finitize the construction.

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