On one-dimensional compressible Navier–Stokes equations for a reacting mixture in unbounded domains

Siran Li

Abstract. In this paper we consider the one-dimensional Navier–Stokes system for a heat-conducting, compressible reacting mixture which describes the dynamic combustion of fluids of mixed kinds on unbounded domains. This model has been discussed on bounded domains by Chen (SIAM J Math Anal 23:609–634, 1992) and Chen–Hoff–Trivisa (Arch Ration Mech Anal 166:321–358, 2003), among others, in which the reaction rate function is a discontinuous function obeying the Arrhenius’ law of thermodynamics. We prove the global existence of weak solutions to this model on one-dimensional unbounded domains with large initial data in H^1 . Moreover, the large-time behaviour of the weak solution is identified. In particular, the uniform-in-time bounds for the temperature and specific volume have been established via energy estimates. For this purpose we utilise techniques developed by Kazhikhov–Shelukhin (cf. Kazhikhov in Siber Math J 23:44–49, 1982; Solonnikov and Kazhikhov in Annu Rev Fluid Mech 13:79–95, 1981) and refined by Jiang (Commun Math Phys 200:181–193, 1999, Proc R Soc Edinb Sect A 132:627–638, 2002), as well as a crucial estimate in the recent work by Li–Liang (Arch Ration Mech Anal 220:1195–1208, 2016). Several new estimates are also established, in order to treat the unbounded domain and the reacting terms.

Mathematics Subject Classification. Primary 35Q30, 35Q35, 35Q79; Secondary 76N10, 76N15.

Keywords. Navier–Stokes equations, Compressible, Reacting mixture, Combustion, Global existence, Uniform estimates, Large-time behaviour.

1. Introduction and main results

The equations of motion for the compressible fluids describing chemical reactions and radiative processes have been a central research topic in fluid dynamics, cf. [1–3, 5, 6, 10, 13, 15] and the references cited therein. In the current work we are concerned with the global existence and large-time behaviour of global solutions to the compressible Navier–Stokes equations for a reacting mixture on one-dimensional unbounded domains. Our system describes the physical process of dynamic combustion, for which the reacting rate function is discontinuous and obeys the *Arrhenius’ law* of molecular thermodynamics.

Following Chen [1], in which the explicit transform from the Euler to the Lagrangian coordinates has been computed, in this paper our analysis for the compressible Navier–Stokes equations will be carried out in the *Lagrangian* coordinates, i.e.,

$$u_t - v_x = 0, \quad (1.1)$$

$$v_t + \left(\frac{a\theta}{u} \right)_x = \left(\frac{\mu v_x}{u} \right)_x, \quad (1.2)$$

$$\left(\theta + \frac{v^2}{2} \right)_t + \left(\frac{av\theta}{u} \right)_x = \left(\frac{\mu v v_x + \kappa \theta_x}{u} \right)_x + qK\phi(\theta)Z, \quad (1.3)$$

$$Z_t + K\phi(\theta)Z = \left(\frac{d}{u^2} Z_x \right)_x. \quad (1.4)$$

In the above system we are solving for the four dynamic variables (u, v, θ, Z) , which represent the *specific volume, velocity, temperature, and mass fraction of the reactant*, respectively. The positive constants μ, κ, q, d, a , and K are the coefficients of bulk viscosity, heat conduction, species diffusion, difference in the internal energy of the reactant and the product, the product of Boltzmann's gas constant and the molecular weight, and the rate of reaction, respectively.

One distinctive feature of the above system (1.1)–(1.4) is the presence of $\phi(\theta)$, known as the *reaction rate function*. Here $\phi : \mathbb{R} \rightarrow [0, \infty)$ is a function of the temperature θ determined by the Arrhenius' law:

$$\phi(\theta) = \theta^\alpha e^{-\frac{A}{\theta}} \mathbb{1}_{\{\theta > \theta_{\text{ignite}}\}}, \quad (1.5)$$

where $\alpha, A > 0$ are thermodynamic constants and $\theta_{\text{ignite}} > 0$ is the threshold temperature which triggers the reaction. In particular, this function is discontinuous at θ_{ignite} . To deal with the reaction rate function ϕ , we first regularise it and derive uniform bounds for the resulting C^1 functions, and then pass to the limits to recover the discontinuous $\phi(\theta)$. Here we need the uniform boundedness of ϕ , which is justified *a posteriori* via the uniform bounds for the other dynamical variables, i.e., (u, v, Z) .

In this work we consider the Cauchy problem on the whole real line $\Omega = \mathbb{R}$. More precisely, the *initial data* are prescribed as follows:

$$(u, v, \theta, Z)|_{t=0} = (u_0, v_0, \theta_0, Z_0), \quad (1.6)$$

and the following *far-field condition* is imposed:

$$\lim_{|x| \rightarrow \infty} (u, v, \theta, Z)(x, t) = (1, 0, 1, 0) \quad \text{for all } t \geq 0. \quad (1.7)$$

Physically, it means that at the endpoints of the reacting system the density is constant (i.e., no formation of vacuum or density-concentration), and so is the temperature. Also, the endpoints are kept fixed for all the time, with no chemical reaction triggered there.

Moreover, the initial data are assumed to satisfy the following conditions:

$$\begin{cases} 0 < m_0 \leq u_0(x), \theta_0(x) \leq M_0 < \infty, & 0 \leq Z_0(x) \leq 1, \\ |v_0(x)| \leq M_0, \\ (u_0 - 1, v_0, \theta_0 - 1, Z_0) \in [H^1(\mathbb{R})]^4, & Z_0 \in L^1(\mathbb{R}), \end{cases} \quad (1.8)$$

where m_0, M_0 are uniform constants. The regularity condition in the last line is referred to as the *large data condition*.

Now, let us introduce the notion of weak solutions to the compressible Navier–Stokes system of the reacting mixture, which is our main object of study in this work:

Definition 1.1. The quadruplet $(u, v, \theta, Z) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^4$ is a **weak solution** to the system (1.1)–(1.8) if it satisfies the equations in the sense of distributions on $[0, T] \times \mathbb{R}$ and satisfies the following regularity conditions:

$$\begin{cases} u - 1 \in L^\infty(0, T; H^1(\mathbb{R})), \\ u_t \in L^2(0, T; L^2(\mathbb{R})), \\ v, \theta - 1, Z \in L^\infty(0, T; H^1(\mathbb{R})) \cap L^2(0, T; H^2(\mathbb{R})), \\ v_t, \theta_t, Z_t \in L^2(0, T; L^2(\mathbb{R})). \end{cases}$$

The main results of the paper are summarised as follows. First, we prove the global existence of weak solutions to Eqs. (1.1)–(1.8). Along the way *uniform bounds* (in space–time) for the temperature and specific volume are established:

Theorem 1.2. *There exists a weak solution (u, v, θ, Z) to Eqs. (1.1)–(1.8) on $[0, T] \times \mathbb{R}$ for all $T > 0$. Moreover, there is a uniform constant*

$$C_0 = C_0\left(a, \mu, \kappa, q, K, d, \phi(\cdot), \|(u_0 - 1, v_0, \theta_0 - 1, Z_0)\|_{H^1(\mathbb{R})}, \|Z_0\|_{L^1(\mathbb{R})}, m_0, M_0\right)$$

such that

$$0 < C_0^{-1} \leq \theta(t, x), u(t, x) \leq C_0 < \infty \quad \text{and} \quad 0 \leq Z(t, x) \leq 1 \quad (1.9)$$

for almost all $(t, x) \in [0, T] \times \mathbb{R}$.

The key point of Theorem 1.2 above is that C_0 is independent of T . Furthermore, the asymptotic states as $t \rightarrow \infty$, i.e., the *large-time behaviour* of the reacting mixture, can be fully determined:

Theorem 1.3. *Let (u, v, θ, Z) be a global weak solution to Eqs. (1.1)–(1.8). Then it converges in H^1 to the equilibrium state in the far field, i.e.,*

$$\left\| \left(u(t, \cdot) - 1, v(t, \cdot), \theta(t, \cdot) - 1, Z(t, \cdot) \right) \right\|_{H^1(\mathbb{R})} \longrightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (1.10)$$

The remaining parts of the paper are organised as follows:

In Sect. 2 we collect several auxiliary conserved quantities and monotonicity formulae for the reacting mixture, which will be used throughout the paper. We also introduce a regularisation of the system (1.1)–(1.8). In Sect. 3 we establish the upper and lower bounds for the specific volume u , adapting the methods by Kazhikhov–Shelukhin [11, 12] and Jiang [8, 9]. Next, in Sect. 4, following the arguments by Li–Liang [13] we derive uniform estimates involving v, θ and their first derivatives. Finally, in Sect. 5 we obtain upper and lower bounds for θ uniformly in space–time, together with the bounds for the higher derivatives of (u, v, θ, Z) , and thus conclude the proof of Theorems 1.2 and 1.3.

Before further development, we point out that the key estimate in this work, i.e., Theorem 4.1, essentially relies on the arguments in the recent paper [13] by Li and Liang, which in turn is motivated by the work of Huang–Li–Wang [7] on a blow-up criterion for compressible Euler equations. The new feature of our work lies in the physical process of dynamic combustions, i.e. the analysis of functions ϕ and Z , as well as the unboundedness of the spatial domains.

2. Regularisation, conserved quantity, and entropy formula

In this section we first introduce the *regularised system* of Eqs. (1.1)–(1.8): we replace the discontinuous function ϕ therein by $\phi^\delta \in C^\infty([0, \infty))$, which is required to satisfy

$$(\clubsuit) \quad \begin{cases} \|\phi^\delta\|_{C^1([0, \infty))} \leq \delta^{-1} < \infty; \\ \phi^\delta(\theta) = \phi(\theta) \quad \text{on } [0, \theta_{\text{ignite}} - \delta] \cup [\theta_{\text{ignite}} + \delta, \infty). \end{cases} \quad (2.1)$$

Then, as in Chen [1], by an application of the contraction mapping principle and Schauder estimates for the linearised parabolic equations, one can prove the local existence of strong solutions in the function space $B^{1+\alpha}(\mathbb{R} \times [0, T_0]) \times [B^{2+\alpha}(\mathbb{R} \times [0, T_0])]^3$ for some uniform constant $\alpha \in (0, 1)$, where for a space–time domain $Q \subset \mathbb{R} \times [0, \infty)$ we define

$$\begin{cases} B^{2+\alpha}(Q) = \{\varphi \in \mathcal{X}^\alpha(Q) : \varphi_t, \varphi_x, \varphi_{xx} \in \mathcal{X}^\alpha(Q)\}, & B^{1+\alpha}(Q) = \{\varphi \in \mathcal{X}^\alpha(Q) : \varphi_t, \varphi_x \in \mathcal{X}^\alpha(Q)\}, \\ \mathcal{X}^\alpha(Q) := \left\{ \varphi \in C^0(Q) : \sup_Q |\varphi| + \sup_{(t,x) \neq (s,y) \in Q} \frac{|\varphi(t,x) - \varphi(s,y)|}{|t-s|^{\alpha/2} + |x-y|^\alpha} < \infty \right\}. \end{cases} \quad (2.2)$$

Remark 2.1. From now on until the end of Sect. 5, we restrict ourselves to the regularised system (still labelled as Eqs. (1.1)–(1.8), with the superscript δ in ϕ^δ dropped), unless otherwise specified.

As the first step of our estimates, let us first verify that Z is indeed a ratio:

Lemma 2.2. *Let (u, v, θ, Z) be a strong solution on $[0, T] \times \mathbb{R}$ for the regularised system. Then $0 \leq Z(t, x) \leq 1$ on $[0, T] \times \mathbb{R}$. Moreover, for any $p \geq 1$, $\|Z(t, \cdot)\|_{L^p(\mathbb{R})}$ is a decreasing function.*

Proof. The proof for $Z \geq 0$ follows from the classical maximum principle: we set

$$Y(t, x) := e^{-\beta t} Z(t, x), \quad (2.3)$$

where $\beta > 0$ is a constant to be determined. Then, in view of Eq. (1.4),

$$Y_t + [\beta + K\phi(\theta)]Y = \left(\frac{d}{v^2} Y_x\right)_x. \quad (2.4)$$

Here the infimum of Y is attained on $\mathbb{R}$, thanks to the far-field condition $\lim_{|x| \rightarrow \infty} Y(\cdot, x) = 0$. Suppose there were $(t_0, x_0) \in [0, T] \times \mathbb{R}$ such that $Y(t_0, x_0) = \inf_{[0, T] \times \mathbb{R}} Y < 0$; then

$$Y_x(t_0, x_0) = 0; \quad Y_t(t_0, x_0) \leq 0; \quad Y_{xx}(t_0, x_0) \geq 0,$$

which contradicts Eq. (2.4). Thus, we get $Y \geq 0$, which is equivalent to $Z \geq 0$.

To prove the upper bound for Z , let us multiply pZ^{p-1} for $p \geq 1$ to obtain

$$\frac{d}{dt} \int_{\mathbb{R}} Z^p dx + \int_{\mathbb{R}} pK\phi(\theta)Z^p dx = - \int_{\mathbb{R}} \frac{dp(p-1)}{u^2} Z^{p-2} (Z_x)^2 dx. \quad (2.5)$$

As $Z \geq 0$, the L^p norm of Z is decreasing in time for all $p \in [1, \infty)$. Thus, using the initial condition $0 \leq Z_0 \leq 1$ and sending $p \rightarrow \infty$, we complete the proof. $\square$

An immediate corollary of the proof of Lemma 2.2 is the following Proposition 2.3. Notice that it also holds for the non-regularised system: to prove Proposition 2.3 we only need the non-negativity of $\phi(\theta)$, rather than any regularity properties.

Proposition 2.3. *Let (u, v, θ, Z) be a weak solution on $[0, T] \times \mathbb{R}$. Then there holds*

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}} Z(t, x) dx + \int_0^T \int_{\mathbb{R}} K\phi(\theta)Z dx dt \leq \int_{\mathbb{R}} Z_0(x) dx =: E_0 < \infty. \quad (2.6)$$

Proof. Send $p \rightarrow 1^+$ in Eq. (2.5) and use the dominated convergence theorem. $\square$

Let us remark that in Proposition 2.3 above we do not have the conservation of total mass or energy, as they may become unbounded. For instance, consider the reacting system of only one type of perfect gas. In this case the total energy is $\int_{\mathbb{R}} \left(\theta(t, x) + \frac{v(t, x)^2}{2} + qZ(t, x) \right) dx$. However, in view of the far-field condition (1.7), $\theta \equiv 1$ is expected to be a steady-state solution, which shall be verified later by the large-time behaviour (Theorem 1.3). Similarly, $u_0 \equiv 1$ gives us infinite total mass.

Now we establish an important *monotonicity formula*, which is interpreted as the entropy/energy formula for the reacting mixture, referred to as the “entropy inequality” or “entropy formula” in the sequel. In physics, the expressions $(u - 1 - \log(u))$ and $(\theta - 1 - \log(\theta))$ consist of the relative entropy, which obeys the *Clausius–Duhem inequality of thermodynamics*. We refer the readers to the appendix in [2] for a discussion on the relevant physical backgrounds.

Proposition 2.4. (Entropy Inequality) *Let (u, v, θ, Z) be a weak solution on $[0, T] \times \mathbb{R}$. Then the following inequality holds:*

$$\sup_{t \in [0, T]} \int_{\mathbb{R}} \left\{ a(u - 1 - \log(u)) + (\theta - 1 - \log(\theta)) + \frac{v^2}{2} \right\} dx + \int_0^T \int_{\mathbb{R}} \left\{ \frac{\mu v_x^2}{u\theta} + \frac{\kappa \theta_x^2}{u\theta^2} \right\} dx dt \leq C, \quad (2.7)$$

where the constant C depends only on $q, m_0, a, E_0, \|u_0 - 1\|_{L^2(\mathbb{R})}, \|\theta_0 - 1\|_{L^2(\mathbb{R})}$, and $\|v_0\|_{L^2(\mathbb{R})}$.

Proof. First we derive an alternative version of the evolution equation for the temperature. Substituting the mass and momentum equations (1.1), (1.2) into Eq. (1.3), one obtains:

$$\theta_t + a \frac{\theta}{u} v_x = \left(\kappa \frac{\theta_x}{u} \right)_x + \mu \frac{v_x^2}{u} + qK\phi(\theta)Z. \quad (2.8)$$

Now, let us multiply $a(1 - \frac{1}{u})$ to Eq. (1.1), v to Eq. (1.2), and $(1 - \frac{1}{\theta})$ to Eq. (2.8). Adding up the resulting expressions together, we deduce that

$$\begin{aligned} & \frac{\partial}{\partial t} \left[a(u - 1 - \log(u)) + (\theta - 1 - \log(\theta)) + \frac{v^2}{2} \right] + \frac{\mu v_x^2}{u\theta} + \frac{\kappa \theta_x^2}{u\theta^2} \\ &= \left(1 - \frac{1}{\theta} \right) qK\phi(\theta)Z + \frac{\partial}{\partial x} \left[\frac{\mu v v_x - a v \theta}{u} + \left(1 - \frac{1}{\theta} \right) \frac{\kappa \theta_x}{u} + a v \right]. \end{aligned} \quad (2.9)$$

Then, for the right-hand side, note that $\int_{\mathbb{R}} \frac{\partial}{\partial x} \left[\frac{\mu v v_x - a v \theta}{u} + \left(1 - \frac{1}{\theta} \right) \frac{\kappa \theta_x}{u} + a v \right] dx = 0$ due to the far-field condition (1.7). In light of Proposition 2.3, we then have

$$\int_0^T \int_{\mathbb{R}} \left(1 - \frac{1}{\theta} \right) qK\phi(\theta)Z \, dx dt \leq q \int_{\mathbb{R}} Z_0(x) \, dx \leq qE_0.$$

Hence, integrating Eq. (2.9) over $[0, T] \times \mathbb{R}$, we get

$$\begin{aligned} & \sup_{t \in [0, T]} \int_{\mathbb{R}} \left\{ a(u - 1 - \log(u)) + (\theta - 1 - \log(\theta)) + \frac{v^2}{2} \right\} dx + \int_0^T \int_{\mathbb{R}} \left\{ \frac{\mu v_x^2}{u\theta} + \frac{\kappa \theta_x^2}{u\theta^2} \right\} dx dt \\ & \leq qE_0 + \int_{\mathbb{R}} \left\{ a(u_0 - 1 - \log u_0) + (\theta_0 - 1 - \log \theta_0) + \frac{1}{2} v_0^2 \right\} dx. \end{aligned} \quad (2.10)$$

Now, by the condition on the initial data, $\int_{\mathbb{R}} v_0^2 dx$ is finite. In view of the assumption that $0 < m_0 \leq u_0(\cdot), \theta_0(\cdot) \leq M_0 < \infty$, it remains to establish the following *claim*: whenever $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $0 < m_0 \leq f(\cdot) \leq M_0 < \infty$, there holds

$$\left| \int_{\mathbb{R}} f(x) - 1 - \log f(x) \, dx \right| \leq C \int_{\mathbb{R}} |f(x) - 1|^2 \, dx. \quad (2.11)$$

Here C depends only on m_0 and M_0 .

Indeed, consider the auxiliary function

$$\Phi(s) := s - 1 - \log(s) - (s - 1)^2 \quad \text{on } [m_0, M_0],$$

whose Taylor expansion around 1 is

$$\begin{aligned} \Phi(s) &= \Phi(1) + \Phi'(1)(s - 1) + \frac{1}{2} \Phi''(1)(s - 1)^2 + \frac{1}{6} \Phi'''(\hat{s})(s - 1)^3 \\ &= -\frac{1}{2}(s - 1)^2 - \frac{1}{3\hat{s}^3}(s - 1)^3 \quad \text{for some } \hat{s} \text{ between 1 and } s. \end{aligned}$$

Thus, we have

$$\left| \int_{\mathbb{R}} \Phi(f(x)) \, dx \right| \leq \left(\frac{1}{2} + \frac{\max\{(m_0 - 1)^2, (M_0 - 1)^2\}}{3 \min\{1, m_0\}^3} \right) \int_{\mathbb{R}} |f(x) - 1|^2 \, dx, \quad (2.12)$$

from which the *claim* follows directly. This proves the entropy inequality. $\square$

As a remark, Proposition 2.4 is valid for both the regularised and non-regularised systems.

3. Uniform bounds for the specific volume u

In this section we establish the uniform (in space–time) upper and lower bounds for u . The proof is an adaptation of the classical argument by Kazhikhov and Shelukhin, cf. [11, 12] and the references cited therein. It relies on an explicit representation formula for u in terms of the other dynamical variables, which are in turn controlled by the entropy formula, i.e., Eq. (2.7).

Before stating and proving further results, let us first explain the notations and conventions adopted in the rest of the paper:

- We use C_i , $i \in \{0, 1, 2, 3, \dots\}$, to denote the positive constants depending only on the initial data and the fluid. More precisely,

$$0 < C_i = C_i\left(a, \mu, \kappa, q, K, d, \phi(\cdot), \|(u_0 - 1, v_0, \theta_0 - 1, Z_0)\|_{H^1(\mathbb{R})}, \inf_{\mathbb{R}} \phi, \sup_{\mathbb{R}} \phi, m_0, M_0\right).$$

It is crucial that the C_i 's are independent of the uniform norm of ϕ' .

- We denote by ϵ the generic small constants that appear in the estimates. They only depend on the constants of the fluid, unless otherwise specified.

The main result of this section is as follows:

Theorem 3.1. *Let (u, v, θ, Z) be a solution to system (1.1)–(1.8) on $[0, T] \times \mathbb{R}$. Then, there exists a uniform constant C_0 such that*

$$0 < C_0^{-1} \leq u(\cdot, \cdot) \leq C_0 < \infty \text{ on } [0, T] \times \mathbb{R}. \quad (3.1)$$

A key ingredient of the proof is the following “localisation trick” in [11, 12] by Kazhikhov–Shelukhin. For self-containedness we include the proof below:

Lemma 3.2. *There exist two universal constants γ_1, γ_2 such that*

$$0 < \gamma_1 \leq \int_{I_k} u(t, x) \, dx, \quad \int_{I_k} \theta(t, x) \, dx \leq \gamma_2 < \infty \quad (3.2)$$

for all $k \in \mathbb{Z}$ and $t > 0$; here, $I_k = [k, k + 1]$. Moreover, given any such t and k , we can find $b_k(t) \in I_k$ so that

$$0 < \gamma_1 \leq u(t, b_k(t)), \quad \theta(t, b_k(t)) \leq \gamma_2 < \infty. \quad (3.3)$$

Proof for Lemma 3.2. Let us denote

$$\psi(s) := s - 1 - \log(s), \quad (3.4)$$

which is a convex function on $[0, \infty)$. Then, on each space interval $I_k = [k, k + 1]$, $k \in \mathbb{Z}$, applying the entropy formula (2.7) and Jensen's inequality we deduce that

$$\begin{cases} \psi\left(\int_{I_k} u(t, x) \, dx\right) \leq \int_{I_k} \psi(u(t, x)) \, dx \leq C, \\ \psi\left(\int_{I_k} \theta(t, x) \, dx\right) \leq \int_{I_k} \psi(\theta(t, x)) \, dx \leq C, \end{cases}$$

where $C = C(q, m_0, M_0, a, E_0, \|u_0 - 1\|_{L^2(\mathbb{R})}, \|\theta_0 - 1\|_{L^2(\mathbb{R})}, \|v_0\|_{L^2(\mathbb{R})})$. Moreover, as ψ is monotonically decreasing from infinity to zero on $(0, 1]$ and monotonically increasing from zero to infinity on $[1, \infty)$, we can find two positive constants γ_1, γ_2 such that, for all $k \in \mathbb{Z}, t > 0$,

$$0 < \gamma_1 \leq \int_{I_k} u(t, x) \, dx, \quad \int_{I_k} \theta(t, x) \, dx \leq \gamma_2 < \infty.$$

This prove the first part of the lemma.

For the second part, we fix a small constant $\epsilon \in (0, 1/2)$. Then, we take any $t > 0$ and consider the “exceptional” set:

$$\mathfrak{S}_k(t) := \{x \in I_k : \theta(t, x) < \gamma_1 \text{ or } \theta(t, x) > \gamma_2 \text{ or } u(t, x) < \gamma_1 \text{ or } u(t, x) > \gamma_2\}. \quad (3.5)$$

By investigating the graph of ψ we note the following: on $\mathfrak{S}_k(t)$, either $\psi(\theta)$ or $a\psi(u)$ is greater than some large number $\tilde{K} = \tilde{K}(\gamma_1, \gamma_2) \geq 1$. Thus, employing Eq. (2.7) and the Chebyshev’s inequality, we deduce that

$$\tilde{K} |\mathfrak{S}_k(t)| \leq \sup_{0 \leq t < T} \int_{\mathbb{R}} [a\psi(u) + \psi(\theta)] dx \leq C, \quad (3.6)$$

where for a Borel set $B \subset \mathbb{R}$ its one-dimensional Lebesgue measure is denoted as $|B|$. Now, we observe that $\tilde{K}$ increases if either γ_2 increases or γ_1 decreases. Hence, by suitably choosing γ_1, γ_2 which depend only on a, q, E_0 , we can bound

$$|\mathfrak{S}_k(t)| \leq 1 - \epsilon \quad (3.7)$$

uniformly in time. Therefore, for each $t \in [0, T]$, let us pick an arbitrary $b_k(t) \in I_k \setminus \mathfrak{S}_k(t)$ to complete the proof.

With Lemma 3.2, we are at the stage of proving our main theorem in this section. The proof is a straightforward adaptation of the estimates in [8, 9] by Jiang. In fact, similar estimates have been obtained in [2, 11, 14] and several other works, but not uniformly in time. The crucial observation in [8, 9] is that, although $\int_0^t \frac{\theta(\tau, x)}{u(\tau, x)} d\tau$ is difficult to be bounded even at a single point $x = b_k(t)$, its spatial average $\int_s^t \int_{I_k} \frac{\theta(\tau, \xi)}{u(\tau, \xi)} d\xi d\tau$ can nevertheless be controlled (here $0 \leq s < t \leq T$). Throughout the proof, N denotes any uniform constant *independent of* t, x, k .

Proof for Theorem 3.1. The proof is divided into three steps:

1. First, we choose a spatial cut-off function $\chi \in C_c^\infty([0, \infty))$, $\chi \equiv 1$ on $[0, k]$, $\chi \equiv 0$ on $(k+1, \infty)$, and $0 \leq \|\chi\|_{C^1} \leq 1$. Testing against the momentum equation (1.2), one obtains:

$$-\int_x^\infty [v(t, \xi)\chi(\xi)]_t d\xi = \sigma(t, x) + \int_{I_k} \chi_x(\xi)\sigma(t, \xi) d\xi \quad \text{for all } x \in I_k. \quad (3.8)$$

Here, σ is the *effective viscous flux*, defined as

$$\sigma := \frac{\mu v_x - a\theta}{u}. \quad (3.9)$$

Starting with Eq. (3.8), the integration over $[0, t]$ gives us:

$$\begin{aligned} & \int_x^\infty (v(t, \xi) - v_0(\xi)) \chi(\xi) d\xi \\ &= \mu \log \frac{u(t, x)}{u_0(x)} - a \int_0^t \frac{\theta}{u}(\tau, x) d\tau + \int_0^t \int_{I_k} \chi_x(\xi) \sigma(\tau, \xi) d\xi d\tau. \end{aligned}$$

Then, we take the exponential of both sides to derive that

$$u(t, x) = u_0(x) \times \frac{\exp \left\{ \frac{1}{\mu} \int_x^\infty [v(t, \xi) - v_0(\xi)] \chi(\xi) d\xi \right\} \exp \left\{ \frac{a}{\mu} \int_0^t \frac{\theta(\tau, x)}{u(\tau, x)} d\tau \right\}}{\exp \left\{ \frac{1}{\mu} \int_0^t \int_{I_k} \chi_x(\xi) \sigma(\tau, \xi) d\xi d\tau \right\}}. \quad (3.10)$$

Now, introduce the following short-hand notations in the above expression:

$$\begin{cases} B(t, x) := u_0(x) \exp \left\{ \frac{1}{\mu} \int_x^\infty (v_0(\xi) - v(t, \xi)) \chi(\xi) d\xi \right\}, \\ Y(t) := \exp \left\{ \frac{1}{\mu} \int_0^t \int_{I_k} \chi_x(\xi) \sigma(\tau, \xi) d\xi d\tau \right\} = \exp \left\{ \frac{1}{\mu} \int_0^t \int_{I_k} \frac{\mu v_x(\tau, \xi) - a\theta(\tau, \xi)}{u(\tau, \xi)} \chi_x(\xi) d\xi d\tau \right\}. \end{cases} \quad (3.11)$$

Thus, we have

$$\frac{1}{Y(t)B(t, x)} = \frac{1}{u(t, x)} \exp \left\{ \frac{a}{\mu} \int_0^t \frac{\theta(\tau, x)}{u(\tau, x)} d\tau \right\}.$$

We multiply the above equation by $a\mu^{-1}\theta(t, x)$ and integrate over t to obtain:

$$\exp \left\{ \frac{a}{\mu} \int_0^t \frac{\theta(\tau, x)}{u(\tau, x)} d\tau \right\} = 1 + \frac{a}{\mu} \int_0^t \frac{\theta(\tau, x)}{Y(\tau)B(\tau, x)} d\tau.$$

This leads to an explicit representation formula for the specific volume, namely

$$u(t, x) = Y(t)B(t, x) + a\mu^{-1} \int_0^t \frac{Y(\tau)B(\tau, x)\theta(\tau, x)}{Y(\tau)B(\tau, x)} d\tau. \quad (3.12)$$

2. In this step we derive the uniform bounds for u , based on the above representation formula. First, since $\sup_{0 \leq t \leq T} \int_{\mathbb{R}} v^2(t, x) dx \leq C$ (which is an immediate consequence of the entropy formula, i.e., Eq. (2.7)), one concludes that

$$0 < N^{-1} \leq B(t, x) \leq N < \infty. \quad (3.13)$$

Next, for any $0 < s < t \leq T$, a lower bound can be derived for $\int_s^t \theta(\tau, x) d\tau$ on I_k *uniformly in* k . For this purpose, we first employ Jensen's inequality to estimate

$$\begin{aligned} \int_s^t \theta(\tau, x) d\tau &\geq (t-s) \exp \left\{ \int_s^t \frac{1}{t-s} \log(\theta) d\tau \right\} \\ &= (t-s) \exp \left\{ \frac{1}{t-s} \int_s^t \left[\int_{b_k(t)}^x \frac{\theta_x(\tau, \xi)}{\theta(\tau, \xi)} d\xi + \log \theta(\tau, b_k(t)) \right] d\tau \right\} \\ &\geq (t-s) \exp \left\{ N - \frac{1}{t-s} \left| \int_s^t \int_{b_k(t)}^x \frac{\theta_x}{\theta}(\tau, \xi) d\xi d\tau \right| \right\} \\ &\geq N(t-s) e^{-\frac{1}{N(t-s)}}, \end{aligned} \quad (3.14)$$

which holds in view of Eq. (2.7), $\int_{I_k} u(t, x) dx \leq \gamma_2$ (due to Lemma 3.2), as well as the concavity of log. Thus, we have:

$$\begin{aligned}
 \int_s^t \int_{I_k} \sigma(\tau, x) dx d\tau &\leq (-a + \epsilon) \int_s^t \int_{I_k} \frac{\theta(\tau, \xi)}{u(\tau, \xi)} d\xi d\tau + C(\epsilon) \int_s^t \int_{I_k} \frac{\mu u_x^2}{u\theta} d\xi d\tau \\
 &\leq N - \frac{a}{2} \int_s^t \int_{I_k} \frac{\theta(\tau, \xi)}{u(\tau, \xi)} d\xi d\tau \\
 &\leq N - N \int_s^t \inf_{I_k} \theta(\tau, \cdot) \left(\int_{I_k} \frac{1}{u(\tau, \xi)} d\xi \right) d\tau \\
 &\leq N - N \int_s^t \inf_{I_k} \theta(\tau, \cdot) \left(\int_{I_k} u(\tau, \xi) d\xi \right)^{-1} d\tau \leq N - N^{-1}(t - s), \quad (3.15)
 \end{aligned}$$

for which one utilises Jensen's inequality, the lower bound for $\int_s^t \theta d\tau$, Eq. (2.7) and Lemma 3.2. Hence, for arbitrary $0 \leq \tau \leq t$ the following holds:

$$0 \leq Y(t) \leq Ne^{-\frac{t}{N}}, \quad \frac{Y(t)}{Y(\tau)} \leq Ne^{-\frac{t-\tau}{N}}. \quad (3.16)$$

3. Using the bounds in Eqs. (3.13) and (3.16), the representation formula (3.12), and the localisation trick (Lemma 3.2), we now conclude that

$$\begin{cases} u(t, x) \leq N + N \int_0^t \theta(\tau, x) e^{-(t-\tau)/N} d\tau, \\ \gamma_1 \leq \int_{I_k} u(t, x) dx \leq Ne^{-t/N} + N \int_0^t Y(t)/Y(\tau) d\tau \end{cases} \quad \text{on } [0, \infty) \times I_k. \quad (3.17)$$

On the other hand, we have a reverse inequality which bounds θ in terms of u :

$$\begin{aligned}
 |\sqrt{\theta}(t, x) - \sqrt{\theta}(t, b_k(t))| &\leq \frac{1}{2} \int_{I_k} \frac{|\theta_x(t, x)|}{\sqrt{\theta}(t, x)} dx \\
 &\leq \frac{1}{2} \left(\int_{I_k} \frac{\theta_x^2}{u\theta^2}(t, x) dx \right)^{\frac{1}{2}} \left(\int_{I_k} u(t, x) \theta(t, x) dx \right)^{\frac{1}{2}} \\
 &\leq \frac{1}{2} \sqrt{\gamma_2} \left(\int_{I_k} \frac{\theta_x^2}{u\theta^2}(t, x) dx \right)^{\frac{1}{2}} \max_{I_k} \sqrt{u(t, \cdot)} \quad \text{on } [0, \infty) \times I_k, \quad (3.18)
 \end{aligned}$$

again due to Lemma 3.2. From here, we can quote verbatim the arguments in Jiang [8] from Eq. (2.20) to the end of Sect. 2 to complete the proof.

4. The crucial estimate for v and θ

In this section we establish a key estimate involving v, θ, v_x, θ_x , and suitable powers of them. This inequality is an adaptation of the key estimate in Li–Liang [13] (Lemma 2.2 therein). However, due to the presence of the chemical reaction processes, extra work needs to be done in order to control the variable Z .

Theorem 4.1. *Let (u, v, θ, Z) be a solution to system (1.1)–(1.8) on $[0, T] \times \mathbb{R}$. Then there exists $C_1 > 0$, depending only on the initial data, such that*

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}} [(\theta - 2)_+^2 + v^4](t, x) dx + \int_0^T \int_{\mathbb{R}} [(1 + \theta + v^2)v_x^2 + \theta_x^2](t, x) dx dt \leq C_1. \quad (4.1)$$

To simplify the presentation, let us collect several simple algebraic identities that shall be used repetitively in the subsequent development:

Lemma 4.2. *Let us denote the spatial level sets by*

$$\Sigma_a(t) := \{x \in \mathbb{R} : \theta(t, x) \geq a\}, \quad (4.2)$$

and write $\psi(s) = s - 1 - \log(s)$ on $\mathbb{R}_+$ as before. Then,

(1) *For any $a > 1$, there exists a universal constant $C = C(a, E_0, q, \|u_0 - 1, v_0, \theta_0 - 1\|_{L^2(\mathbb{R})})$, such that*

$$\sup_{0 \leq t < \infty} \int_{\Sigma_a(t)} \theta(t, x) dx \leq C \sup_{0 \leq t < \infty} \int_{\mathbb{R}} \psi(\theta(t, x)) dx \leq C. \quad (4.3)$$

(2) *For $a > 1$ there exists $C = C(a, E_0, q, \|u_0 - 1, v_0, \theta_0 - 1\|_{L^2(\mathbb{R})})$ such that*

$$\sup_{0 \leq t < \infty} \int_{\mathbb{R} \setminus \Sigma_a(t)} (\theta(t, x) - 1)^2 dx \leq C \sup_{0 \leq t < \infty} \int_{\mathbb{R}} \psi(\theta(t, x)) dx \leq C. \quad (4.4)$$

(3) *We have the algebraic inequalities (where $B > 0$ is a constant)*

$$\begin{cases} \theta^2 \chi_{\Sigma_2(t)} \leq 16(\theta - 3/2)_+^2, \\ \theta(\theta - 2)_+ \leq 2(\theta - 3/2)_+^2, \\ (\theta - 1)^2 \chi_{\Sigma_2(t)} \leq B(\theta - 3/2)_+^2. \end{cases} \quad (4.5)$$

(4) *For any $\psi \in H^1(\mathbb{R}) = W^{1,2}(\mathbb{R})$, we have*

$$\sup_{x \in \mathbb{R}} |\psi(x)|^2 \leq \|\psi'\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \leq \|\psi\|_{H^1(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})}. \quad (4.6)$$

Proof. (1)–(3) follow from straightforward algebraic computations; we omit the details here. Let us only comment that in (1), the following choice of constant

$$C(a) = \frac{a}{\psi(a)} = \frac{a}{a - 1 - \log(a)}$$

satisfies the requirement, as $\psi(s)$ has a double zero at 1; also, in (3) any $B > \frac{4}{3}$ works. Finally, (4) is the standard Sobolev inequality corresponding to the embedding $H^1(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$. $\square$

Proof for Theorem 4.1. We divide our arguments in four steps.

1. We start by deriving an energy estimate for the temperature equation, in the form of Eq. (2.8). The aim is to bound the L^2 norm of θ in the “high-temperature region”, in terms of other dynamical variables.

For this purpose let us multiply $(\theta - 2)_+$ to Eq. (2.8). This gives us

$$\begin{aligned} & (\theta - 2)_+ \theta_t + \kappa \frac{\theta_x}{u} \left[(\theta - 2)_+ \right]_x \\ &= \frac{1}{2} \left[(\theta - 2)_+^2 \right]_t + \kappa \frac{1}{2} \left(\frac{\theta_x^2}{u} \right)_x \mathbb{1}_{\Sigma_2(t)} \\ &= \left[\frac{\kappa(\theta - 2)_+ \theta_x}{u} \right]_x + \mu \frac{v_x^2(\theta - 2)_+}{u} - a \frac{\theta}{u} u_x (\theta - 2)_+ + qK\phi(\theta)Z(\theta - 2)_+. \end{aligned} \quad (4.7)$$

Noticing that $(\theta(t, x) - 2)_+ \rightarrow 0$ as $|x| \rightarrow \infty$, we integrate over $[0, T] \times \mathbb{R}$ to derive:

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}} [(\theta(T, x) - 2)_+]^2 dx + \int_0^T \int_{\Sigma_2(t)} \kappa \frac{\theta_x^2}{u} dx dt \\ &= \frac{1}{2} \int_{\mathbb{R}} [(\theta_0(x) - 2)_+]^2 dx + \int_0^T \int_{\mathbb{R}} \mu \frac{v_x^2}{u} (\theta - 2)_+ dx dt \\ &\quad - a \int_0^T \int_{\mathbb{R}} \frac{\theta v_x}{u} (\theta - 2)_+ dx dt + \int_0^T \int_{\mathbb{R}} qK\phi(\theta)Z(\theta - 2)_+ dx dt. \end{aligned}$$

On the other hand, multiplying $2v(\theta - 2)_+$ to Eq. (1.2) yields that

$$\begin{aligned} \left[v^2(\theta - 2)_+ \right]_t + 2\mu \frac{v_x^2}{u} (\theta - 2)_+ &= 2a \frac{\theta}{u} v_x (\theta - 2)_+ + 2a \frac{v\theta}{u} \left[(\theta - 2)_+ \right]_x - 2\mu \frac{v_x^2}{u} (\theta - 2)_+ - 2\mu \frac{vv_x}{u} \left[(\theta - 2)_+ \right]_x \\ &\quad + 2 \left[\frac{\mu vv_x (\theta - 2)_+ - av\theta (\theta - 2)_+}{u} \right]_x. \end{aligned} \quad (4.8)$$

Hence, integrating over $[0, T] \times \mathbb{R}$, we obtain

$$\begin{aligned} & \int_{\mathbb{R}} \left\{ v^2(\theta - 2)_+(T, x) \right\} dx + \int_0^T \int_{\mathbb{R}} \frac{2\mu(\theta - 2)_+}{u} v_x^2 dx dt \\ &= \int_{\mathbb{R}} \left\{ v_0^2(x) (\theta_0(x) - 2)_+ \right\} dx + 2a \int_0^T \int_{\mathbb{R}} \frac{\theta v_x}{u} (\theta - 2)_+ dx dt + 2a \int_0^T \int_{\Sigma_2(t)} \frac{v\theta\theta_x}{u} dx dt \\ &\quad - 2\mu \int_0^T \int_{\Sigma_2(t)} \frac{vv_x}{u} \theta_x dx dt + \int_0^T \int_{\Sigma_2(t)} v^2 [(\theta - 2)_+]_t dx dt. \end{aligned}$$

Adding the above two integral expressions together, evaluating $[(\theta - 2)_+]_t$ on the level set $\Sigma_2(t)$, and employing the evolution equation (2.8) for θ , we now arrive at:

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}} \left\{ (\theta - 2)_+^2 + v^2(\theta - 2)_+ \right\} (T, x) dx + \mu \int_0^T \int_{\mathbb{R}} \frac{(\theta - 2)_+}{u} v_x^2 dx dt + \kappa \int_0^T \int_{\Sigma_2(t)} \frac{\theta_x^2}{u} dx dt \\ &= \frac{1}{2} \int_{\mathbb{R}} \left\{ (\theta_0 - 2)_+^2 + v_0^2(\theta_0 - 2)_+ \right\} (x) dx + a \int_0^T \int_{\mathbb{R}} \left\{ \frac{\theta}{u} v_x (\theta - 2)_+ \right\} dx dt \end{aligned}$$

$$\begin{aligned}
& + 2a \int_0^T \int_{\Sigma_2(t)} \frac{v\theta\theta_x}{u} dxdt - 2\mu \int_0^T \int_{\Sigma_2(t)} \frac{vv_x}{u} \theta_x dxdt + \int_0^T \int_{\Sigma_2(t)} v^2 \left(\frac{\kappa\theta_x}{u} \right)_x dxdt \\
& + \int_0^T \int_{\Sigma_2(t)} \left\{ v^2 \frac{\mu v_x^2 - a\theta v_x}{u} \right\} dxdt + \int_0^T \int_{\mathbb{R}} \left\{ qK\phi(\theta)Z[(\theta-2)_+ + v^2] \right\} dxdt \\
& =: \frac{1}{2} \int_{\mathbb{R}} \left\{ (\theta_0 - 2)_+^2 + v_0^2(\theta_0 - 2) \right\}(x) dx + \sum_{j=1}^6 I_j.
\end{aligned} \tag{4.9}$$

2. Our task in the sequel is to estimate $I_1, I_2, \dots, I_6$ term by term. To this end, we use Young's inequality (or Cauchy–Schwarz) repeatedly to separate each I_j into “small” and “large” parts: the small part can be absorbed into the left-hand sides, and the large part can be controlled via the uniform bounds established in §2, and also the uniform boundedness of u (Theorem 3.1).

- For I_1 , using Eqs. (4.3) and (4.5), we estimate as follows:

$$\begin{aligned}
|I_1| & \leq \epsilon_1 \int_0^T \int_{\mathbb{R}} \frac{(\theta-2)_+}{u} v_x^2 dxdt + C(\epsilon_1) \int_0^T \int_{\mathbb{R}} \theta^2 (\theta-2)_+ dxdt \\
& \leq \epsilon_1 \int_0^T \int_{\mathbb{R}} \frac{(\theta-2)_+}{u} v_x^2 dxdt + C(\epsilon_1) \int_0^T \sup_{\mathbb{R}} \left[(\theta(t, \cdot) - \frac{3}{2})_+ \right]^2 dt.
\end{aligned} \tag{4.10}$$

- For I_2 , notice that

$$|I_2| \leq \epsilon_2 \int_0^T \int_{\Sigma_2(t)} \frac{\theta_x^2}{u} dxdt + C(\epsilon_2) \int_0^T \int_{\Sigma_2(t)} v^2 \theta^2 dxdt.$$

Again, we use Eqs. (4.5) and (2.7) to derive that

$$|I_2| \leq \epsilon_2 \int_0^T \int_{\Sigma_2(t)} \frac{\theta_x^2}{u} dxdt + C(\epsilon_2) \int_0^T \sup_{\mathbb{R}} \left[(\theta(t, \cdot) - \frac{3}{2})_+ \right]^2 dt. \tag{4.11}$$

- For I_3 , let us directly bound

$$|I_3| \leq \epsilon_3 \int_0^T \int_{\Sigma_2(t)} \frac{\theta_x^2}{u} dxdt + C(\epsilon_3) \int_0^T \int_{\Sigma_2(t)} v^2 v_x^2 dxdt. \tag{4.12}$$

- $I_4 := \int_0^T \int_{\Sigma_2(t)} v^2 \left(\frac{\kappa\theta_x}{u} \right)_x dxdt$ is a term with special structure. By a standard trick, we integrate against a test function $\varphi(\theta)$:

$$\begin{aligned}
\int_0^T \int_{\Sigma_2(t)} v^2 \varphi(\theta) \left[\frac{\kappa\theta_x}{u} \right]_x dxdt & = \int_0^T \int_{\Sigma_2(t)} \left[\frac{\kappa v^2 \varphi(\theta) \theta_x}{u} \right]_x dxdt \\
& - 2\kappa \int_0^T \int_{\Sigma_2(t)} \frac{vv_x \theta_x}{u} \varphi(\theta) dxdt - \kappa \int_0^T \int_{\Sigma_2(t)} \frac{v^2 \varphi'(\theta) \theta_x^2}{u} dxdt.
\end{aligned}$$

Hence, choosing a sequence of test functions $\varphi_\eta \in C^\infty[0, \infty)$ such that $\varphi_\eta(\theta) \equiv 0$ for $\theta \leq 2$, $\varphi_\eta(\theta) \equiv 1$ for $\theta \geq 2 + \eta$, and $\varphi'_\eta(\theta) \geq 0$, we immediately get

$$\begin{aligned} I_4 &= \lim_{\eta \searrow 0} \int_0^T \int_{\Sigma_2(t)} v^2 \varphi_\eta(\theta) \left[\frac{\kappa \theta_x}{u} \right]_x dx dt = \lim_{\eta \searrow 0} -2\kappa \int_0^T \int_{\Sigma_2(t)} \frac{v v_x \theta_x}{u} \varphi_\eta(\theta) dx dt \\ &\leq \epsilon_4 \int_0^T \int_{\Sigma_2(t)} \frac{\theta_x^2}{u} dx dt + C(\epsilon_4) \int_0^T \int_{\Sigma_2(t)} v^2 v_x^2 dx dt. \end{aligned} \quad (4.13)$$

- I_5 is simple: by Eqs. (2.7) and (4.5),

$$\begin{aligned} |I_5| &\leq C \int_0^T \int_{\Sigma_2(t)} (v^2 v_x^2 + v^2 \theta^2) dx dt \\ &\leq C \int_0^T \int_{\Sigma_2(t)} v^2 v_x^2 dx dt + C \int_0^T \sup_{\mathbb{R}} \left\{ \left(\theta(t, \cdot) - \frac{3}{2} \right)_+ \right\}^2 dt. \end{aligned} \quad (4.14)$$

- Finally, let us deal with I_6 , which is the term involving Z . In view of the boundedness of ϕ in the C^0 -topology, $\sup_{0 \leq t \leq T} \int_{\mathbb{R}} Z(t, x) dx \leq C$ (cf. Proposition 2.3), and that $(\theta - 2)_+ \leq (\theta - 3/2)_+^2$ (cf.

Lemma 4.2), we achieve the following:

$$\left| \int_0^T \int_{\mathbb{R}} qK\phi(\theta)Z(\theta - 2)_+ dx dt \right| \leq C \int_0^T \sup_{\mathbb{R}} \left\{ \left(\theta(t, \cdot) - \frac{3}{2} \right)_+ \right\}^2 dt. \quad (4.15)$$

On the other hand, by Eq. (2.7) and the identity in Eq. (4.6), we have:

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{R}} qK\phi(\theta)Zv^2 dx dt \right| &\leq C \int_0^T \|v(t, \cdot)\|_{L^2(\mathbb{R})} \|v_x(t, \cdot)\|_{L^2(\mathbb{R})} dt \\ &\leq C \int_0^T \int_{\mathbb{R}} v_x^2 dx dt. \end{aligned} \quad (4.16)$$

Thus,

$$|I_6| \leq C \int_0^T \int_{\Sigma_2(t)} v_x^2 dx dt + C \int_0^T \sup_{\mathbb{R}} \left\{ \left(\theta(t, \cdot) - \frac{3}{2} \right)_+ \right\}^2 dt. \quad (4.17)$$

Now we combine the previous estimates in Eqs. (4.10)–(4.15) to control the right-hand side of Eq. (4.9). Indeed, selecting $\epsilon_1 = \frac{1}{2}\mu$ and $\epsilon_2 = \epsilon_3 = \epsilon_4 = \frac{1}{4}\kappa$ proves the existence of a constant $C_2 > 0$, depending only on the initial data, $\mu, \kappa, a, q, K, \|\phi\|_{L^\infty}$ and C_0 in Theorem 3.1, so that for all $0 \leq t \leq T$ the following holds:

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}} \{(\theta - 2)_+^2 + v^2(\theta - 2)_+\} (t, x) dx + \mu \int_0^T \int_{\mathbb{R}} \frac{\theta}{u} v_x^2 dx dt + \kappa \int_0^T \int_{\Sigma_2(t)} \frac{\theta_x^2}{u} dx dt \\ &\leq C_2 + C_2 \int_0^T \sup_{\mathbb{R}} \left\{ \left(\theta(t, \cdot) - \frac{3}{2} \right)_+ \right\}^2 dt + C_2 \int_0^T \int_{\mathbb{R}} (1 + v^2) v_x^2 dx dt. \end{aligned} \quad (4.18)$$

3. In the third step we estimate the term

$$\int_0^T \sup_{\mathbb{R}} \left\{ \left(\theta(t, \cdot) - \frac{3}{2} \right)_+ \right\}^2 dt \quad (4.19)$$

on the right-hand side of Eq. (4.18). Indeed, (4.3) and the entropy formula (2.7) imply that

$$\begin{aligned} \int_0^T \sup_{\mathbb{R}} \left\{ \left(\theta(t, \cdot) - \frac{3}{2} \right)_+ \right\}^2 dt &\leq \int_0^T \sup_{x \in \mathbb{R}} \left\{ \int_x^\infty -\partial_x \left(\theta(t, \xi) - \frac{3}{2} \right)_+ d\xi \right\}^2 dt \\ &\leq \int_0^T \left(\int_{\Sigma_{3/2}(t)} |\theta_x| dx \right)^2 dt \\ &\leq \int_0^T \left(\int_{\Sigma_{3/2}(t)} \frac{|\theta_x|^2}{\theta} dx \right) \left(\int_{\Sigma_{3/2}(t)} \theta dx \right) dt \\ &\leq C \int_0^T \int_{\Sigma_{3/2}(t)} \frac{|\theta_x|^2}{\theta} dx dt \\ &\leq C(\epsilon_5) \int_0^T \int_{\mathbb{R}} \frac{|\theta_x|^2}{u \theta^2} dx dt + \epsilon_5 \int_0^T \int_{\Sigma_{3/2}(t)} |\theta_x|^2 dx dt, \end{aligned} \quad (4.20)$$

where Cauchy–Schwarz and the uniform boundedness of u are used in the last line. Moreover, observe that the first term on right-hand side can be bounded by Eq. (2.7), and by choosing $\epsilon_5 = \frac{\kappa}{4}$, the second term can be absorbed into the left-hand side of Eq. (4.18). Thus, there is a universal constant $C_3 > 0$ such that for all $0 \leq t \leq T$ we have:

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}} \{ (\theta - 2)_+^2 + v^2(\theta - 2)_+ \} (t, x) dx + \mu \int_0^T \int_{\mathbb{R}} \frac{\theta}{u} v_x^2 dx dt + \kappa \int_0^T \int_{\Sigma_2(t)} \frac{\theta_x^2}{u} dx dt \\ &\leq C_3 \left(1 + \int_0^T \int_{\mathbb{R}} (1 + v^2) v_x^2 dx dt \right). \end{aligned} \quad (4.21)$$

4. Finally, it remains to bound the right-hand side of Eq. (4.21). For this purpose, we multiply v^3 to the momentum equation (1.2) and investigate the evolution of the L^4 norm of v , as in Kazhikhov–Shelukhin [12]. In this manner we obtain:

$$\frac{1}{4} (v^4)_t + 3\mu \frac{v^2 v_x^2}{u} = \left(\frac{\mu v^3 v_x - a v^3 \theta}{u} \right)_x + 3a \frac{\theta}{u} v^2 v_x. \quad (4.22)$$

Hence, integrating over $[0, T] \times \mathbb{R}$, we find that

$$\sup_{0 \leq t \leq T} \frac{1}{4} \int_{\mathbb{R}} v^4(t, x) dx + 3\mu \int_0^T \int_{\mathbb{R}} \frac{v^2 v_x^2}{u} dx dt = \frac{1}{4} \int_{\mathbb{R}} v_0^4(x) dx + 3a \int_0^T \int_{\mathbb{R}} \frac{\theta}{u} v^2 v_x dx dt. \quad (4.23)$$

To estimate the last term on the right-hand side, one makes use of the following observation in [13]: $(u - 1)$ is square-integrable due to the boundedness of u and the integrability of $\psi(u) = u - 1 - \log u$ (cf. Theorem 3.1 and Lemma 4.2). Hence, let us consider

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} \frac{\theta}{u} v^2 v_x \, dx dt &= \int_0^T \int_{\mathbb{R}} \frac{(1-u)v^2 v_x}{u} \, dx dt + \int_0^T \int_{\Sigma_2(t)} \frac{(\theta-1)v^2 v_x}{u} \, dx dt \\ &\quad + \int_0^T \int_{\mathbb{R} \setminus \Sigma_2(t)} \frac{(\theta-1)v^2 v_x}{u} \, dx dt =: K_1 + K_2 + K_3, \end{aligned}$$

and estimate K_1, K_2, K_3 as follows:

- For K_1 , we bound

$$\begin{aligned} |K_1| &\leq C \int_0^T \left\{ \left(\sup_{\mathbb{R}} v^2(t, \cdot) \right) \|1 - u(t, \cdot)\|_{L^2(\mathbb{R})} \|v_x(t, \cdot)\|_{L^2(\mathbb{R})} \right\} dt \\ &\leq C \int_0^T \left\{ \|v(t, \cdot)\|_{L^2(\mathbb{R})} \|1 - u(t, \cdot)\|_{L^2(\mathbb{R})} \|v_x(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right\} dt \\ &\leq C \sup_{0 \leq t \leq T} \|v(t, \cdot)\|_{L^2(\mathbb{R})} \sup_{0 \leq t \leq T} \|1 - u(t, \cdot)\|_{L^2(\mathbb{R})} \int_0^T \int_{\mathbb{R}} v_x^2(t, x) \, dx dt \leq C \int_0^T \int_{\mathbb{R}} v_x^2(t, x) \, dx dt, \end{aligned}$$

thanks to items (1) and (4) in Lemma 4.2 and the entropy formula, namely Eq. (2.7). On the other hand, by Cauchy–Schwarz one has

$$\int_0^T \int_{\mathbb{R}} v_x^2 \, dx dt \leq \epsilon_6 \int_0^T \int_{\mathbb{R}} \theta v_x^2 \, dx dt + C(\epsilon_6) \int_0^T \int_{\mathbb{R}} \frac{v_x^2}{\theta u} \, dx dt;$$

hence,

$$|K_1| \leq \epsilon_6 \int_0^T \int_{\mathbb{R}} \theta v_x^2 \, dx dt + C(\epsilon_6). \quad (4.24)$$

- Similarly, to deal with K_2 , Lemma 4.2 gives us

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R} \setminus \Sigma_2(t)} (\theta - 1)^2 dx dt \leq C;$$

thus, one readily derives

$$|K_2| \leq \epsilon_7 \int_0^T \int_{\mathbb{R}} \theta v_x^2 \, dx dt + C(\epsilon_7) \quad (4.25)$$

via analogous arguments.

- Finally, K_3 is bounded as follows:

$$\begin{aligned}
|K_3| &\leq \epsilon_8 \int_0^T \int_{\Sigma_2(t)} v^2 v_x^2 \, dx \, dt + C(\epsilon_8) \left(\sup_{0 \leq t \leq T} \int_{\mathbb{R}} v^2 \, dx \right) \left(\int_0^T \sup_{\Sigma_2(t)} (\theta - 1)^2 \, dt \right) \\
&\leq \epsilon_8 \int_0^T \int_{\Sigma_2(t)} v^2 v_x^2 \, dx \, dt + C(\epsilon_8) \left(\sup_{0 \leq t \leq T} \int_{\mathbb{R}} v^2 \, dx \right) \left(\int_0^T \sup_{\mathbb{R}} \left[\theta(t, \cdot) - \frac{3}{2} \right]_+^2 \, dt \right) \\
&\leq \epsilon_8 \int_0^T \int_{\Sigma_2(t)} v^2 v_x^2 \, dx \, dt + \epsilon_9 \int_0^T \int_{\Sigma_{\frac{3}{2}}(t)} \theta_x^2(t, x) \, dx \, dt + C(\epsilon_8, \epsilon_9),
\end{aligned}$$

where in the final line one utilises Eq. (4.20).

Finally, we select $\epsilon_6, \epsilon_7, \epsilon_8, \epsilon_9$ so small that the corresponding terms get absorbed into the left-hand side of Eq. (4.21). The proof is completed by putting K_1, K_2, K_3 together. $\square$

5. Completion of the Proof of Theorems 1.2 and 1.3

With the above preparations, we finally arrive at the stage of proving the main results of the paper (Theorems 1.2 and 1.3), concerning the global existence and large-time behaviour of Eqs. (1.1)–(1.8).

This final section is organised as follows. First, let us derive some uniform bounds for the higher derivatives of (u, v, θ, Z) . As a by-product, the temperature θ is uniformly bounded from the above. Then, employing these bounds and investigating the limiting process $T \rightarrow \infty$, we are able to deduce the large-time behaviour, i.e., Theorem 1.3. Thus, the uniform lower bound for θ can be deduced, which agrees with the physical law that the absolute zero temperature cannot be reached. As both the upper and the lower bounds for θ are at hand, our local (in time) estimates can be extended globally. Finally, the global existence of weak solutions is derived as a corollary of the estimates aforementioned.

Lemma 5.1. *There exists a uniform constant C_5 such that the following estimate holds for the solutions on $[0, T) \times \mathbb{R}$, for any $T > 0$:*

$$\begin{aligned}
&\sup_{0 \leq t \leq T} \int_{\mathbb{R}} (u_x^2 + v_x^2 + \theta_x^2 + Z_x^2) \, dx \\
&\quad + \int_0^T \int_{\mathbb{R}} (\theta u_x^2 + u_{xt}^2 + v_{xx}^2 + \theta_{xx}^2 + Z_{xx}^2 + v_t^2 + \theta_t^2 + Z_t^2) \, dx \, dt \leq C_5.
\end{aligned} \tag{5.1}$$

Moreover, θ is uniformly bounded from above:

$$\sup_{[0, T] \times \mathbb{R}} \theta \leq C_5. \tag{5.2}$$

Proof. Before carrying out the estimates, we notice that the terms in Eq. (5.1) involving $u_{xt}^2, v_t^2, \theta_t^2, Z_t^2$ are bounded by the other terms in the same equation: this is an immediate consequence of Eqs. (1.1)–(1.4). Therefore, we only need to bound the *spatial* derivatives.

1. First of all, let us estimate the derivatives of u . Substituting the mass equation (1.1) in the momentum equation (1.2), one deduces that

$$v_t + a \left(\frac{\theta}{u} \right)_x = \mu (\log(u))_{tx}.$$

Then, multiplying $(\log(u))_x$ to both sides, we obtain

$$\frac{\mu}{2} \left((\log u)_x^2 \right)_t + a \frac{\theta u_x^2}{u^3} = \left(v(\log u)_t \right)_x + \frac{u_x \theta_x}{u^2} + \left(v \frac{u_x}{u} \right)_t - \frac{v_x^2}{u}. \quad (5.3)$$

In view of Theorem 4.1 and the entropy formula (2.7), we integrate over $[0, T] \times \mathbb{R}$ to get

$$\begin{aligned} & \sup_{0 \leq t \leq T} \int_{\mathbb{R}} \frac{u_x^2}{u^2} dx + \int_0^T \int_{\mathbb{R}} \theta u_x^2 dx dt \\ & \leq \epsilon_{10} \int_0^T \int_{\mathbb{R}} \theta u_x^2 dx dt + C(\epsilon_{10}) \int_0^T \int_{\mathbb{R}} \frac{\theta_x^2}{\theta^2} dx dt + C(\epsilon_{10}) \int_0^T \int_{\mathbb{R}} \theta^2 dx dt \\ & \quad + C(\epsilon_{10}) \int_0^T \int_{\mathbb{R}} \frac{v_x^2}{\theta} dx dt + \epsilon_{10} \sup_{0 \leq t \leq T} \int_{\mathbb{R}} \frac{u_x^2}{u^2} dx + C(\epsilon_{10}) \sup_{0 \leq t \leq T} \int_{\mathbb{R}} v^2 dx dt \\ & \leq \epsilon_{10} \sup_{0 \leq t \leq T} \int_{\mathbb{R}} \frac{u_x^2}{u^2} dx + \epsilon_{10} \int_0^T \int_{\mathbb{R}} \theta u_x^2 dx dt + C(\epsilon_{10}). \end{aligned}$$

So, by choosing suitably small ϵ_{10} , the above estimates give us

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}} u_x^2 dx + \int_0^T \int_{\mathbb{R}} \theta u_x^2 dx dt \leq C_6. \quad (5.4)$$

2. Now we estimate the derivatives of v by multiplying v_{xx} to the momentum equation (1.2). This gives us

$$\frac{1}{2} (v_x^2)_t + \frac{\mu}{u} (v_{xx})^2 = (v_x v_t)_x + \mu \frac{v_x u_x v_{xx}}{u^2} - a \frac{v_{xx} \theta_x}{u} + a \frac{\theta u_x v_{xx}}{u^2}, \quad (5.5)$$

from which we obtain that

$$\begin{aligned} & \sup_{0 \leq t \leq T} \int_{\mathbb{R}} v_x^2 dx + \int_0^T \int_{\mathbb{R}} v_{xx}^2 dx dt \\ & \leq \epsilon_{11} \int_0^T \int_{\mathbb{R}} v_{xx}^2 dx dt + C(\epsilon_{11}) \int_0^T \int_{\mathbb{R}} \frac{v_x^2}{\theta} dx dt \\ & \quad + 2C(\epsilon_{11}) \left\{ \sup_{[0, T] \times \mathbb{R}} \theta(\cdot, \cdot) \right\} \int_0^T \int_{\mathbb{R}} \theta u_x^2 dx dt + C(\epsilon_{11}) \int_0^T \int_{\mathbb{R}} \theta^2 dx dt. \end{aligned}$$

The last three terms on the right-hand side are bounded by the entropy formula (2.7), Theorem 4.1, and Eq. (5.4) in Step 1 of the same proof. Thus, choosing ϵ_{11} suitably small, we get

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}} v_x^2 dx + \int_0^T \int_{\mathbb{R}} v_{xx}^2 dx dt \leq C_7 \left(1 + \sup_{[0, T] \times \mathbb{R}} \theta \right). \quad (5.6)$$

3. Next, let us estimate the derivatives of Z , which is specific to our problem of the reacting mixture. We multiply Z_{xx} to Eq. (1.4) to get

$$\frac{(Z_x^2)_t}{2} + \frac{d}{u^2} Z_{xx}^2 = [(Z_t + K\phi(\theta)Z)Z_x]_x - 2d \frac{u_x Z Z_x Z_{xx}}{u^3}. \quad (5.7)$$

We recall that $0 \leq Z \leq 1$ (Lemma 2.2); so, thanks to the Sobolev inequality in Eq. (4.6), the following estimates are valid:

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \int_{\mathbb{R}} Z_x^2(t, x) dx + \int_0^T \int_{\mathbb{R}} Z_{xx}^2 dx dt \\
& \leq \epsilon_{12} \int_0^T \int_{\mathbb{R}} Z_{xx}^2 dx dt + C(\epsilon_{12}) \left\{ \sup_{0 \leq t \leq T} \int_{\mathbb{R}} |u_x|^2 dx \right\} \left\{ \sup_{x \in \mathbb{R}} \int_0^T |Z_x|^2 dt \right\} \\
& \leq \epsilon_{12} \int_0^T \int_{\mathbb{R}} Z_{xx}^2 dx dt + C(\epsilon_{12}) \int_0^T \|Z_x(t, \cdot)\|_{L^2(\mathbb{R})} \|Z_{xx}(t, \cdot)\|_{L^2(\mathbb{R})} dt \\
& \leq \epsilon_{12} \int_0^T \int_{\mathbb{R}} Z_{xx}^2 dx dt + \epsilon_{13} \int_0^T \int_{\mathbb{R}} Z_{xx}^2(t, x) dx + C(\epsilon_{12}, \epsilon_{13}) \int_0^T \int_{\mathbb{R}} Z_x^2(t, x) dx dt. \tag{5.8}
\end{aligned}$$

On the other hand, multiplying Z to Eq. (1.4) leads to:

$$\frac{1}{2}(Z^2)_t + K\phi(\theta)Z^2 + \frac{d}{u^2}(Z_x)^2 = \left(\frac{d}{u^2} Z Z_x \right)_x.$$

Thus, integrating over space-time, we obtain

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}} Z_x^2 dx dt & \leq \frac{1}{2} \sup_{0 \leq t \leq T} \int_{\mathbb{R}} Z^2(t, x) dx + \int_0^T \int_{\mathbb{R}} K\phi(\theta) Z^2(t, x) dx dt \\
& \leq \frac{1}{2} \sup_{0 \leq t \leq T} \int_{\mathbb{R}} Z(t, x) dx + \int_0^T \int_{\mathbb{R}} K\phi(\theta) Z(t, x) dx dt \leq E_0, \tag{5.9}
\end{aligned}$$

thanks to Lemma 2.2 and Proposition 2.3. This leads to the conclusion:

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}} Z_x^2 dx + \int_0^T \int_{\mathbb{R}} Z_{xx}^2 dx dt + \int_0^T \int_{\mathbb{R}} Z_x^2 dx dt \leq C_8. \tag{5.10}$$

4. In this step we establish the bounds for derivatives of θ . As before, multiplying θ_{xx} to the temperature equation (2.8) yields:

$$\frac{1}{2}(\theta_x^2)_t + \frac{\kappa}{u}\theta_{xx}^2 = (\theta_t\theta_x)_x + \kappa \frac{\theta_x u_x \theta_{xx}}{u^2} - qK\phi(\theta)Z_x\theta_x + a \frac{\theta}{u}v_x\theta_{xx} - \mu \frac{v_x^2\theta_{xx}}{u}. \tag{5.11}$$

Now, we integrate over $[0, T] \times \mathbb{R}$ and repetitively use Eq. (2.7), Theorem 4.1, Eq. (4.6), Young's inequality, as well as Eqs. (5.4) (5.6) and (5.10) in the previous steps of the same proof, to derive the following inequality:

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \int_{\mathbb{R}} \theta_x^2 dx + \int_0^T \int_{\mathbb{R}} \theta_{xx}^2 dx dt \\
& \leq C \left\{ \int_0^T \int_{\mathbb{R}} |\theta_x u_x \theta_{xx}| dx dt + \int_0^T \|Z_x\|_{L^2(\mathbb{R})} \|\theta_x\|_{L^2(\mathbb{R})} dt \right. \\
& \quad \left. + \int_0^T \int_{\mathbb{R}} |\theta v_x \theta_{xx}| dt dx + \int_0^T \|v_x\|_{L^2(\mathbb{R})} \|v_x\|_{L^\infty(\mathbb{R})} \|\theta_{xx}\|_{L^2(\mathbb{R})} dt \right\}.
\end{aligned}$$

In the sequel let us bound each of the four terms on the right-hand side of the preceding expression. For the first term, we consider

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}} |\theta_x u_x \theta_{xx}| \, dx dt \\
& \leq \epsilon_{14} \int_0^T \int_{\mathbb{R}} |\theta_{xx}|^2 \, dx dt + C(\epsilon_{14}) \left\{ \sup_{0 \leq t \leq T} \int_{\mathbb{R}} |u_x(t, x)|^2 \, dx \right\} \left\{ \sup_{x \in \mathbb{R}} \int_0^T |\theta_x(t, x)|^2 \, dt \right\} \\
& \leq \epsilon_{14} \int_0^T \int_{\mathbb{R}} |\theta_{xx}|^2 \, dx dt + C(\epsilon_{14}) C_6 \left\{ \sup_{x \in \mathbb{R}} \int_0^T |\theta_x(t, x)|^2 \, dt \right\} \\
& \leq \epsilon_{14} \int_0^T \int_{\mathbb{R}} |\theta_{xx}|^2 \, dx dt + 2C(\epsilon_{14}) C_6 \int_0^T \int_{\mathbb{R}} |\theta_x \theta_{xx}| \, dx dt \\
& \leq 2\epsilon_{14} \int_0^T \int_{\mathbb{R}} |\theta_{xx}|^2 \, dx dt + C'(\epsilon_{14}, C_6) \int_0^T \int_{\mathbb{R}} |\theta_x(t, x)|^2 \, dx dt \\
& \leq 2\epsilon_{14} \int_0^T \int_{\mathbb{R}} |\theta_{xx}|^2 \, dx dt + C'(\epsilon_{14}, C_6) C_1.
\end{aligned}$$

Here, in the first line we use the Cauchy–Schwarz inequality and the obvious interpolation; in the second Eq. (5.4); in the third the identity $|\theta_x(t, x)|^2 = \left| \int_x^\infty 2\theta_x \theta_{xx}(t, x') \, dx' \right|$, in the fourth Cauchy–Schwarz again, and in the last line we invoke Theorem 4.1.

The second term is easily bounded as follows:

$$\int_0^T \|Z_x\|_{L^2(\mathbb{R})} \|\theta_x\|_{L^2(\mathbb{R})} \, dt \leq \epsilon_{15} \sup_{0 \leq t \leq T} \int_{\mathbb{R}} \theta_x^2 \, dx + C(\epsilon_{15}).$$

For the third term, we compute:

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}} |\theta v_x \theta_{xx}| \, dx dt \\
& \leq \epsilon_{16} \int_0^T \int_{\mathbb{R}} |\theta_{xx}(t, x)|^2 \, dx dt + C(\epsilon_{16}) \int_0^T \int_{\mathbb{R}} \theta^2 v_x^2 \, dx dt \\
& \leq \epsilon_{16} \int_0^T \int_{\mathbb{R}} |\theta_{xx}(t, x)|^2 \, dx dt + C(\epsilon_{16}) \left\{ \frac{2}{3} \sup_{[0, T] \times \mathbb{R}} \theta^{\frac{3}{2}} + \frac{1}{3} \left(\int_0^T \int_{\mathbb{R}} \theta v_x^2(t, x) \, dx dt \right)^3 \right\} \\
& \leq \epsilon_{16} \int_0^T \int_{\mathbb{R}} |\theta_{xx}(t, x)|^2 \, dx dt + C(\epsilon_{16}) \left\{ \frac{2}{3} \sup_{[0, T] \times \mathbb{R}} \theta^{\frac{3}{2}} + \frac{1}{3} (C_1)^3 \right\},
\end{aligned}$$

where we have used the Cauchy–Schwarz inequality, the Young’s inequality $ab \leq \frac{2}{3}a^{3/2} + \frac{1}{3}b^3$ for $a, b \geq 0$, and Theorem 4.1 in each line, respectively.

Finally, for the fourth term, we employ again Eq. (4.6) to derive that

$$\begin{aligned}
& \int_0^T \left\{ \|v_x\|_{L^2(\mathbb{R})} \|v_x\|_{L^\infty(\mathbb{R})} \|\theta_{xx}\|_{L^2(\mathbb{R})} \right\} dt \\
& \leq C \int_0^T \left\{ \|v_x\|_{L^2(\mathbb{R})} \|v_x\|_{H^1(\mathbb{R})} \|\theta_{xx}\|_{L^2(\mathbb{R})} \right\} dt
\end{aligned}$$

$$\begin{aligned}
&\leq C \left(\sup_{0 \leq t \leq T} \|v_x\|_{L^2(\mathbb{R})} \right) \left(\int_0^T \|v_x\|_{H^1(\mathbb{R})} \|\theta_{xx}\|_{L^2(\mathbb{R})} dt \right) \\
&\leq C \sup_{0 \leq t \leq T} \int_{\mathbb{R}} v_x^2(t, x) dx + C \left(\int_0^T \|v_x\|_{H^1(\mathbb{R})} \|\theta_{xx}\|_{L^2(\mathbb{R})} dx dt \right)^2 \\
&\leq C \sup_{0 \leq t \leq T} \int_{\mathbb{R}} v_x^2(t, x) dx + C(\epsilon_{17}) \int_0^T \int_{\mathbb{R}} (v_x^2 + v_{xx}^2) dx dt + \epsilon_{17} \int_0^T \int_{\mathbb{R}} \theta_{xx}^2 dx dt \\
&\leq \epsilon_{17} \int_0^T \int_{\mathbb{R}} \theta_{xx}^2 dx dt + C(\epsilon_{17}) \left\{ 1 + \sup_{[0, T] \times \mathbb{R}} \theta \right\},
\end{aligned}$$

which again is based on the Cauchy–Schwarz inequality and the entropy formula (2.7).

Therefore, using the previous estimates, we choose suitable $\epsilon_{14}, \epsilon_{15}, \epsilon_{16}$, and ϵ_{17} to get:

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}} \theta_x^2 dx + \int_0^T \int_{\mathbb{R}} \theta_{xx}^2 dx dt \leq C_9 \left(1 + \sup_{[0, T] \times \mathbb{R}} \theta + \sup_{[0, T] \times \mathbb{R}} \theta^3 \right). \quad (5.12)$$

5. Finally, we conclude the uniform upper boundedness of θ in space–time. Notice that, by the Sobolev inequality (4.6), for any $0 \leq t \leq T$ there holds

$$\|(\theta - 2)_+(t, \cdot)\|_{C(\mathbb{R})}^2 \leq \|(\theta - 2)_+(t, \cdot)\|_{L^2(\mathbb{R})} \|\theta_x(t, \cdot)\|_{L^2(\mathbb{R})}. \quad (5.13)$$

Then, using Theorem 4.1, it can be deduced that $\|(\theta - 2)_+(t, \cdot)\|_{L^2(\mathbb{R})} \leq \sqrt{C_1}$ for any t , while $\|\theta_x(t, \cdot)\|_{L^2(\mathbb{R})}$ is estimated by Eq. (5.12). Hence, we get

$$\|(\theta - 2)_+(t, \cdot)\|_{C(\mathbb{R})}^2 \leq C_{10} \left(1 + \sup_{[0, T] \times \mathbb{R}} \theta^{\frac{1}{2}} + \sup_{[0, T] \times \mathbb{R}} \theta^{\frac{3}{2}} \right).$$

In particular, by comparing the growth rate at infinity, we get:

$$\sup_{[0, T] \times \mathbb{R}} \theta(\cdot, \cdot) \leq C_{11}. \quad (5.14)$$

Thus, putting together Eqs. (5.4), (5.6), (5.10), (5.12), (5.14), we complete the proof. $\square$

Based on Lemma 5.1, we are now ready to establish the global existence and the large-time behaviour of the weak solutions, which are the main results of the paper.

Proof of Theorems 1.2 and 1.3. The arguments are divided into four steps.

1. First, let us prove the large-time behaviour under the temporary assumption ($\clubsuit$) introduced in §2, i.e., the regularisation. We can easily deduce that

$$\int_0^\infty \left\{ \left| \frac{d}{dt} \|v_x(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right| + \left| \frac{d}{dt} \|\theta_x(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right| + \left| \frac{d}{dt} \|Z_x(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right| \right\} dt \leq C_{12}. \quad (5.15)$$

Indeed, by the Cauchy–Schwarz inequality and integration by parts, there holds

$$\int_0^T \left| \frac{d}{dt} \|v_x(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right| dt \leq \int_0^T \int_{\mathbb{R}} |v_{xx} v_t| dx dt = \left(\int_0^T \int_{\mathbb{R}} v_{xx}^2 dx dt \right)^{1/2} \left(\int_0^T \int_{\mathbb{R}} v_t^2 dx dt \right)^{1/2},$$

which is bounded by C_5 in Lemma 5.1; then, we send $T \rightarrow \infty$. The treatment for the other two terms is similar.

2. Thanks to Eq. (5.15), we have

$$\|(v_x, \theta_x, Z_x)(t, \cdot)\|_{L^2(\mathbb{R})} \longrightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (5.16)$$

This is because the function $t \mapsto \|v_x(t, \cdot)\|_{L^2}^2 + \|\theta_x(t, \cdot)\|_{L^2}^2 + \|Z_x(t, \cdot)\|_{L^2}^2$ lies in $W^{1,1}(\mathbb{R}_+)$, hence decays to zero near infinity. From here we immediately deduce that

$$v^2(t, x) \leq \|v_x(t, \cdot)\|_{L^2(\mathbb{R})} \|v(t, \cdot)\|_{L^2(\mathbb{R})} \leq \sqrt{2qE_0} \|v_x(t, \cdot)\|_{L^2(\mathbb{R})} \longrightarrow 0, \quad (5.17)$$

which is valid for any $x \in \mathbb{R}$, in view of the Sobolev inequality (4.6).

Next, the asymptotic for θ is obtained similarly: using that $\sup_{[0,T] \times \mathbb{R}} \theta \leq 3 + C_{10}$ (cf. Step 5 in the proof of Lemma 5.1) and Eq. (4.4), we have

$$(\theta(t, x) - 1)^2 \leq \|(\theta(t, \cdot) - 1)\mathbb{1}_{\mathbb{R} \setminus \Sigma_2(t)}\|_{L^2(\mathbb{R})} \|\theta_x(t, \cdot)\|_{L^2(\mathbb{R})} \longrightarrow 0, \quad (5.18)$$

in which the dominated convergence theorem is used. Also, Lemma 5.1 leads to

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}} u_x^2 \, dx + \int_0^T \int_{\mathbb{R}} \theta u_x^2 \, dx \, dt \leq C_5.$$

As we have already established the uniform boundedness of θ , it follows that

$$\int_0^\infty \left| \frac{d}{dt} \|u_x(t, \cdot)\|_{L^2(\mathbb{R})} \right| \, dt \leq C_{12}.$$

Moreover, the subsequent result holds:

$$\{u(t, x) - 1\} \longrightarrow 0 \quad \text{uniformly in space–time.} \quad (5.19)$$

Indeed, by the entropy inequality (2.7) and the uniform bound on u (cf. Theorem 3.1),

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}} (u - 1)^2 \, dx \leq C \sup_{0 \leq t \leq T} \int_{\mathbb{R}} (u - 1 - \log(u)) \, dx \leq C_{13}. \quad (5.20)$$

Thus, via precisely the same arguments for v and θ as above (5.19) is proved.

Finally, to control the combustion term Z specific to the combustion flow, we integrate by parts to derive that

$$\begin{aligned} Z^{\frac{3}{2}}(t, x) &= - \int_x^\infty \left[Z^{\frac{3}{2}} \right]_x(t, \xi) \, d\xi \\ &\leq \frac{3}{2} \int_{\mathbb{R}} Z^{\frac{1}{2}}(t, \xi) |Z_x(t, \xi)| \, d\xi \\ &\leq \frac{3}{2} \|Z_x\|_{L^2(\mathbb{R})} \left(\int_{\mathbb{R}} Z(t, \xi) \, d\xi \right)^{\frac{1}{2}} \longrightarrow 0, \end{aligned} \quad (5.21)$$

thanks to Eqs. (5.16) and (2.6). Therefore, collecting the estimates in Eqs. (5.17)(5.18)(5.19)(5.21), the proof for Theorem 1.3 is now complete, under the regularisation $\phi \equiv \phi^\delta \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

2. In this step we establish the uniform lower bound for θ , based on the large-time behaviour established in Step 1 of the same proof for C^1 reaction rate functions.

For this purpose, we first obtain a lower bound for θ up to some given time $T_* > 0$ on the compact domain $[-L, L]$ for some finite number $L > 0$. Let us denote by $\zeta := \theta^{-1}$. Then, multiplying $(-\theta^{-2})$ to the temperature equation (1.3), we arrive at the following evolution equation for ζ :

$$\zeta_t + 2\kappa \frac{\zeta_x^2}{u\zeta} + \mu \frac{\zeta^2 v_x^2}{u} = \kappa \frac{\zeta_{xx}}{u} - \kappa \frac{\zeta_x u_x}{u^2} + a \frac{\zeta}{u} v_x - \zeta^2 q K \phi(\theta) Z. \quad (5.22)$$

Completing the squares and writing the first two terms on right-hand side in the full divergence form, we obtain that

$$\zeta_t + 2\kappa \frac{\zeta_x^2}{u\zeta} + \frac{\mu}{u} \left[\zeta v_x - \frac{a}{2\mu} \right]^2 + \zeta^2 q K \phi(\theta) Z = \kappa \left(\frac{\zeta_x}{u} \right)_x + \frac{a^2}{4\mu u}.$$

Then, we restrict to the finite spatial interval $[-L, L]$ and multiply $(2p\zeta^{2p-1})$ to the previous equation with $p > \frac{3}{2}$. Integrating by parts and using the periodic boundary condition on $[-L, L]$, we arrive at

$$\frac{d}{dt} \int_{-L}^L \zeta^{2p}(t, x) dx \leq \frac{a^2 p}{2\mu} \int_{-L}^L \frac{\zeta(t, x)^{2p-1}}{u(t, x)} dx.$$

Now, applying Hölder's inequality together with the uniform lower bound $u \geq C_0^{-1}$ in Theorem 3.1, one deduces

$$2p \|\zeta\|_{L^{2p}([-L, L])}^{2p-1} \times \frac{d}{dt} \|\zeta\|_{L^{2p}([-L, L])} \leq 2p \frac{a^2}{4\mu C_0} \|\zeta^{2p-1}\|_{L^{\frac{2p}{2p-1}}([-L, L])} (2L)^{\frac{1}{2p}}. \quad (5.23)$$

Here it is crucial to choose L depending on p : indeed, taking $L = 2^{2p-1}$, then $L \rightarrow \infty$ as $p \rightarrow \infty$, while $(2L)^{\frac{1}{2p}} = 2$. The previous estimate thus becomes

$$\frac{d}{dt} \|\zeta\|_{L^{2p}([-L, L])} \leq \frac{a^2}{2\mu C_0},$$

which is uniform in L and p . Thus, for any fixed $T_* > 0$, we can send p, L to infinity and apply the Grönwall lemma to conclude that

$$\zeta(t, x) \leq C e^{CT_*}, \quad (5.24)$$

which is equivalent to

$$\inf_{[0, T_*] \times \mathbb{R}} \theta \geq C^{-1} e^{-CT_*}, \quad (5.25)$$

which is a space-time uniform lower bound for θ up to time T_* .

Finally, to promote the local (in time) bound to a global bound, we make use of the result in Step 1 above: there we have shown that $\theta \rightarrow 1$ uniformly as $t \rightarrow \infty$. As a result, choose a $T_* \in (0, \infty)$ such that $0.99 \leq \theta(t, x) \leq 1.01$ whenever $t \geq T_*$ and $x \in \mathbb{R}$. Thus, together with the *local* lower bound of θ in Eq. (5.25), we can conclude the *global* lower bound for θ . Now we are able to conclude the proof of Theorems 1.2 and 1.3, subject to condition (♣).

3. Finally, let us remove condition (♣) and establish the theorems for generic discontinuous reaction rate functions ϕ obeying the Arrhenius' law. For this purpose, we notice that all the previous bounds are independent of δ (recall that δ^{-1} is the upper bound for the first derivative of the regularised function ϕ^δ). For each $\delta > 0$, we have obtained a global strong solution $(u_\delta, v_\delta, \theta_\delta, Z_\delta)$ to Eqs. (1.1)–(1.8), with $\phi(\theta)$ replaced by $\phi^\delta(\theta)$. The solutions $(u_\delta, v_\delta, \theta_\delta, Z_\delta)$ verify all the uniform estimates in Sects. 3–5, independent of δ . Hence, by a standard compactness argument in $[H^1(\mathbb{R})]^4$, we can select a subsequence converging to a global weak solution to Eqs. (1.1)–(1.8), where the discontinuous reaction rate function $\phi(\theta)$ satisfies the Arrhenius' law. Therefore, the proof of Theorems 1.2 and 1.3 is now complete.

At the end of the paper we make several concluding remarks:

- (1) First of all, the physical meaning of the results in this paper is natural: for a one-dimensional reacting mixture on unbounded domains, if the far-field condition is imposed as in Eq. (1.7), then the chemical reaction will occur and proceed towards completion as time approaches infinity, regardless of the detailed structure of the reaction rate function $\phi(\theta)$. In this process, the density and temperature of the reacting mixture will be uniformly bounded away from zero and infinity. We also note that even in the case $\theta_{\text{ignite}} = 1$, the large-time behaviour of the temperature is still $\theta \rightarrow 1$ as $t \rightarrow \infty$.

- (2) The results in the paper can be extended to several other types of boundary conditions. For example, let us consider the domain to be the half line $\Omega = [0, \infty)$, with the same far-field condition at ∞ :

$$\lim_{x \rightarrow \infty} (u, v, \theta, Z) = (1, 0, 1, 0) \quad \text{for all } t \geq 0.$$

At $x = 0$ we can impose the **impermeability + thermal insulation condition (I)**:

$$v(t, 0) = 0; \quad \theta_x(t, 0) = 0; \quad Z(t, x) = 0 \text{ or } Z_x(t, 0) = 0, \quad (5.26)$$

or the **impermeability + constant source condition (II)**:

$$v(t, 0) = 0; \quad \theta(t, 0) = 1; \quad Z(t, x) = 0 \text{ or } Z_x(t, 0) = 0. \quad (5.27)$$

These boundary conditions are also considered in [13] for one-dimensional heat-conducting compressible fluids without reaction terms.

Here, we claim that the same statements for Theorems 1.2 and 1.3 remain valid, subject to boundary condition (I) or (II). This can be proved by the same arguments, as long as the integration by parts arguments still hold. Indeed, Eqs. (4.7), (4.8), (4.20), (4.22), (5.3), (5.5), and (5.7) remain valid; also, under the condition (I), Eq. (5.11) stays the same, while subject to condition (II) we can make simple modifications to recover Eq. (5.12).

- (3) In the end, we emphasise that the arguments in [12] for the lower boundedness of temperature are not valid on unbounded domains ($u^{-1} \notin L^p(\mathbb{R})$ for $p \geq 1$) and the arguments in [13] for the large-time behaviour cannot be applied without modifications in the presence of the Z term. In our work, new estimates have been developed to cope with unbounded domains and the dynamic combustion process.

Acknowledgements

Siran Li's research was supported in part by the UK EPSRC Science and Innovation award to the Oxford Centre for Nonlinear PDE (EP/E035027/1). The author would like to thank Professors Sir John M. Ball, Gui-Qiang G. Chen, Lawrence C. Evans, and Mr. Francis Hounkpe for their careful reading of the draft and the insightful discussions. The author would also like to thank the referees for their valuable suggestions.

Open Access. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

- [1] Chen, G.-Q.: Global solutions to the compressible Navier–Stokes equations for a reacting mixture. *SIAM J. Math. Anal.* **23**, 609–634 (1992)
- [2] Chen, G.-Q., Hoff, D., Trivisa, K.: Global solutions to a model for exothermically reacting, compressible flows with large discontinuous initial data. *Arch. Ration. Mech. Anal.* **166**, 321–358 (2003)
- [3] Chen, G.-Q., Kratka, M.: Global solutions to the Navier–Stokes equations for compressible heat-conducting flow with symmetry and free boundary. *Commun. PDE* **27**, 907–943 (2002)
- [4] Chen, G.-Q., Wagner, D.H.: Global entropy solutions to exothermically reacting, compressible Euler equations. *J. Differ. Equ.* **191**, 277–322 (2003)
- [5] Donatelli, D., Trivisa, K.: On the motion of a viscous compressible radiative-reacting gas. *Commun. Math. Phys* **265**, 463–491 (2006)
- [6] Documet, B., Zlotnik, A.: Lyapunov functional method for 1D radiative and reactive viscous gas dynamics. *Arch. Ration. Mech. Anal.* **177**, 185–229 (2005)

- [7] Huang, X., Li, J., Wang, Y.: Serrin-type blowup criterion for full compressible Navier–Stokes system. *Arch. Ration. Mech. Anal.* **207**, 303–316 (2013)
- [8] Jiang, S.: Large-time behavior of solutions to the equations of a one-dimensional viscous polytropic ideal gas in unbounded domains. *Commun. Math. Phys.* **200**, 181–193 (1999)
- [9] Jiang, S.: Remarks on the asymptotic behaviour of solutions to the compressible Navier–Stokes equations in the half-line. *Proc. R. Soc. Edinb. Sect. A* **132**, 627–638 (2002)
- [10] Kanel', Y.I.: On a model system of equations of one-dimensional gas motion. *Differ. Equ.* **4**, 374–380 (1968)
- [11] Kazhikhov, A.V.: Cauchy problem for viscous gas equations. *Siber. Math. J.* **23**, 44–49 (1982)
- [12] Kazhikhov, A.V., Shelukhin, V.V.: Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas, (PMM vol. 41, n=2, 1977, pp. 282–291). *J. Appl. Math. Mech.* **41**, 273–282 (1977)
- [13] Li, J., Liang, Z.: Some uniform estimates and large-time behavior of solutions to one-dimensional compressible Navier–Stokes system in unbounded domains with large data. *Arch. Ration. Mech. Anal.* **220**, 1195–1208 (2016)
- [14] Solonnikov, V.A., Kazhikhov, A.V.: Existence theorems for the equations of motion of a compressible viscous fluid. *Ann. Rev. Fluid Mech.* **13**, 79–95 (1981)
- [15] Wang, D.: Global solutions for the mixture of real compressible reacting flows in combustion. *Commun. Pure Appl. Anal.* **3**, 775–790 (2004)

Siran Li
Mathematical Institute
University of Oxford
Oxford OX2 6GG
UK
e-mail: Siran.Li@rice.edu

(Received: October 22, 2016)