Erratum to: Square Functions and Vector-Valued Calderón-Zygmund Operators

Jarod Hart

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The purpose of this note is to acknowledge and correct an error in Remark 2.1 of [1]. The *condition set 1* does not necessarily imply *condition set 2* for all $y, y' \in \mathbb{R}^n$, but only when y and y' are close (respectively when z and z' are close). In this note, we actually strengthen Theorems 1.1–1.4 in [1] by weakening the hypotheses from *condition set 2* to a weaker replacement *condition set 2'*.

Define *condition set 2'* to be the following: There exists $N_2 > M$ and $0 < \gamma_2 \leq 1$ such that (2.5), (2.6), and (2.7) in [1] hold for all $k \in \mathbb{Z}$ and $x, y, y', z, z' \in \mathbb{R}^n$ whenever $|y - y'| < (2^{-k} + |x - y|)/2$ and $|z - z'| < (2^{-k} + |x - z|)/2$. After restricting the values of x, y, y', z, z' for which these estimates hold, *condition set 2'* is equivalent to *condition set 1*, and gives the appropriate replacement. We now replace *condition set 2* by *condition set 2'* in the hypotheses of Theorems 1.1–1.4 and Propositions 3.1 and 3.2, i.e. we require that the estimates hold only when y and y' (respectively z and z') are close. Moreover the appropriate modification in Theorem 1.4 to (1.14) is to restrict to the case where $|x - x'| \leq (2^{-k} + \max(|x - y|, |x - z|))/2$. The corresponding modifications to the arguments in [1] are standard and they only need the regularity estimate (2.6) when y and y' are close (respectively (2.7) when z and z' are close). We provide the details here.

Modifications to Remark 2.1 Assume *condition set 1* and define η , N_2 , and γ_2 in the same way as in the original work. Estimate $|\theta_t(x, y, z) - \theta_t(x, y', z)|$ by the triangle

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J. Hart (✉)

Department of Mathematics, University of Kansas, 1460 Jayhawk Blvd., Lawrence, KS 66044, USA
e-mail: jhart@math.ku.edu

inequality and estimate (2.2) for each term. Use the geometric mean of this estimate and (2.3) to bound

$$\begin{aligned} & |\theta_k(x, y, z) - \theta_k(x, y', z)| \\ & \lesssim \frac{2^{kn}(2^k|y - y'|)^{\gamma_2}}{(1 + 2^k|x - z|)^{N_2 + \gamma_2}} \left(\frac{2^{kn}}{(1 + 2^k|x - y|)^{N_2 + \gamma_2}} + \frac{2^{kn}}{(1 + 2^k|x - y'|)^{N_2 + \gamma_2}} \right). \quad (\text{A}) \end{aligned}$$

With the additional condition $|y - y'| \leq (2^{-k} + |x - y|)/2$, we have $1 + 2^k|x - y'| \geq (1 + 2^k|x - y|)/2$, and hence estimate (2.6) holds for such $x, y, y', z \in \mathbb{R}^n$. A similar argument verifies (2.7) when $|z - z'| \leq (2^{-k} + |x - z|)/2$.

Conversely assuming *condition set 2'*, we need only deal with the case that $|y - y'| > (2^{-k} + |x - y|)/2$. In particular this implies that $2^k|y - y'| > 1/2$. Then using (2.2), it follows that

$$|\theta_k(x, y, z) - \theta_k(x, y', z)| \lesssim 2^{2kn} \lesssim 2^{2kn}(2^k|y - y'|)^{\gamma_1}.$$

Again, a similar argument verifies (2.4).

Modification to Propositions 3.1 and 3.2 We verify that Proposition 3.1 still holds assuming *condition set 2'*. In the first computation in the proof of Proposition 3.1 where $k \leq j$, the bound holds if we restrict the integral in u to the set $|u - y| < (2^{-k} + \max(|x - u|, |x - y|))/2$. The necessary modification for where $|u - y| \geq (2^{-k} + \max(|x - u|, |x - y|))/2$ is made using the size condition on $\theta_k^{(0)}$: First bound $|\theta_k^{(0)}(x, u, v) - \theta_k^{(0)}(x, y, v)| \leq |\theta_k^{(0)}(x, u, v)| + |\theta_k^{(0)}(x, y, v)|$ using the size estimate (1.6) for each term, and bound $|\theta_j^{(1)}(u, y)|$ and $|\theta_j^{(2)}(v, z)|$ using (3.1). Then the bound can be reduced to the original computations by using that for $|u - y| > 2^{-k}$

$$\frac{2^{jn}}{(1 + 2^j|u - y|)^{N + \gamma}} \leq \frac{2^{jn}}{(1 + 2^j|u - y|)^N (2^j 2^{-k})^\gamma} = 2^{-\gamma|k - j|} \frac{2^{jn}}{(1 + 2^j|u - y|)^N}.$$

The same computations arise in the case that $j \leq k$ and in Proposition 3.2 with the weakened kernel conditions.

Modification to Theorem 1.4 We also verify that the weakened regularity conditions for θ_k are sufficient for θ_k to define an ℓ^2 -valued Calderón-Zygmund kernel. We modify the argument at the end of Sect. 5, proving that $\{\theta_k\}_{k \in \mathbb{Z}}$ satisfies (5.3) with $\mathcal{B} = \ell^2$. For the first term of (5.3), note that $|h| < \max(|x - y|, |x - z|)/2 \leq (\max(|x - y|, |x - z|) + 2^{-k})/2$, so we can apply (1.14) and hence the estimate holds by the argument in the [1]. For the regularity in y , we apply estimate (A) to $|\theta_k(x, y, z) - \theta_k(x, y + h, z)|$, and bound both terms from estimate (A) as in [1]. This yields that $\|\theta_k(x, y, z) - \theta_k(x, y + h, z)\|_{\mathcal{BL}}$ is bounded by a constant times

$$\begin{aligned}
& \sum_{k \in \mathbb{Z}} \frac{2^{2kn} (2^k |h|)^\gamma}{(1 + 2^k |x - y|)^N (1 + 2^k |x - z|)^N} + \frac{2^{2kn} (2^k |h|)^\gamma}{(1 + 2^k |x - (y + h)|)^N (1 + 2^k |x - z|)^N} \\
& \lesssim \frac{|h|^\gamma}{(|x - y| + |x - z|)^{n+\gamma}} + \frac{|h|^\gamma}{(|x - y - h| + |x - z|)^{n+\gamma}} \\
& \lesssim \frac{|h|^\gamma}{(|x - y| + |x - z|)^{n+\gamma}}
\end{aligned}$$

In the last inequality we use that $|x - y - h| + |x - z| \geq |x - y| + |h| + |x - z| \geq (|x - y| + |x - z|)/2$ since $|h| \leq \max(|x - y|, |x - z|)/2$.

Modification to Final Remarks The final remarks are correct as they are stated, but the regularity conditions on $\partial_y^\mu \theta_k$ and $\partial_z^\nu \theta_k$ can be replaced by weaker conditions in the same way as above.

References

1. Hart, J.: Bilinear square functions and vector-valued Calderón-Zygmund operators. *J. Fourier Anal. Appl.* **18**(6), 1291–1313 (2012)