Noncooperative elliptic systems

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Abstract. We show how monotonicity methods combined with infinite dimensional sandwich pairs can be used to solve very general systems of equations that are not semibounded.

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1. Introduction

The purpose of our investigation is to solve systems of equations of the form

$$\mathcal{A}v = f(x, v, w) \quad (1)$$

$$\mathcal{B}w = g(x, v, w), \quad (2)$$

where $\mathcal{A}, \mathcal{B}$ are linear partial differential operators. We assume that there is a Carathéodory function $F(x, v, w)$ on $\Omega \times \mathbb{R}^2$ such that

$$f(x, v, w) = \partial F / \partial v, \quad g(x, v, w) = \partial F / \partial w. \quad (3)$$

If $\mathcal{A}, \mathcal{B}$ are positive and we assume

$$2F(x, s, t) \leq \lambda(x)s^2 + \mu(x)t^2 + W(x), \quad x \in \Omega, s, t \in \mathbb{R}, \quad (4)$$

where $W(x) \in L^1(\Omega)$,

$$\lambda(x) \leq \lambda_0, \quad \mu(x) \leq \mu_0, \quad x \in \Omega, \quad (5)$$

and $\lambda_0(\mu_0)$ is the lowest eigenvalue of $\mathcal{A}(\mathcal{B})$, then the problem (1), (2) is called cooperative. It can be solved by a minimization process. In fact, the functional corresponding to the system is coercive.

On the other hand, if one operator is positive and the other is negative (i.e., the system is noncooperative), then serious problems arise. The functional is unbounded from above and below on infinite dimensional subspaces. This situation has been attacked before under quite strong hypotheses (cf., e.g., [1–7, 11, 13, 19–21, 24, 25, 28, 30, 31] and the references contained there). The purpose of the present paper is to weaken the hypotheses considerably. In particular, we do not assume that the functions $f(x, v, w)$, $g(x, v, w)$ are differentiable in any sense, nor do we assume any asymptotic limits for them. Moreover, we do not assume that the system (1), (2) satisfies any Palais–Smale or Cerami condition. However, there is a price to pay. If $\mathcal{A}, \mathcal{B}$ are positive, we consider the problem

$$-\mathcal{A}v = f(x, v, w) \quad (6)$$

$$\mathcal{B}w = g(x, v, w), \quad (7)$$

and embed it in a family of systems of the form

$$-\mathcal{A}v = f(x, v, w) \quad (8)$$

$$\lambda \mathcal{B}w = g(x, v, w), \quad (9)$$

where λ is a positive parameter. We show, under minimal assumptions, that this system has a solution (or a nontrivial solution) for almost all values of λ in a specified interval containing $\lambda = 1$. These theorems do not show that the system is solvable for $\lambda = 1$. In order to show that the system is solvable for $\lambda = 1$, we must add more hypotheses. We present such theorems as a contrast.

Our theorems for elliptic systems are stated and proved in Sect. 4. They are based on abstract theorems presented in Sect. 2 as well as the monotonicity trick introduced by Struwe in [26, 27] for minimization problems. This trick was also used by others to solve Landesman–Lazer type problems [8], bifurcation problems [9], superlinear problems [21], Hamiltonian systems [29], and Schrödinger equations [31]. The monotonicity method for our situation is introduced in Sect. 3 and proved in Sect. 6. Proofs of the abstract theorems will be given in Sect. 5. Contrasting theorems that require solutions for $\lambda = 1$ are stated and proved in Sect. 7. Related material can be found in [10], [12], [14–18] and [22].

2. Flows

Let E be a Banach space, and let Σ be the set of all continuous maps $\sigma = \sigma(t)$ from $[0, 1] \times E$ to E such that

1. $\sigma(0)$ is the identity map,
2. for each $t \in [0, 1]$, $\sigma(t)$ is a homeomorphism of E onto E ,
3. $\sigma'(t)$ is piecewise continuous on $[0, 1]$ and satisfies

$$\|\sigma'(t)u\| \leq \text{const.}, \quad u \in E. \quad (10)$$

The mappings in Σ are called *flows*. We note the following.

Remark 1. If σ_1, σ_2 are in Σ , define $\sigma_3 = \sigma_1 \circ \sigma_2$ by

$$\sigma_3(s) = \begin{cases} \sigma_1(2s), & 0 \leq s \leq \frac{1}{2}, \\ \sigma_2(2s - 1)\sigma_1(1), & \frac{1}{2} < s \leq 1. \end{cases}$$

Then, $\sigma_1 \circ \sigma_2 \in \Sigma$.

Let N be a closed, separable subspace of a Hilbert space E . We can define a new norm $|v|_w$ satisfying $|v|_w \leq \|v\|$, $\forall v \in N$ and such that the topology induced by this norm is equivalent to the weak topology of N on bounded subsets of N . This can be done as follows: Let $\{e_k\}$ be an orthonormal basis for N . Define

$$|v|_w = \sum_{k=1}^{\infty} \frac{|(v, e_k)|}{2^k}, \quad v \in N.$$

Then, $|v|_w$ is a norm on N and satisfies $|v|_w \leq \|v\|$, $v \in N$. If $v_j \rightarrow v$ weakly in N , then there is a $C > 0$ such that

$$\|v_j\|, \|v\| \leq C, \quad \forall j > 0.$$

For any $\varepsilon > 0$, there exist $K > 0, M > 0$, such that $1/2^K < \varepsilon/(4C)$ and $|(v_j - v, e_k)| < \varepsilon/2$ for $1 \leq k \leq K, j > M$. Therefore,

$$\begin{aligned} |v_j - v|_w &= \sum_{k=1}^{\infty} \frac{|v_j - v, e_k|}{2^k} \\ &\leq \sum_{k=1}^K \frac{\varepsilon/2}{2^k} + \sum_{k=K+1}^{\infty} \frac{2C}{2^k} \\ &\leq \frac{\varepsilon}{2} \sum_{k=1}^{\infty} \frac{1}{2^k} + \frac{2C}{2^K} \sum_{k=1}^{\infty} \frac{1}{2^k} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}. \end{aligned}$$

Therefore, $v_j \rightarrow v$ weakly in N implies $|v_j - v|_w \rightarrow 0$.

Conversely, let $\|v_j\|, \|v\| \leq C$ for all $j > 0$ and $|v_j - v|_w \rightarrow 0$. Let $\varepsilon > 0$ be given. If $h = \sum_{k=1}^{\infty} \alpha_k e_k \in N$, take K so large that $\|h_K\| < \varepsilon/(4C)$, where $h_K = \sum_{k=K+1}^{\infty} \alpha_k e_k$. Take M so large that $|v_j - v|_w < \varepsilon/(2 \max_{1 \leq k \leq K} 2^k |\alpha_k|)$ for all $j > M$. Then,

$$\begin{aligned} |(v_j - v, h - h_K)| &= \left| \sum_{k=1}^K \alpha_k (v_j - v, e_k) \right| \\ &\leq \max_{1 \leq k \leq K} 2^k |\alpha_k| \sum_{k=1}^K \frac{|(v_j - v, e_k)|}{2^k} \\ &< \varepsilon/2 \end{aligned}$$

for $j > M$. Also, $|(v_j - v, h_K)| \leq 2C\|h_K\| < \varepsilon/2$. Therefore,

$$|(v_j - v, h)| < \varepsilon, \quad \forall j > M,$$

that is, $v_j \rightarrow v$ weakly in N .

For $u = v + w \in E = N \oplus N^\perp$ with $v \in N, w \in N^\perp$, we define $|u|_w^2 = |v|_w^2 + \|w\|^2$. Thus, $|u|_w \leq \|u\|, \forall u \in E$. We denote E equipped with this norm by E_w . In particular, if $u_n = v_n + w_n$ is $|\cdot|_w$ -bounded and $u_n \xrightarrow{|\cdot|_w} u$, then $v_n \rightharpoonup v$ weakly in N , $w_n \rightarrow w$ strongly in $N^\perp$, $u_n \rightharpoonup v + w$ weakly in E .

For $G \in C^1(E, \mathbb{R})$, we define Σ_w to be the set of all $\sigma(t) \in \Sigma : [0, 1] \times E \mapsto E$ such that

1. $\sigma(t)$ is $|\cdot|_w$ -continuous.
2. For each compact subset L of E_w , there is a finite dimensional subspace E_f of E such that $\sigma(t)u - u \in E_f, u \in L, t \in I$.

Here, we use E_f to denote various finite-dimensional subspaces of E when exact dimensions are irrelevant. Note that $\sigma(t) \equiv 1$ is in Σ_w .

We have

Lemma 2. *If L is compact in E_w and $\sigma \in \Sigma_w$, then*

$$\tilde{L} = \{\sigma(t)L : t \in I\}$$

is compact in E_w .

Proof. Suppose $\{t_k\} \subset I, \{u_k\} \subset L$ are sequences. Then, there are renamed subsequences such that

$$t_j \rightarrow t_0, \quad |u_k - u_0|_w \rightarrow 0.$$

Since,

$$\sigma(t_j)u_k - \sigma(t_0)u_0 = \sigma(t_j)u_k - \sigma(t_0)u_k + \sigma(t_0)u_k - \sigma(t_0)u_0,$$

we have $|\sigma(t_j)u_k - \sigma(t_0)u_0|_w \rightarrow 0$. $\square$

Lemma 3. *If $\sigma_1, \sigma_2 \in \Sigma_w$, then $\sigma_3 = \sigma_1 \circ \sigma_2 \in \Sigma_w$.*

Let $\mathcal{K}_w$ be the collection of sets

$$K = K_\sigma = \{\sigma(1)N : \sigma \in \Sigma_w\}.$$

Theorem 4. *Let N be a closed separable subspace of a Hilbert space E . Let G be a continuously differentiable functional on E such that*

$$v_n = Pu_n \rightarrow v \quad \text{weakly in } E, \quad w_n = (I - P)u_n \rightarrow w \quad \text{strongly in } E \quad (11)$$

implies

$$G'(v_n + w_n) \rightarrow G'(v + w) \quad \text{weakly in } E, \quad (12)$$

where P is the projection of E onto N . Assume

$$a := \inf_{K \in \mathcal{K}_w} \sup_K G \quad (13)$$

is finite. Assume, in addition, that there is a constant C_0 such that for each $\delta > 0$, there is a $K \in \mathcal{K}_w$ satisfying

$$\sup_K G \leq a + \delta, \quad (14)$$

such that the inequality

$$G(u) \geq a - \delta, \quad u \in K, \quad (15)$$

implies $\|u\| \leq C_0$. Then, there is a sequence $\{u_k\} \subset E$ such that

$$\|u_k\| \leq C_0 + 1, \quad G(u_k) \rightarrow a, \quad \|G'(u_k)\| \rightarrow 0. \quad (16)$$

Theorem 5. *Let $\mathcal{K}_w$ be as above, and let $G(u)$ be a C^1 functional on E . Assume that there are subsets A, B of E such that*

$$a_0 := \sup_A G < \infty, \quad b_0 := \inf_B G > -\infty, \quad (17)$$

$A \in \mathcal{K}_w$ and

$$B \cap K \neq \emptyset, \quad K \in \mathcal{K}_w. \quad (18)$$

Assume, in addition, that there is a constant C_0 such that for each $\delta > 0$, there is a $K \in \mathcal{K}_w$ satisfying (14) such that the inequality (15) implies $\|u\| \leq C_0$. Then, there is a bounded sequence $\{u_k\} \subset E$ such that

$$G(u_k) \rightarrow c, \quad \|G'(u_k)\| \rightarrow 0, \quad (19)$$

where c satisfies $b_0 \leq c \leq a_0$.

Theorem 6. *Let F be a continuous map of E onto N such that $F|_N = I$. Let p be any point of E . Assume that for each finite dimensional subspace S of E containing p such that $FS \neq \{0\}$, there is a finite dimensional subspace $S_0 \neq \{0\}$ of N containing p such that*

$$v \in S_0, \quad w \in S \implies F(v + w) \in S_0. \quad (20)$$

Then, $B = F^{-1}(p)$ satisfies (18).

Corollary 7. *Let p be a fixed point of N . Let F be a continuous map from E onto N satisfying*

- $F|_N = I$;
- *there exists a fixed finite-dimensional subspace E_0 of E such that*
 $F(u - v) - (F(u) - F(v)) \in E_0 \quad \forall u, v \in E$;
- *F maps finite-dimensional subspaces of E to finite-dimensional subspaces of N ;*

Then, $B = F^{-1}(p)$ satisfies (18).

Proof. We show that F satisfies the hypotheses of Theorem 6. Clearly, we can take $E_0 \subset N$. Let S be a finite dimensional subspace of E such that $FS \neq \{0\}$. Let S_0 be a finite dimensional subspace of N containing $FS + E_0$. If $v \in S_0$, $w \in S$, then

$$F(v + w) - F(w) - F(v) \in E_0.$$

Thus, $F(v + w) \in S_0$. □

Definition 8. We shall say that a pair of subsets A, B of a Banach space E forms a **weak sandwich pair** if $A \in \mathcal{K}_w$ and B satisfies (18).

We have

Theorem 9. *Let N be a separable subspace of a Banach space E , and let p be any point of N . Let F be a continuous map of E onto N satisfying the hypotheses of Theorem 6. Then, $A = N$ and $B = F^{-1}(p)$ form a weak sandwich pair.*

Corollary 10. *Let N be a closed subspace of a Hilbert space E and let $M = N^\perp$. Assume that at least one of the subspaces M, N is separable. Then, M, N form a weak sandwich pair.*

Corollary 11. *Let N be a closed, separable subspace of a Hilbert space E with complement $M' = M \oplus \{v_0\}$, where v_0 is an element in E having unit norm. Let δ be any positive number, and let $\varphi(t) \in C^1(\mathbb{R})$ be such that*

$$0 \leq \varphi(t) \leq 1, \quad \varphi(0) = 1,$$

and

$$\varphi(t) = 0, \quad |t| \geq 1.$$

Let

$$F(v + w + sv_0) = v + [s + \delta - \delta\varphi(\|w\|^2/\delta^2)]v_0, \quad v \in N, \quad w \in M, \quad s \in \mathbb{R}. \quad (21)$$

Then, $A = N' = N \oplus \{v_0\}$, $B = F^{-1}(\delta v_0)$ form a weak sandwich pair.

Proof. Define

$$J(v + w + sv_0) = v + w + [s - \delta + \delta\varphi(\|w\|^2/\delta^2)]v_0, \quad v \in N, \quad w \in M, \quad s \in \mathbb{R}.$$

Then, J is a diffeomorphism on E . Moreover, $A = JN'$ and $B = J[M + \delta v_0]$. Since N' and $M + \delta v_0$ form a weak sandwich pair by Corollary 10, we see that A, B also form a weak sandwich pair. □

3. The parameter problem

Let E be a reflexive Banach space with norm $\|\cdot\|$, and let A, B be two closed subsets of E . Let G be a continuously differentiable functional on E such that

$$v_n = Pu_n \rightarrow v \quad \text{weakly in } E, \quad w_n = (I - P)u_n \rightarrow w \quad \text{strongly in } E \quad (22)$$

implies

$$G'(v_n + w_n) \rightarrow G'(v + w) \quad \text{weakly in } E, \quad (23)$$

where P is the projection of E onto N . Suppose that G is of the form: $G(u) := I(u) + J(u)$, $u \in E$, where $I, J \in C^1(E, \mathbb{R})$ map bounded sets to bounded sets. Define

$$G_\lambda(u) = \lambda I(u) + J(u), \quad \lambda \in \Lambda,$$

where Λ is an open interval contained in $(0, +\infty)$. Assume one of the following alternatives holds.

(H₁) $I(u) \geq 0$ for all $u \in E$ and $I(u) + |J(u)| \rightarrow \infty$ as $\|u\| \rightarrow \infty$.

(H₂) $I(u) \leq 0$ for all $u \in E$ and $|I(u)| + |J(u)| \rightarrow \infty$ as $\|u\| \rightarrow \infty$.

Furthermore, we suppose that

(H₃) $a(\lambda) := \inf_{K \in \mathcal{K}_w} \sup_K G_\lambda$ is finite for any $\lambda \in \Lambda$.

Theorem 12. *Assume that **(H₁)** (or **(H₂)**) and **(H₃)** hold. Then, we have*

- (1) *For almost all $\lambda \in \Lambda$, there exists a constant $k_0(\lambda) := k_0$ (depending only on λ) such that for each $\delta > 0$, there exists a $K \in \mathcal{K}_w$ such that*

$$\sup_K G_\lambda \leq a(\lambda) + \delta$$

and

$$\|u\| \leq k_0 \quad \text{whenever} \quad u \in K \quad \text{and} \quad G_\lambda(u) \geq a(\lambda) - \delta. \quad (24)$$

- (2) *For almost all $\lambda \in \Lambda$, there exists a bounded sequence $u_k(\lambda) \in E$ such that*

$$\|G'_\lambda(u_k)\| \rightarrow 0, \quad G_\lambda(u_k) \rightarrow a(\lambda) := \inf_{K \in \mathcal{K}_w} \sup_K G_\lambda, \quad \text{as } k \rightarrow \infty.$$

Corollary 13. *The conclusions of Theorem 12 hold if we replace hypothesis **(H₃)** with **(H'₃)**. There is a weak sandwich pair A, B such that*

$$a_\lambda := \sup_A G_\lambda < \infty, \quad b_\lambda := \inf_B G_\lambda > -\infty \quad (25)$$

for each $\lambda \in \Lambda$. Thus, for a.e. $\lambda \in \Lambda$, there is a bounded sequence $\{u_k\} \subset E$ such that

$$G_\lambda(u_k) \rightarrow c_\lambda, \quad \|G'_\lambda(u_k)\| \rightarrow 0, \quad (26)$$

where c_λ satisfies $b_\lambda \leq c_\lambda \leq a_\lambda$.

4. The system

Let $\mathcal{A}, \mathcal{B}$ be positive, self-adjoint operators on $L^2(\Omega)$ with compact resolvents, where $\Omega \subset \mathbb{R}^n$. Let $F(x, v, w)$ be a Carathéodory function on $\Omega \times \mathbb{R}^2$ such that

$$f(x, v, w) = \partial F / \partial v, \quad g(x, v, w) = \partial F / \partial w \quad (27)$$

are also Carathéodory functions satisfying

$$|f(x, v, w)| + |g(x, v, w)| \leq C_0(|v| + |w| + 1), \quad v, w \in \mathbb{R}. \quad (28)$$

We wish to solve the system

$$-\mathcal{A}v = f(x, v, w) \quad (29)$$

$$\lambda \mathcal{B}w = g(x, v, w). \quad (30)$$

(The reason for the negative sign is that a positive sign leads to a simple minimization problem when $\lambda > 0$.) Let $\lambda_0(\mu_0)$ be the lowest eigenvalue of $\mathcal{A}(\mathcal{B})$. These are assumed positive.

Our first result is

Theorem 14. *Assume*

$$2F(x, s, 0) \geq -\lambda_0 s^2 - W_1(x), \quad x \in \Omega, s \in \mathbb{R}, \quad (31)$$

and

$$2F(x, s, t) \leq \lambda(x)s^2 + \mu(x)t^2 + W_2(x), \quad x \in \Omega, s, t \in \mathbb{R}, \quad (32)$$

where $W_i(x) \in L^1(\Omega)$ and

$$\lambda(x) \leq \lambda_0, \quad \mu(x) \leq \mu_0, \quad x \in \Omega. \quad (33)$$

Then, the system (29), (30) has a solution for a.e. $\lambda \geq \beta/\mu_0$, where $\beta = \sup \mu(x)$.

Proof. Let $D = D(\mathcal{A}^{1/2}) \times D(\mathcal{B}^{1/2})$. Then, D becomes a Hilbert space with norm given by

$$\|u\|_D^2 = (\mathcal{A}v, v) + (\mathcal{B}w, w), \quad u = (v, w) \in D. \quad (34)$$

We define

$$G_\lambda(u) = \lambda b(w) - a(v) - 2 \int_{\Omega} F(x, v, w) dx, \quad u \in D \quad (35)$$

where

$$a(v) = (\mathcal{A}v, v), \quad b(w) = (\mathcal{B}w, w). \quad (36)$$

Then, $G_\lambda \in C^1(D, \mathbb{R})$ and

$$(G'_\lambda(u), h)/2 = \lambda b(w, h_2) - a(v, h_1) - (f(u), h_1) - (g(u), h_2) \quad (37)$$

where we write $f(u), g(u)$ in place of $f(x, v, w), g(x, v, w)$, respectively. It is readily seen that the system (29), (30) is equivalent to

$$G'_\lambda(u) = 0. \quad (38)$$

We let N be the set of those $(v, 0) \in D$ and M the set of those $(0, w) \in D$. Then, M, N are orthogonal closed subspaces such that

$$D = M \oplus N. \quad (39)$$

If we define

$$L_\lambda u = 2(-v, \lambda w), \quad u = (v, w) \in D \quad (40)$$

then L_λ is a self-adjoint bounded operator on D . Also

$$G'_\lambda(u) = L_\lambda u + c_0(u) \quad (41)$$

where

$$c_0(u) = -(A^{-1}f(u), B^{-1}g(u)) \quad (42)$$

is compact on D . This follows from (28) and the fact that $\mathcal{A}$ and $\mathcal{B}$ have compact resolvents. It also follows that G'_λ has weak-to-weak continuity. For if $u_k \rightarrow u$ weakly, then $L_\lambda u_k \rightarrow L_\lambda u$ weakly and $c_0(u_k)$ has a convergent subsequence. Now by (32)

$$G_\lambda(0, w) \geq \lambda b(w) - \int_{\Omega} \mu(x)w^2 + W_2(x) dx, \quad (0, w) \in M. \quad (43)$$

Thus,

$$\inf_M G_\lambda \geq - \int_{\Omega} W_2(x) dx \equiv b_0. \quad (44)$$

On the other hand, (31) implies

$$G_\lambda(v, 0) \leq -a(v) + \lambda_0 \|v\|^2 + \int_{\Omega} W_1(x) \, dx, \quad (v, 0) \in N. \quad (45)$$

Thus,

$$\sup_N G_\lambda \leq \int_{\Omega} W_1(x) \, dx \equiv a_0. \quad (46)$$

Define

$$I(u) = b(w), \quad J(u) = -a(v) - 2 \int_{\Omega} F(x, v, w) \, dx, \quad u \in D. \quad (47)$$

Then, $G_1(u) = I(u) + J(u)$ and

$$I(u) - J(u) = b(w) + a(v) + 2 \int_{\Omega} F(x, v, w) \, dx, \quad u \in D.$$

Thus,

$$I(u) - J(u) \rightarrow \infty \text{ as } \|u\|_D \rightarrow \infty. \quad (48)$$

To see this, let N' be the orthogonal complement of $N_0 = E(\lambda_0)$ in N . Then, $N = N' \oplus N_0$. Let $M_0 = E(\mu_0)$, and let M' be its orthogonal complement in M . Assume (48) is not true. Then, there would be a sequence $\{v_k, w_k\}$ such that $a(v_k) + b(w_k) \rightarrow \infty$ and $b(w_k) - \int_{\Omega} \mu(x) w_k^2 + a(v_k) - \int_{\Omega} \lambda(x) v_k^2$ is bounded. Write $v_k = v'_k + y_k$, $v'_k \in N'$, $y_k \in N_0$ and $w_k = w'_k + h_k$, $w'_k \in M'$, $h_k \in M_0$. If $b_k^2 = b(w_k) \rightarrow \infty$, let $\tilde{w}_k = w_k/b_k$. Then, $b(\tilde{w}_k) = 1$, and there is a renamed subsequence such that $\tilde{w}_k \rightarrow \tilde{w}$, $\tilde{w}'_k \rightarrow \tilde{w}'$, $\tilde{h}_k \rightarrow \tilde{h}$ weakly in D , strongly in $L^2(\Omega)$, and a.e. in Ω . Then,

$$\begin{aligned} \left[b(w_k) - \int_{\Omega} \mu(x) w_k^2 \right] / b_k^2 &= 1 - \int_{\Omega} \mu(x) \tilde{w}_k(x)^2 \\ &\rightarrow 1 - \int_{\Omega} \mu(x) \tilde{w}(x)^2 \\ &= [1 - b(\tilde{w})] + [b(\tilde{w}) - \mu_0 \|\tilde{w}\|^2] \\ &\quad + \int_{\Omega} [\mu_0 - \mu(x)] \tilde{w}^2 \\ &= A + B + C, \end{aligned}$$

with A, B, C nonnegative. If $A = 0$, then $b(\tilde{w}) = 1$. If $B = 0$ also, then $\tilde{w}' = 0$, $\tilde{w} = \tilde{h}$. If, in addition, $C = 0$, then

$$\int_{\Omega} [\mu_0 - \mu(x)] \tilde{h}^2 = 0.$$

Consequently, $\tilde{h} = 0$ on a set of positive measure. By hypothesis, $\tilde{h} \equiv 0$. This means that $\tilde{w} \equiv 0$, contradicting the fact that $b(\tilde{w}) = 1$. Thus, $b(w_k) \rightarrow \infty$ implies

$$b(w_k) - \int_{\Omega} \mu(x) w_k^2 \rightarrow \infty.$$

Similarly, $a(v_k) \rightarrow \infty$ implies

$$a(v_k) - \int_{\Omega} \lambda(x) v_k^2 \rightarrow \infty.$$

Thus, (48) holds.

We can now apply Corollary 13 to conclude that for a.e. $\lambda \geq \beta/\mu_0$, there is a bounded sequence $\{u_k\} \subset D$ such that (26) holds. Once this is known, we can use the usual procedures to show that there is a renamed subsequence such that $u_k \rightarrow u$ in D , and u satisfies (38). $\square$

Theorem 15. *In addition, assume that the eigenfunctions of λ_0 and μ_0 are bounded and $\neq 0$ a.e. in Ω , and there is a $q > 2$ such that*

$$\|w\|_q^2 \leq Cb(w), \quad w \in M. \quad (49)$$

Assume that for some $\delta > 0$,

$$2F(x, s, t) \leq \beta t^2 - \lambda_0 s^2, \quad |t| + |s| \leq \delta, \quad (50)$$

where $\beta = \sup \mu(x)$. Then, the system (29)–(30) has a nontrivial solution for a.e. $\lambda \geq \beta/\mu_0$.

Proof. Let N' be the orthogonal complement of $N_0 = \{\varphi_0\}$ in N , where φ_0 is the eigenfunction of A corresponding to λ_0 . Then, $N = N' \oplus N_0$. Let M_0 be the subspace of M spanned by the eigenfunctions of B corresponding to μ_0 , and let M' be its orthogonal complement in M . Since N_0 and M_0 are contained in $L^\infty(\Omega)$, there is a positive constant ρ such that

$$a(y) \leq \rho^2 \Rightarrow \|y\|_\infty \leq \delta/4, \quad y \in N_0 \quad (51)$$

$$b(h) \leq \rho^2 \Rightarrow \|h\|_\infty \leq \delta/4, \quad h \in M_0 \quad (52)$$

where δ is the number given in (50). If

$$a(y) \leq \rho^2, \quad b(w) \leq \rho^2, \quad |y(x)| + |w(x)| \geq \delta \quad (53)$$

we write $w = h + w'$, $h \in M_0$, $w' \in M'$ and

$$\delta \leq |y(x)| + |w(x)| \leq |y(x)| + |h(x)| + |w'(x)| \leq (\delta/2) + |w'(x)|. \quad (54)$$

Thus,

$$|y(x)| + |h(x)| \leq \delta/2 \leq |w'(x)| \quad (55)$$

and

$$|y(x)| + |w(x)| \leq 2|w'(x)|. \quad (56)$$

Now by (50) and (56)

$$G_\lambda(y, w) = \lambda b(w) - a(y) - 2 \int_{\Omega} F(x, y, w) dx \quad (57)$$

$$\geq \lambda b(w) - a(y) - \int_{|y|+|w|<\delta} \{\beta w^2 - \lambda_0 y^2\} dx \quad (58)$$

$$- c_0 \int_{|y|+|w|>\delta} (|y| + |w| + 1)^2 dx \quad (59)$$

$$\geq \lambda b(w) - a(y) - \beta \|w\|^2 + \lambda_0 \|y\|^2 - c_1 \int_{2|w'|>\delta} |w'|^q dx \quad (60)$$

$$\geq \lambda b(w') - \beta \|w'\|^2 - c_2 b(w')^{q/2} \quad (61)$$

$$\geq \left(\lambda - \frac{\beta}{\mu_1} - c_2 b(w')^{(q/2)-1} \right) b(w'), \quad a(y) \leq \rho^2, \quad b(w) \leq \rho^2 \quad (62)$$

where μ_1 is the next eigenvalue of B after μ_0 . If we reduce ρ accordingly, we can find a positive constant ν such that

$$G_\lambda(y, w) \geq \nu b(w'), \quad a(y) \leq \rho^2, \quad b(w) \leq \rho^2. \quad (63)$$

I claim that either (29) (30) has a nontrivial solution or there is an $\epsilon > 0$ such that

$$G_\lambda(y, w) \geq \epsilon, \quad a(y) + b(w) = \rho^2. \quad (64)$$

If suppose (64) did not hold, then there would be a sequence $\{y_k, w_k\}$ such that $a(y_k) + b(w_k) = \rho^2$ and $G_\lambda(y_k, w_k) \rightarrow 0$. If we write $w_k = w'_k + h_k$, $w'_k \in M'$, $h_k \in M_0$, then (63) tells us that $b(w'_k) \rightarrow 0$. Thus, $a(y_k) + b(h_k) \rightarrow \rho^2$. Since N_0, M_0 are finite dimensional, there is a renamed subsequence such that $y_k \rightarrow y$ in N_0 and $h_k \rightarrow h$ in M_0 . By (51) and (52), $\|y\|_\infty \leq \delta/4$ and $\|h\|_\infty \leq \delta/4$. Consequently, (50) implies

$$2F(x, y, h) \leq \lambda \mu_0 h^2 - \lambda_0 y^2. \quad (65)$$

Since

$$G_\lambda(y, h) = \lambda b(h) - a(y) - 2 \int_{\Omega} F(x, y, h) dx = 0 \quad (66)$$

we have

$$\int_{\Omega} \{2F(x, y, h) + \lambda_0 y^2 - \lambda \mu_0 h^2\} dx = 0. \quad (67)$$

In view of (65), this implies

$$2F(x, y, h) \equiv \lambda \mu_0 h^2 - \lambda_0 y^2. \quad (68)$$

For $\zeta \in C_0^\infty(\Omega)$ and $t > 0$ small, we have

$$2[F(x, y + t\zeta, h) - F(x, y, h)]/t \leq -\lambda_0[(y + t\zeta)^2 - y^2]/t. \quad (69)$$

Taking $t \rightarrow 0$, we have

$$f(x, y, h)\zeta \leq -\lambda_0 y\zeta. \quad (70)$$

Since this is true for all $\zeta \in C_0^\infty(\Omega)$, we have

$$f(x, y, h) = -\lambda_0 y = -Ay. \quad (71)$$

Similarly,

$$2[F(x, y, h + t\zeta) - F(x, y, h)]/t \leq \lambda\mu_0[(h + t\zeta)^2 - h^2]/t \quad (72)$$

and consequently

$$g(x, y, h)\zeta \leq \lambda\mu_0 h\zeta \quad (73)$$

and

$$g(x, y, h) = \lambda\mu_0 h = \lambda Bh \quad (74)$$

We see from (71) and (74) that (29) (30) has a nontrivial solution. Thus, we may assume that (64) holds. Next, we note that there is an $\varepsilon > 0$ depending on ρ and λ such that

$$G_\lambda(0, w) \geq \varepsilon, \quad b(w) \geq \rho > 0, \quad w \in M.$$

To see this, suppose that $\{w_k\} \subset M$ is a sequence such that

$$G_\lambda(0, w_k) \rightarrow 0, \quad b(w_k) \geq \rho.$$

If

$$b_k^2 = b(w_k) \leq C,$$

this implies

$$b(w_k) - \mu_0 \|w_k\|^2 \rightarrow 0$$

and

$$\int [\lambda\mu_0 - \mu(x)] w_k^2 dx \rightarrow 0,$$

since

$$G_\lambda(0, w) \geq \lambda[b(w) - \mu_0 \|w\|^2] + \int [\lambda\mu_0 - \mu(x)] w^2 dx, \quad w \in M.$$

If we write $w_k = w'_k + h_k$, $w'_k \in M'$, $h_k \in M_0$ as before, then this tells us that $b(w'_k) \rightarrow 0$. Since M_0 is finite dimensional, there is a renamed subsequence such that $h_k \rightarrow h$. But the two conclusions above tell us that $h = 0$. Since $b(h) \geq \rho$, we see that $\varepsilon > 0$ exists for any constant C . If the sequence $\{b_k\}$ is not bounded, we take $\tilde{w}_k = w_k/b_k$. Then,

$$G_\lambda(0, w_k)/b_k^2 \geq \lambda[b(\tilde{w}_k) - \mu_0 \|\tilde{w}_k\|^2] + \int [\lambda\mu_0 - \mu(x)] \tilde{w}_k^2 dx,$$

and the inequality is true in this case as well. Next we note that there is a $\nu > 0$ such that

$$G_\lambda(0, w) \geq \nu b(w), \quad w \in M. \quad (75)$$

Assuming this for the moment, we see that

$$\inf_B G_\lambda \geq \varepsilon_1 > 0 \quad (76)$$

where

$$B = \{w \in M : b(w) \geq \rho^2\} \cup \{u = (s\varphi_0, w) : s \geq 0, w \in M, \|u\|_D = \rho\}, \quad (77)$$

and $\varepsilon_1 = \min\{\varepsilon, \nu\rho^2\}$. By (46), there is an $R > \rho$ such that

$$\sup_A G_\lambda = a_\lambda < \infty, \quad (78)$$

where $A = N$. By Corollary 11, A, B form a weak sandwich pair. Hence, for a.e. $\lambda > \beta/\mu_0$, there is a bounded sequence $\{u_k\} \subset D$ such that (26) holds with $c_\lambda \geq \varepsilon_1$. Arguing as in the proof of Theorem 14, we see that there is a $u \in D$ such that $G_\lambda(u) = c_\lambda \geq \varepsilon_1 > 0$, $G'_\lambda(u) = 0$. Since $c_\lambda \neq 0$ and $G_\lambda(0) = 0$, we see that $u \neq 0$, and we have a nontrivial solution of the system (29) (30).

It therefore remains only to prove (75). Clearly $\nu \geq 0$. If $\nu = 0$, then there is a sequence $\{w_k\} \subset M$ such that

$$G_\lambda(0, w_k) \rightarrow 0, \quad b(w_k) = 1. \quad (79)$$

Thus, there is a renamed subsequence such that $w_k \rightarrow w$ weakly in M , strongly in $L^2(\Omega)$ and a.e. in Ω . Consequently,

$$\int_{\Omega} [\lambda\mu_0 - \mu(x)] w_k^2 dx \leq \lambda - \int_{\Omega} \mu(x) w_k^2 dx \leq G_\lambda(0, w_k) \rightarrow 0 \quad (80)$$

and

$$\lambda = \int_{\Omega} \mu(x) w^2 dx \leq \lambda\mu_0 \|w\|^2 \leq \lambda b(w) \leq \lambda \quad (81)$$

which means that we have equality throughout. It follows that we must have $w \in E(\mu_0)$, the eigenspace of μ_0 . Since $w \not\equiv 0$, we have $w \neq 0$ a.e. But

$$\int_{\Omega} [\lambda\mu_0 - \mu(x)] w^2 dx = 0 \quad (82)$$

implies that the integrand vanishes identically on Ω , and consequently $\beta = \lambda\mu_0$, violating the hypothesis of the theorem. This establishes (75) and completes the proof of the theorem. $\square$

5. Finding the sequences

We proceed to the proof of Theorem 4. Let $M = C_0 + 1$. Then,

$$\|\sigma(1)v\| \leq M$$

whenever $\sigma \in \Sigma_w$ satisfies $\|\sigma'(t)\| \leq 1$ and $v \in E$ satisfies $\|v\| \leq C_0$. If the theorem were false, then there would be a $\delta > 0$ such that

$$\|G'(u)\| \geq 3\delta \quad (83)$$

when

$$u \in \tilde{E} = \{u \in E : \|u\| \leq M + 1, |G(u) - a| \leq 3\delta\}. \quad (84)$$

Take $\delta < 1/3$. For $u \in \tilde{E}$, let $q(u) = G'(u)/\|G'(u)\|$. Then, by (83)

$$(G'(u), q(u)) \geq 2\delta, \quad u \in \tilde{E}. \quad (85)$$

For each $u \in \tilde{E}$, there is a E_w neighborhood $W(u)$ of u such that

$$(G'(h), q(u)) > \delta, \quad h \in W(u) \cap \tilde{E}. \quad (86)$$

For otherwise, there would be a sequence $\{h_k\} \subset \tilde{E}$ such that

$$|h_k - u|_w \rightarrow 0 \quad \text{and} \quad (G'(h_k), q(u)) \leq \delta. \quad (87)$$

Since $\tilde{E}$ is bounded in E , $Ph_k \rightarrow Pu$ weakly in N and $(I - P)h_k \rightarrow (I - P)u$ strongly in M . Hence, by hypothesis,

$$(G'(h_k), q(u)) \rightarrow (G'(u), q(u)) \geq 2\delta$$

in view of (85). This contradicts (87). Let $\tilde{E}_w$ be the set $\tilde{E}$ with the inherited topology of E_w . It is a metric space and $W(u) \cap \tilde{E}$ is an open set in this space. Thus, $\{W(u) \cap \tilde{E}\}, u \in \tilde{E}$, is an open covering of the paracompact space $\tilde{E}_w$. Consequently, there is a locally finite refinement $\{W_\tau\}$ of this cover. For each τ , there is an element u_τ such that $W_\tau \subset W(u_\tau)$. Let $\{\psi_\tau\}$ be a partition of unity subordinate to

this covering. Each ψ_τ is locally Lipschitz continuous with respect to the norm $|u|_w$ and consequently with respect to the norm of E . Let

$$Y(u) = \sum \psi_\tau(u)q(u_\tau), \quad u \in E. \quad (88)$$

Then, $Y(u)$ is locally Lipschitz continuous with respect to both norms. Moreover,

$$\|Y(u)\| \leq \sum \psi_\tau(u)\|q(u_\tau)\| \leq 1 \quad (89)$$

Let

$$\begin{aligned} Q_0 &= \{u \in E : \|u\| \leq M + 1, |G(u) - a| \leq 2\delta\}, \\ Q_1 &= \{u \in E : \|u\| \leq M, |G(u) - a| \leq \delta\}, \\ Q_2 &= E \setminus Q_0, \\ \eta(u) &= d(u, Q_2)/[d(u, Q_1) + d(u, Q_2)]. \end{aligned}$$

It is easily checked that $\eta(u)$ is locally Lipschitz continuous on E (in both norms) and satisfies

$$\begin{cases} \eta(u) = 1, & u \in Q_1, \\ \eta(u) = 0, & u \in Q_2, \\ \eta(u) \in (0, 1), & \text{otherwise.} \end{cases} \quad (90)$$

Let

$$W(u) = -\eta(u)Y(u).$$

Then,

$$\|W(u)\| \leq 1, \quad u \in E.$$

By Theorem 4.5 of [19], for each $v \in E$ there is a unique solution $\sigma(t)v$ of

$$\sigma'(t) = W(\sigma(t)), \quad t \in \mathbb{R}^+, \quad \sigma(0) = v. \quad (91)$$

We have

$$\begin{aligned} dG(\sigma(t)v)/dt &= -\eta(\sigma(t)v)(G'(\sigma(t)v), Y(\sigma(t)v)) \\ &\leq -\theta\eta(\sigma)\|G'(\sigma)\| \\ &\leq -3\theta\delta\eta(\sigma). \end{aligned} \quad (92)$$

We note that $\sigma(t)$ is locally Lipschitz continuous with respect to the weak norm. Since

$$\sigma(t)u - \sigma(t)u_0 = u - u_0 + \int_0^t [W(\sigma(s)u) - W(\sigma(s)u_0)] \, ds,$$

we have

$$|\sigma(t)u - \sigma(t)u_0|_w \leq |u - u_0|_w + \int_0^t |W(\sigma(s)u) - W(\sigma(s)u_0)|_w \, ds,$$

which implies

$$|\sigma(t)u - \sigma(t)u_0|_w \leq |u - u_0|_w + C \int_0^t |\sigma(s)u - \sigma(s)u_0|_w \, ds$$

and

$$|\sigma(t)u - \sigma(t)u_0|_w \leq C|u - u_0|_w$$

for $|u - u_0|_w$ sufficiently small. The other properties are easily checked. Now $K = \tilde{\sigma}(1)N$, where $\tilde{\sigma} \in \Sigma_w$. By Lemma 2, $\sigma \circ \tilde{\sigma} \in \Sigma_w$. Consequently, $\tilde{K} = \sigma(1)K \in \mathcal{K}_w$, and

$$G(w) < a - \delta, \quad w \in \tilde{K}. \quad (93)$$

But this contradicts the definition (13) of a . Thus, (83) cannot hold for u satisfying (84). This proves the theorem.

Proof of Theorem 5. Since $A \in \mathcal{K}_w$, clearly $a \leq a_0$. Moreover, for any $K \in \mathcal{K}_w$, we have

$$b_0 = \inf_B G \leq \inf_{B \cap K} G \leq \sup_{B \cap K} G \leq \sup_K G.$$

Hence, $b_0 \leq a$. Thus, a is finite. Apply Theorem 4. $\square$

Proof of Theorem 6. Let $B = F^{-1}(p)$. If we can show that B satisfies (18), then the result will follow from Theorem 5. Now (18) is equivalent to

$$F^{-1}(p) \cap \sigma(1)N \neq \emptyset, \quad \sigma \in \Sigma_w. \quad (94)$$

Let $\Omega_R(p)$ be a ball in N with radius R and center p , and let $\sigma(t)$ be any flow in Σ_w . Since

$$\sigma(t)u - u = \int_0^t \sigma'(\tau)u \, d\tau, \quad (95)$$

we have

$$\|\sigma(t)u - \sigma(s)u\| \leq C|t - s|.$$

If $u \in A_R = \partial\Omega_R(p)$, and $v \in B$, we have

$$h(s) := d(\sigma(s)u, B) \leq \|\sigma(s)u - v\| \leq \|\sigma(t)u - v\| + C|t - s|.$$

This implies,

$$h(s) \leq h(t) + C|t - s|. \quad (96)$$

Moreover, by Lemmas 4.3 and 4.8 of [19], $h(s)$ satisfies

$$h(s) \geq m(R) \rightarrow \infty \text{ as } R \rightarrow \infty, \quad 0 \leq s \leq 1, \quad u \in A_R = \partial\Omega_R(p).$$

Thus,

$$\|\sigma(s)u - F^{-1}(p)\| \geq h(s) \geq m(R) \rightarrow \infty, \quad u \in A_R.$$

Consequently,

$$F^{-1}(p) \cap \sigma(t)A_R = \emptyset, \quad \sigma \in \Sigma_w, \quad t \in I, \quad (97)$$

for R sufficiently large. Since $\overline{\Omega_R(p)}$ is compact in E_w , there is a finite dimensional subspace $S \subset E$ such that

$$\sigma(t)v - v \in S, \quad v \in \overline{\Omega_R(p)}.$$

Enlarge S to contain p and satisfy $FS \neq \{0\}$. By hypothesis, there is a finite dimensional subspace $S_0 \neq \{0\}$ of N containing p such that

$$v \in S_0, \quad w \in S \implies F(v + w) \in S_0. \quad (98)$$

Take $w = \sigma(t)v - v \in S$. Then,

$$F\sigma(t)v \in S_0, \quad v \in \overline{\Omega_R(p)} \cap S_0.$$

Let

$$\varphi_t(v) = \varphi(v, t) = F\sigma(t)v, \quad v \in \overline{\Omega_R(p)}, \quad 0 \leq t \leq 1. \quad (99)$$

Then, $\varphi(v, t)$ maps $\overline{\Omega_R(p)} \cap S_0 \times [0, 1]$ into S_0 . Thus,

$$\varphi_t(v) \neq 0, \quad v \in \partial(\overline{\Omega_R(p)} \cap S_0) = \partial\overline{\Omega_R(p)} \cap S_0, \quad 0 \leq t \leq 1, \quad (100)$$

in view of (97). Consequently, the Brouwer degree $d(\varphi_t, \Omega_R(p) \cap S_0, p)$ can be defined. Since $\varphi_t(v)$ is continuous, we have

$$d(\varphi_1, \Omega_R(p) \cap S_0, p) = d(\varphi_0, \Omega_R(p) \cap S_0, p) = d(I, \Omega_R(p) \cap S_0, p) = 1.$$

Hence, there is a $v \in \Omega_R(p)$ such that $F\sigma(1)v = p$. Consequently, $\sigma(1)v \in B$. Thus, (94) holds. This completes the proof. $\square$

Proof of Corollary 10. Take $F = P$, the projection onto N in Theorem 6. Note that $F = P$ satisfies the hypotheses of Corollary 7, and consequently the hypotheses of Theorem 6. $\square$

6. The monotonicity trick

We now give the proof of Theorem 12.

Proof. We prove conclusion (1) assuming the first alternative hypothesis $(\mathbf{H}_1)$.

By $(\mathbf{H}_1)$, the map $\lambda \mapsto a(\lambda)$ is nondecreasing. Hence, $a'(\lambda) := da(\lambda)/d\lambda$ exists for almost every $\lambda \in \Lambda$. From this point on, we consider those λ where $a'(\lambda)$ exists. For fixed $\lambda \in \Lambda$, let $\lambda_n \in (\lambda, 2\lambda) \cap \Lambda$, $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$. Then, there exists $\bar{n}(\lambda)$ such that

$$a'(\lambda) - 1 \leq \frac{a(\lambda_n) - a(\lambda)}{\lambda_n - \lambda} \leq a'(\lambda) + 1 \quad \text{for } n \geq \bar{n}(\lambda). \quad (101)$$

Next, we note that there exist $K_n \in \mathcal{K}_Q$, $k_0 := k_0(\lambda) > 0$ such that

$$\|u\| \leq k_0 \quad \text{whenever} \quad G_\lambda(u) \geq a(\lambda) - (\lambda_n - \lambda). \quad (102)$$

In fact, by the definition of $a(\lambda_n)$, there exists K_n such that

$$\sup_{K_n} G_\lambda(u) \leq \sup_{K_n} G_{\lambda_n}(u) \leq a(\lambda_n) + (\lambda_n - \lambda). \quad (103)$$

If $G_\lambda(u) \geq a(\lambda) - (\lambda_n - \lambda)$ for some $u \in K_n$, then, by (101) and (103), we have that

$$\begin{aligned} I(u) &= \frac{G_{\lambda_n}(u) - G_\lambda(u)}{\lambda_n - \lambda} \\ &\leq \frac{a(\lambda_n) + (\lambda_n - \lambda) - a(\lambda) + (\lambda_n - \lambda)}{\lambda_n - \lambda} \\ &\leq a'(\lambda) + 3, \end{aligned} \quad (104)$$

and it follows that

$$\begin{aligned} J(u) &= \lambda_n I(u) - G_{\lambda_n}(u) \\ &\leq \lambda_n(a'(\lambda) + 3) - G_\lambda(u) \\ &\leq \lambda_n(a'(\lambda) + 3) - a(\lambda) + (\lambda_n - \lambda) \\ &\leq 2\lambda(a'(\lambda) + 3) - a(\lambda) + \lambda. \end{aligned} \quad (105)$$

On the other hand, by $(\mathbf{H}_1)$, (101), and (103),

$$\begin{aligned} J(u) &= \lambda_n I(u) - G_{\lambda_n}(u) \\ &\geq -G_{\lambda_n}(u) \\ &\geq -(a(\lambda_n) + (\lambda_n - \lambda)) \\ &\geq -(a(\lambda) + (\lambda_n - \lambda)(a'(\lambda) + 2)) \\ &\geq -a(\lambda) - \lambda|a'(\lambda) + 2|. \end{aligned} \quad (106)$$

Combining (104)–(106), and $(\mathbf{H}_1)$, we see that there exists $k_0(\lambda) := k_0$ (depending only on λ) such that (102) holds.

By the choice of K_n and (2.1), we see that

$$\begin{aligned} G_\lambda(u) &\leq G_{\lambda_n}(u) \\ &\leq \sup_{K_n} G_{\lambda_n}(u) \\ &\leq a(\lambda_n) + (\lambda_n - \lambda) \\ &\leq (a'(\lambda) + 1)(\lambda_n - \lambda) + a(\lambda) + (\lambda_n - \lambda) \\ &\leq a(\lambda) + (a'(\lambda) + 2)(\lambda_n - \lambda) \end{aligned}$$

for all $u \in K_n$. We take n sufficiently large to ensure that $|a'(\lambda) + 2|(\lambda_n - \lambda) < \delta$. This proves conclusion (1). Conclusion (2) now follows from Theorem 4. The proof under hypothesis $(\mathbf{H}_2)$ is similar is omitted. $\square$

Proof of Corollary 13. Use Theorem 5. $\square$

7. Contrast

We now exhibit theorems corresponding to Theorems 14 and 15 which require the solvability of system (29), (30) for $\lambda = 1$.

Theorem 16. *In addition to (31) and (32), assume*

$$f(x, ty, tz)/t \rightarrow \alpha_+(x)v^+ - \alpha_-(x)v^- + \beta_+(x)w^+ - \beta_-(x)w^- \quad (107)$$

$$g(x, ty, tz)/t \rightarrow \gamma_+(x)v^+ - \gamma_-(x)v^- + \delta_+(x)w^+ - \delta_-(x)w^- \quad (108)$$

as $t \rightarrow +\infty, y \rightarrow v, z \rightarrow w$, where $a^\pm = \max(\pm a, 0)$. We also assume that the only solution of

$$-\mathcal{A}v = \alpha_+v^+ - \alpha_-v^- + \beta_+w^+ - \beta_-w^- \quad (109)$$

$$\mathcal{B}w = \gamma_+v^+ - \gamma_-v^- + \delta_+w^+ - \delta_-w^- \quad (110)$$

is $v = w = 0$. Then, the system (6), (7) has a solution.

Proof. We follow the proof of Theorem 14 for the case $\lambda = 1$. We conclude that there is a sequence $\{u_k\} \subset D$ such that

$$G_1(u_k) \rightarrow c_1, \quad \|G'_1(u_k)\| \rightarrow 0. \quad (111)$$

Let $u_k = (v_k, w_k)$. I claim that

$$\rho_k^2 = a(v_k) + b(w_k) \leq C. \quad (112)$$

To see this, assume that $\rho_k \rightarrow \infty$, and let $\tilde{u}_k = u_k/\rho_k$. Then, there is a renamed subsequence such that $\tilde{u}_k \rightarrow \tilde{u}$ weakly in D , strongly in $L^2(\Omega)$ and a.e. in Ω . If $h = (h_1, h_2) \in D$, then

$$(G'(u_k), h)/\rho_k = 2b(\tilde{w}_k, h_2) - 2a(\tilde{v}_k, h_1) - 2(f(u_k), h_1)/\rho_k - 2(g(u_k), h_2)/\rho_k. \quad (113)$$

Taking the limit and applying (28), (107), and (108), we see that $\tilde{u} = (\tilde{v}, \tilde{w})$ is a solution of (109) and (110). Hence, $\tilde{u} = 0$ by hypothesis. On the other hand, since $a(\tilde{v}_k) + b(\tilde{w}_k) = 1$, there is a renamed subsequence such that $a(\tilde{v}_k) \rightarrow \tilde{a}$, $b(\tilde{w}_k) \rightarrow \tilde{b}$ with $\tilde{a} + \tilde{b} = 1$. Thus, by (107), (108), and (37)

$$\begin{aligned} (G'(u_k), (\tilde{v}_k, 0))/2\rho_k &= -a(\tilde{v}_k) - (f(u_k), \tilde{v}_k)/\rho_k \\ &\rightarrow -\tilde{a} - \int_{\Omega} (\alpha_+ \tilde{v}^+ - \alpha_- \tilde{v}^- + \beta_+ \tilde{w}^+ - \beta_- \tilde{w}^-) \tilde{v} \, dx \end{aligned}$$

and

$$\begin{aligned} (G'(u_k), (0, \tilde{w}_k))/2\rho_k &= b(\tilde{w}_k) - (g(u_k), \tilde{w}_k)/\rho_k \\ &\rightarrow \tilde{b} - \int_{\Omega} (\gamma_+ \tilde{v}^+ - \gamma_- \tilde{v}^- + \delta_+ \tilde{w}^+ - \delta_- \tilde{w}^-) \tilde{w} \, dx. \end{aligned}$$

Thus, by (111),

$$\tilde{a} = - \int_{\Omega} (\alpha_+ \tilde{v}^+ - \alpha_- \tilde{v}^- + \beta_+ \tilde{w}^+ - \beta_- \tilde{w}^-) \tilde{v} \, dx \quad (114)$$

and

$$\tilde{b} = \int_{\Omega} (\gamma_+ \tilde{v}^+ - \gamma_- \tilde{v}^- + \delta_+ \tilde{w}^+ - \delta_- \tilde{w}^-) \tilde{w} \, dx. \quad (115)$$

Since one of the two numbers $\tilde{a}, \tilde{b}$ is not zero, we see that we cannot have $\tilde{u} \equiv 0$. This contradiction proves (112). Once this is known we can use the usual procedures to show that there is a renamed subsequence such that $u_k \rightarrow u$ in D , and u satisfies (38). $\square$

Theorem 17. *In addition, assume that the eigenfunctions of λ_0 and μ_0 are bounded and $\neq 0$ a.e. in Ω , and there is a $q > 2$ such that*

$$\|w\|_q^2 \leq Cb(w), \quad w \in M. \quad (116)$$

Assume that for some $\delta > 0$,

$$2F(x, s, t) \leq \mu_0 t^2 - \lambda_0 s^2, \quad |t| + |s| \leq \delta. \quad (117)$$

Then, the system (6), (7) has a nontrivial solution.

Proof. We follow the proof of Theorem 15 for the case $\lambda = 1$. We have

$$\inf_B G \geq \varepsilon_1 > 0 \quad (118)$$

where

$$B = \{w \in M : b(w) \geq \rho^2\} \cup \{u = (s\varphi_0, w) : s \geq 0, w \in M, \|u\|_D = \rho\}, \quad (119)$$

and $\varepsilon_1 = \min\{\varepsilon, \nu\rho^2\}$. By (46), there is an $R > \rho$ such that

$$\sup_A G = a_0 < \infty, \quad (120)$$

where $A = N$. By Proposition 11, A, B form a weak sandwich pair. Moreover, G satisfies (17) with $\varepsilon_1 \geq b_0$. Hence, there is a sequence $\{u_k\} \subset D$ such that (19) holds with $c \geq \varepsilon_1$. Arguing as in the proof of Theorem 16, we see that there is a $u \in D$ such that $G(u) = c \geq \varepsilon_1 > 0$, $G'(u) = 0$. Since $c \neq 0$ and $G(0) = 0$, we see that $u \neq 0$, and we have a nontrivial solution of the system (6), (7). $\square$

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