Non-abelian Vortices: Existence, Uniqueness and Asymptotics

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Abstract. We provide a unified approach in the (analytical) study of non-abelian vortices for the models proposed in [4, 7]. We also obtain the necessary estimates to describe the asymptotic behavior of a vortex configuration under a regime of physical interest.

Mathematics Subject Classification (2010). Primary 35J20, 35J50, 35J60; Secondary 58J38.

Keywords. Non-abelian vortices, variational methods, elliptic systems.

1. Introduction

In recent years much attention has been devoted to the understanding of non-abelian vortices, also in connection with the delicate issue of confinement. In this respect several models have been proposed (see [1, 2, 5, 8, 10, 14, 15, 17] and references therein) and analyzed in relation with the well understood abelian-Higgs or $U(1)$ -model and the corresponding ANO-vortices [9].

Non-abelian vortices concern models that include both abelian and non abelian gauge fields, which may be considered in the framework of a general $G \times U(1)$ gauge theory (typically $G = SO(2N)$ or $G = USp(2N)$) as in [7, 2, 3, 13, 16], or embedded in a supersymmetric theory as in [4, 5, 8].

In all cases, the “essential” structure of the abelian model is preserved, and for a special choice of the physical parameters, the second order field equations decouple into first order (selfdual) equations of the BPS-type in matrix form. The desired (non-abelian) vortices are obtained as the static bi-dimensional solutions of the BPS-equations.

However, in contrast to the abelian situation, the non-abelian BPS-vortex equations are difficult to solve analytically, and beside some numerical result, they have been handled only under some a priori ansatz on the structure of the solution (see e.g. [3], [4], [7]).

This work was completed with the support of the FIRB-Ideas project: Analysis and Beyond.

Here, we shall pursue further the analytical study of non-abelian vortices and provide a unified approach that includes in particular the models introduced in [4, 5, 6, 8] and in [7, 2, 3, 13, 16].

In this way we obtain a simple and direct proof of the existence and uniqueness results in [11, 12], in addition to some sharp a priori estimates. Those estimates will allow us to determine the asymptotic behavior of the non-abelian vortex solution for limiting values of certain relevant parameters, as useful in the physical applications.

In particular for the models in [4, 5, 6, 8], involving $SU(N)$ -gauge fields (see below), we shall be able to describe the asymptotic regime: $N \rightarrow \infty$.

2. Preliminaries

In this section we derive the elliptic system of equations that governs non-abelian BPS-vortices (satisfying suitable ansatz) relative to [4, 7].

We start with the model in [4] (see also [5, 6]), which is formulated in terms of two gauge fields $(W)_{\mu=0,1,2,3}$ and $(w)_{\mu=0,1,2,3}$ over the groups $U(N)$ and $U(1)$ respectively.

The corresponding N -Higgs fields can be identified, with a $N \times N$ complex matrix H (in the fundamental representation). The parameters involved are the $SU(N)$ coupling parameter g , the $U(1)$ coupling parameter e and the Fayet-Iliopoulos constant c , in terms of which the mass spectra m_g and m_e for the non-abelian and abelian fields are given respectively as follows:

$$m_g = g\sqrt{c} \text{ and } m_e = e\sqrt{2Nc}. \quad (2.1)$$

Using standard notation, where I_N denotes the $N \times N$ identity matrix and $< H >$ denotes the traceless part of the matrix H , the BPS-equations relative to static vortex (string) configurations (along the x_3 -axis) take the form:

$$\frac{1}{2}(D_1 H + iD_2 H) = 0, \quad F_{12} = \frac{m_g^2}{2c} < HH^\dagger >; \quad f_{12} = \frac{m_g^2}{2c} \text{Tr}(HH^\dagger - cI_N) \quad (2.2)$$

where: $D_j H = \partial_j H + i(W_j + w_j I_N)H$, $j = 1, 2$; the non-abelian flux $F_{12} = \partial_1 W_2 - \partial_2 W_1 + i[W_1, W_2]$ and the abelian flux $f_{12} = \partial_1 w_2 - \partial_2 w_1$. All the fields above are supposed to be nontrivial only with respect to the index $\mu = 1, 2$; see [4, 5] for details.

In spite of their “formal” analogy with the abelian case (cfr. [9]), the non-abelian equations (2.2) are much harder to handle analytically, and so far a (radial) solution is obtained only under the ansatz:

$$H = \sqrt{c} \text{diag}(h_1, h_2, \dots, h_2) \quad (2.3)$$

with $h_1 = h_1(x_1, x_2) \in \mathbb{C}$ and $0 < h_2 = h_2(x_1, x_2) \in \mathbb{R}$ two smooth functions satisfying (away from the zeroes of h_1):

$$\begin{cases} -\Delta \log |h_1|^2 = m_e^2 - \left(\frac{m_e^2}{N} + m_g^2 \frac{N-1}{N} \right) |h_1|^2 - \frac{N-1}{N} (m_e^2 - m_g^2) h_2^2 \\ -\Delta \log h_2^2 = m_e^2 - \left(m_e^2 \frac{N-1}{N} + \frac{m_g^2}{N} \right) h_2^2 - \frac{1}{N} (m_e^2 - m_g^2) |h_1|^2. \end{cases} \quad (2.4)$$

The complex function h_1 is allowed to vanish at an assigned set of (vortex) points $\{p_1, \dots, p_n\}$ (repeated with multiplicity), so that

$$\log |h_1|^2 - \sum_{j=1}^n \log |x - p_j|^2 \in C^\infty. \quad (2.5)$$

In this way one can check that, by taking $(W)_{\mu=1,2}$ and $(w)_{\mu=1,2}$ satisfying

$$\frac{1}{2}[(W_1 + w_1 I_N) + i(W_2 + w_2 I_N)] = -iS^{-1}\bar{\partial}S$$

with

$$cSS^\dagger = \text{diag} \left(\left(\frac{|h_1|^2}{\prod_j |x - p_j|^2} \right)^{-1}, h_2^{-2}, \dots, h_2^{-2} \right),$$

then (2.2) holds, (see [4]).

Furthermore we can recover the fluxes as follows:

$$\begin{cases} f_{1,2} = \frac{1}{2N} \Delta (\log |h_1|^2 + (N-1) \log h_2^2) \\ F_{1,2} = \frac{N-1}{2N} \Delta (\log |h_1|^2 - \log h_2^2) \text{diag} \left(1, -\frac{1}{N-1}, \dots, -\frac{1}{N-1} \right). \end{cases} \quad (2.6)$$

Obviously, when $N = 1$ the solution behaves exactly as an abelian vortex with the Higgs particle described by a complex function h satisfying (away from the vortex points):

$$-\Delta \log |h| = m_e^2 (1 - |h|^2). \quad (2.7)$$

A similar approach allows one to reduce the search of non-abelian vortices (in the x_3 -direction) for the model proposed in [7] to the study of an analogous elliptic system.

Indeed in [7] the authors show that, when the number of matter flavors equals $2N$, then it is still possible to describe (non-abelian) vortex configurations as solutions of a (self-dual) BPS equations analogous to (2.2), expressed in terms of the mixed color-flavor $2N \times 2N$ matrix q (representing the scalar quarks fields) and the abelian and non-abelian fluxes, see [7].

Actually in [7], to obtain a solution for such BPS-equations, the authors require that the quark fields only wind in the abelian subgroup $(U(1))^N$. In this way the matrix q takes the following diagonal form:

$$q = \text{diag}(\phi I_N, \psi I_N)$$

with $\phi = \phi(x_1, x_2)$ and $\psi = \psi(x_1, x_2)$ complex functions; and the gauge field $(A)_{\mu=1,2}$ takes the form:

$$A_j = a_j I_N + b_j T; \quad T = \text{diag}(I_N, -I_N)$$

with a_j and b_j real smooth functions, $j = 1, 2$. Thus, setting $a = a_1 + ia_2$ and $b = b_1 + ib_2$, the relative BPS-equations are satisfied provided that

$$\bar{\partial}\phi = \frac{i}{2}(a+b)\phi; \quad \bar{\partial}\psi = \frac{i}{2}(a-b)\psi \quad (2.8)$$

and

$$\begin{cases} \partial_1 a_2 - \partial_2 a_1 = -\frac{e^2}{4} \left(|\phi|^2 + |\psi|^2 - \frac{v_0^2}{N} \right) \\ \partial_1 b_2 - \partial_2 b_1 = -\frac{g^2}{4} (|\phi|^2 - |\psi|^2), \end{cases} \quad (2.9)$$

where again e and g denote respectively the abelian and non-abelian coupling parameter, and v_0^2 relates to the symmetry breaking parameter.

On the other hand, we can use (2.8) to express the fluxes $a_{12} = \partial_1 a_2 - \partial_2 a_1$ and $b_{12} = \partial_1 b_2 - \partial_2 b_1$ as follows:

$$a_{12} = \frac{1}{4} \Delta (\log |\phi|^2 + \log |\psi|^2); \quad b_{12} = \frac{1}{4} \Delta (\log |\phi|^2 - \log |\psi|^2). \quad (2.10)$$

So away from the zeroes of ϕ and ψ , we can formulate (2.8)–(2.10) in the following form:

$$\begin{cases} -\Delta \log |\phi|^2 = \frac{e^2 + g^2}{2} |\phi|^2 + \frac{e^2 - g^2}{2} |\psi|^2 - \frac{v_0^2}{2N} e^2 \\ -\Delta \log |\psi|^2 = \frac{e^2 + g^2}{2} |\psi|^2 + \frac{e^2 - g^2}{2} |\phi|^2 - \frac{v_0^2}{2N} e^2. \end{cases} \quad (2.11)$$

The equation (2.8) implies that ϕ and ψ may vanish only at single points (i.e. the vortex points) with integral multiplicity.

Following [7], we assume (for simplicity) that ψ never vanishes, while ϕ vanishes exactly at the (given) points $\{p_1, \dots, p_n\}$ (repeated according to multiplicity). In other words:

$$\log |\psi|^2 \in C^\infty(\mathbb{R}^2); \quad \left(\log |\phi|^2 - \sum_{j=1}^n \log |x - p_j|^2 \right) \in C^\infty(\mathbb{R}^2). \quad (2.12)$$

As a matter of fact, by taking into account (2.5) and (2.12), we can formulate both (2.4) and (2.11) into the following elliptic system of equations:

$$\begin{cases} \Delta v_1 = (\tau \alpha + (1 - \tau) \beta) e^{v_1} + (1 - \tau) (\alpha - \beta) e^{v_2} - \alpha + \sum_{j=1}^n \delta_{p_j} \\ \Delta v_2 = (\tau \beta + (1 - \tau) \alpha) e^{v_2} + \tau (\alpha - \beta) e^{v_1} - \alpha \end{cases} \quad (2.13)$$

with $\alpha, \beta > 0$ and $\tau \in (0, 1)$.

Indeed (2.13) reduces to (2.4) simply by setting:

$$v_1 = \log |h_1|^2, \quad v_2 = \log |h_2|^2, \quad \alpha = m_e^2, \quad \beta = m_g^2, \quad \tau = \frac{1}{N} \quad (2.14)$$

while (2.13) reduces to (2.11) when

$$v_i = \log |\phi_i|^2 - \log \frac{v_0^2}{2N}, \quad i = 1, 2, \quad \alpha = \frac{e^2 v_0^2}{2N}, \quad \beta = \frac{g^2 v_0^2}{2N}, \quad \tau = \frac{1}{2}. \quad (2.15)$$

Other models for non-abelian vortices can be brought into the elliptic system (2.13). Notice also that for $\alpha = \beta$, the first equation in (2.13) reduces to the familiar ANO-vortex equation (cfr. [9]):

$$\Delta v = \beta (e^v - 1) + 4\pi \sum_{j=1}^n \delta_{p_j}. \quad (2.16)$$

In the next two sections we shall describe the solvability of (2.13) under periodic and topological boundary conditions. In particular, we shall recover the results in [11, 12].

3. Periodic solutions of (2.13)

Let $\Omega \subset \mathbb{R}^2$ be a given periodic cell domain. In this section we discuss the solvability of (2.13) subject to periodic boundary conditions on $\partial\Omega$. More precisely, denote by $G(x, p)$ the Green's function of Δ under periodic boundary conditions and such that: $\int_{\Omega} G(x, p) = 0$. Let

$$u_0(x) = 4\pi \sum_{j=1}^n G(x, p_j).$$

In the new variables (u_1, u_2) : $v_1 = u_0 + u_1$ and $v_2 = u_2$, periodic solution of (2.13) are obtained by solving:

$$\begin{cases} \Delta u_1 = (\tau\alpha + (1-\tau)\beta)e^{u_0+u_1} + (1-\tau)(\alpha-\beta)e^{u_2} - \alpha + \frac{4\pi N}{|\Omega|} & \text{in } \Omega \\ \Delta u_2 = (\tau\beta + (1-\tau)\alpha)e^{u_2} + \tau(\alpha-\beta)e^{u_0+u_1} - \alpha & \text{in } \Omega \\ u_1 \text{ and } u_2 \text{ doubly periodic over } \partial\Omega. \end{cases} \quad (3.1)$$

Integrating the above equations over Ω and by performing simple algebra, we derive that

$$\int_{\Omega} e^{u_0+u_1} = |\Omega| - \left(\frac{1-\tau}{\beta} + \frac{\tau}{\alpha} \right) 4\pi n := \rho_1; \quad \int_{\Omega} e^{u_2} = |\Omega| - \tau \left(\frac{1}{\alpha} - \frac{1}{\beta} \right) 4\pi n := \rho_2 \quad (3.2)$$

from which we deduce the following *necessary* condition for the solvability of (3.1):

$$\frac{1-\tau}{\beta} + \frac{\tau}{\alpha} < \frac{4\pi n}{|\Omega|} \quad (3.3)$$

On the other hand, by assuming (3.3) and by setting: $u_j = w_j + c_j$ with $\int_{\Omega} w_j = 0$ and $c_j = \int_{\Omega} w_j$; $j = 1, 2$, we find that,

$$e^{c_1} = \frac{\rho_1}{\int_{\Omega} e^{u_0+w_1}}; \quad e^{c_2} = \frac{\rho_2}{\int_{\Omega} e^{w_2}} \quad \text{with } \rho_j > 0 \text{ in (3.2)} \quad (3.4)$$

$j = 1, 2$ and we can formulate (3.1) only in terms of the unknowns (w_1, w_2) as follows:

$$\begin{cases} \Delta w_1 = (\tau\alpha + (1-\tau)\beta)\rho_1 \left(\frac{e^{u_0+w_1}}{\int_{\Omega} e^{u_0+w_1}} - \frac{1}{|\Omega|} \right) + (1-\tau)(\alpha-\beta)\rho_2 \left(\frac{e^{w_2}}{\int_{\Omega} e^{w_2}} - \frac{1}{|\Omega|} \right) & \text{in } \Omega \\ \Delta w_2 = (\tau\beta + (1-\tau)\alpha)\rho_2 \left(\frac{e^{w_2}}{\int_{\Omega} e^{w_2}} - \frac{1}{|\Omega|} \right) + \tau(\alpha-\beta)\rho_1 \left(\frac{e^{u_0+w_1}}{\int_{\Omega} e^{u_0+w_1}} - \frac{1}{|\Omega|} \right) & \text{in } \Omega \\ \int_{\Omega} w_1 = 0, \int_{\Omega} w_2 = 0, w_1 \text{ and } w_2 \text{ doubly periodic over } \partial\Omega. \end{cases} \quad (3.5)$$

We are going to obtain an equivalent formulation of (3.5) that has the advantage to admit a variational formulation in the space $E = \mathcal{H}(\Omega) \times \mathcal{H}(\Omega)$, where the Hilbert space: $\mathcal{H}(\Omega) = \{w \in H_{loc}^1(\mathbb{R}^2) : w \text{ is periodic with cell domain } \Omega \text{ and } \int_{\Omega} w = 0\}$ is equipped with the usual scalar product.

Observe that by straightforward calculations we have:

$$\begin{cases} \Delta(\tau w_1 + (1-\tau)w_2) = \alpha \left[\tau \rho_1 \left(\frac{e^{u_0+w_1}}{\int_{\Omega} e^{u_0+w_1}} - \frac{1}{|\Omega|} \right) + (1-\tau) \rho_2 \left(\frac{e^{w_2}}{\int_{\Omega} e^{w_2}} - \frac{1}{|\Omega|} \right) \right] & \text{in } \Omega \\ \Delta(w_1 - w_2) = \beta \left[\rho_1 \left(\frac{e^{u_0+w_1}}{\int_{\Omega} e^{u_0+w_1}} - \frac{1}{|\Omega|} \right) - \rho_2 \left(\frac{e^{w_2}}{\int_{\Omega} e^{w_2}} - \frac{1}{|\Omega|} \right) \right] & \text{in } \Omega \\ (w_1, w_2) \in E. \end{cases} \quad (3.6)$$

Thus, in terms of the new unknowns:

$$z_1 = \tau w_1 + (1-\tau)w_2 \text{ and } z_2 = w_1 - w_2 \quad (3.7)$$

we easily check that (weak) solutions of (3.6) correspond to critical points of the following functional:

$$\begin{aligned} I(z_1, z_2) &= \frac{1}{2\alpha} \|\nabla z_1\|_2^2 + \frac{\tau(1-\tau)}{2\beta} \|\nabla z_2\|_2^2 \\ &\quad + \tau \rho_1 \log \int_{\Omega} \left(e^{u_0+z_1+(1-\tau)z_2} \right) + (1-\tau) \rho_2 \log \int_{\Omega} \left(e^{z_1-\tau z_2} \right) \end{aligned} \quad (3.8)$$

with $(z_1, z_2) \in E$.

Proposition 3.1. *If $\tau \in (0, 1)$, $\alpha > 0$, $\beta > 0$ and $\rho_j > 0$, $j = 1, 2$, then the functional I admits a unique critical point corresponding to its global minimizer.*

Proof. By Jensen inequality we have:

$$\int_{\Omega} \left(e^{u_0+z_1+(1-\tau)z_2} \right) \geq 1 \text{ and } \int_{\Omega} \left(e^{z_1-\tau z_2} \right) \geq 1 \quad (3.9)$$

Thus, we find that I is coercive, bounded from below, lower semicontinuous and strictly convex. Therefore it attains its minimum value at (exactly) one point that defines its only possible critical point. $\square$

Notice in particular that if $(z_1, z_2): I(z_1, z_2) = \min_E I$, then $I(z_1, z_2) \leq I(0, 0)$. As a consequence we obtain:

$$\tau(1-\tau) \|\nabla z_2\|_2^2 + \|\nabla z_1\|_2^2 \leq 2\tau \max\{\alpha, \beta\} \rho_1 \log \left(\int_{\Omega} e^{u_0} \right), \quad (3.10)$$

$$0 \leq \log \int_{\Omega} e^{z_1-\tau z_2} \leq \frac{\tau \rho_1}{(1-\tau) \rho_2} \log \left(\int_{\Omega} e^{u_0} \right), \quad (3.11)$$

$$\int_{\Omega} e^{u_0+z_1+(1-\tau)z_2} \leq \int_{\Omega} e^{u_0}.$$

In particular, in terms of the periodic problem (3.1) we may conclude the following:

Theorem 3.2. *The condition (3.3) is necessary and sufficient for the solvability of (3.1), and if (3.3) holds, then (3.1) admits a unique solution (u_1, u_2) . Moreover letting: $c_j = \int_{\Omega} u_j$ and $w_j = u_j - c_j$, $j = 1, 2$, then $c_j = c_j(w_j)$ satisfies (3.4), and there exists a constant $C > 0$, independent of τ, α, β such that:*

$$\text{i) } \tau \|\nabla w_1\|_2^2 + (1-\tau) \|\nabla w_2\|_2^2 \leq C \tau \max\{\alpha, \beta\} \rho_1; \quad \|\nabla(w_1 - w_2)\|_2^2 \leq \frac{C \beta \rho_1}{1-\tau}$$

$$\text{ii) } 1 \leq \int_{\Omega} e^{u_0+w_1} \leq C; \quad \log \left(\int_{\Omega} e^{w_2} \right) \leq C \frac{\tau \rho_1}{(1-\tau)\rho_2}$$

with $\rho_1 > 0$, $\rho_2 > 0$ defined by (3.2).

Clearly the estimates in Theorem 3.2 allow us to handle easily the asymptotic regime:

$$\gamma = \frac{\beta}{\alpha} \rightarrow 1, \quad \tau \in (0, 1) \text{ fixed}; \quad (3.12)$$

where, without surprise, we find that: $w_1 \rightarrow w_A$, $w_2 \rightarrow 0$ in $\mathcal{H}(\Omega)$ (in any relevant norm), and w_A is the unique solution of the abelian periodic vortex equation:

$$\begin{cases} \Delta w = \left(\beta - \frac{4\pi n}{|\Omega|} \right) \left(\frac{e^{u_0+w}}{\int_{\Omega} e^{u_0+w}} - 1 \right) & \text{in } \Omega \\ w \in \mathcal{H}(\Omega) \end{cases} \quad (3.13)$$

with $\beta \in \left(\frac{4\pi n}{|\Omega|}, +\infty \right)$ (this expresses the necessary condition (3.3) when $\alpha = \beta$).

Notice in particular that the parameter $\tau \in (0, 1)$ plays no role in the asymptotic regime (3.12).

More interestingly, we can show an analogous behavior for the less obvious asymptotic regime:

$$\beta \in \left(\frac{4\pi n}{|\Omega|}, +\infty \right) \text{ and } \alpha\tau = m_0 > 0 \text{ fixed, but } \alpha \rightarrow +\infty, \tau \rightarrow 0 \quad (3.14)$$

that in the context of (2.4) (see (2.1) and (2.14)) corresponds to the asymptotic situation of physical interest: $N \rightarrow \infty$ and $\gamma \rightarrow 0$.

In view of the estimates i) and ii) of Theorem 3.2, in the asymptotic regime (3.14), we still find:

$$w_1 \rightarrow w_A \text{ and } w_2 \rightarrow 0. \quad (3.15)$$

Indeed (3.15) certainly holds along a sequence, and the convergence is weak in $\mathcal{H}(\Omega)$, pointwise a.e. in Ω and strongly in $L^p(\Omega)$, $\forall p \geq 1$, with w_A satisfying (3.13), as it follows by using the second equation in (3.6).

The uniqueness of the limit ensures that the convergence actually holds for the full solution set (not only along sequences) and in strong norms. To check the strong convergence, we use the second equation in (3.5), that we multiply by $(w_2 - \log \int_{\Omega} e^{w_2})$ and integrate over Ω to obtain:

$$\begin{aligned} \|\nabla w_2\|_2^2 &= ((1-\tau)\alpha + \tau\beta) \frac{\rho_2}{|\Omega|} \left(1 - e^{w_2 - \log \int_{\Omega} e^{w_2}} \right) \left(w_2 - \log \int_{\Omega} e^{w_2} \right) \\ &\quad + \tau(\alpha - \beta) \frac{\rho_1}{|\Omega|} \left(1 - \frac{e^{u_0+w_1}}{\int_{\Omega} e^{u_0+w_1}} \right) \left(w_2 - \log \int_{\Omega} e^{w_2} \right) \\ &\leq \tau\alpha \left| 1 - \frac{\beta}{\alpha} \right| \frac{\rho_1}{|\Omega|} \left(\|w_2 - \log \int_{\Omega} e^{w_2}\|_1 + \frac{\|e^{u_0+w_1}\|_2}{\int_{\Omega} e^{u_0+w_1}} \|w_2 - \log \int_{\Omega} e^{w_2}\|_2 \right). \end{aligned}$$

Thus, in the asymptotic regime (3.14), the estimates i) and ii) together with the Moser-Trudinger inequality (cfr. e.g. [18]), allow us to obtain a constant $C > 0$: $\|\nabla w_2\|_2^2 \leq C \|w_2 - \log \int_{\Omega} e^{w_2}\|_2 \rightarrow 0$. We can use this information in the second equation of (3.6) to get also the strong convergence of w_1 towards w_A .

4. Topological solutions of (2.13)

In this section we are going to study (2.13) in $\mathbb{R}^2$, under the *topological* boundary condition:

$$v_j(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \quad j = 1, 2. \quad (4.1)$$

To this purpose, for given $\mu > 0$, we introduce the function:

$$u_0(x) = \sum_{j=1}^n \log \left(\frac{|x - p_j|^2}{\mu + |x - p_j|^2} \right)$$

so that in the new (smooth) variables:

$$u_1 = v_1 - u_0, \quad u_2 = v_1$$

we are reduced to analyze the following problem:

$$\begin{cases} -\Delta u_1 = (\tau\alpha + (1-\tau)\beta)(1 - e^{u_0+u_1}) + (1-\tau)(\alpha - \beta)(1 - e^{u_2}) - g_0 \\ -\Delta u_2 = (\tau\beta + (1-\tau)\alpha)(1 - e^{u_2}) + \tau(\alpha - \beta)(1 - e^{u_0+u_1}) \\ u_j(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty \end{cases} \quad (4.2)$$

with $g_0(x) = 4 \sum_{j=1}^n \frac{\mu}{(\mu + |x - p_j|^2)^2} \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$.

As above, when $\alpha = \beta$ the solution (u_1, u_2) reduces simply to: $u_1 = u_A$, $u_2 = 0$; where u_A is the (*unique*) topological solution of the *abelian* vortex problem:

$$\begin{cases} -\Delta u = \beta(1 - e^{u_0+u}) - g_0 \\ u(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty. \end{cases} \quad (4.3)$$

Actually, most of the properties valid for (4.3) are satisfied by solution of (4.2).

Lemma 4.1. *Let (u_1, u_2) be a solution for (4.2). For $\alpha \neq \beta$ we have:*

$$a) \quad u_0 + u_1 < 0 \text{ and } (\alpha - \beta)u_2 > 0 \text{ in } \mathbb{R}^2 \quad (4.4)$$

$$b) \quad (u_1, u_2) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2); \quad (1 - e^{u_0+u_1}), (1 - e^{u_2}) \in L^1(\mathbb{R}^2)$$

and the following holds:

$$\int_{\mathbb{R}^2} (1 - e^{u_0+u_1}) = \frac{4\pi n}{\alpha\beta} ((1-\tau)\alpha + \tau\beta) := \lambda_1, \quad \int_{\mathbb{R}^2} |1 - e^{u_2}| = \frac{4\pi n\tau}{\alpha\beta} |\alpha - \beta| := \lambda_2 \quad (4.5)$$

c) (u_1, u_2) decay exponentially fast to zero at infinity.

Proof. To establish a) we only need to show that $u_0 + u_1 \leq 0$ in $\mathbb{R}^2$, since the strong maximum principle and the second equation in (4.2) will readily yield to the full statement in (4.4). We start with the case $\alpha > \beta$, and argue by contradiction. So we assume there exists $x_0 \in \mathbb{R}^2$ such that: $v_1(x_0) = u_0(x_0) + u_1(x_0) > 0$, hence necessarily $x_0 \notin \{p_1, \dots, p_n\}$.

This fact and (4.1) ensure that there exists $x_1 \in \mathbb{R}^2 \setminus \{p_1, \dots, p_n\}$: $v_1(x_1) = \max_{\mathbb{R}^2} v_1 > 0$. Then from the first equation in (2.13) we find $v_2(x_1) < 0$. This allows to obtain $x_2 \in \mathbb{R}^2$: $v_2(x_2) = \min_{\mathbb{R}^2} v_2 < 0$ and from the second equation in (2.13), deduce $v_1(x_2) > 0$.

As a consequence we also know that $v_1 - v_2$ takes positive values and we can define $\bar{x} \in \mathbb{R}^2 \setminus \{p_1, \dots, p_n\}$: $v_1(\bar{x}) - v_2(\bar{x}) = \max_{\mathbb{R}^2} v_1 - v_2$.

Since $v_1(\bar{x}) - v_2(\bar{x}) \geq v_1(x_1) - v_2(x_1) = \max_{\mathbb{R}^2} v_1 - v_2(x_1)$, we get $v_2(\bar{x}) \leq v_2(x_1) < 0$. Similarly, $v_1(\bar{x}) - v_2(\bar{x}) \geq v_1(x_2) - v_2(x_2) = v_1(x_2) - \min_{\mathbb{R}^2} v_2$ and so $v_1(\bar{x}) \geq v_1(x_2) > 0$.

But this yield to the following contradiction:

$$0 \leq -\Delta(v_1 - v_2)(\bar{x}) = \beta(1 - e^{v_1(\bar{x})}) - \beta(1 - e^{v_2(\bar{x})}) < 0.$$

In case $\beta > \alpha$ we proceed in analogous way only that now our contradiction assumption yields to the existence of $x_1, x_2 \in \mathbb{R}^2$: $v_1(x_1) = \max_{\mathbb{R}^2} v_1 > 0$ and $v_2(x_1) > 0$; $v_2(x_2) = \max_{\mathbb{R}^2} v_2 > 0$ and $v_1(x_2) > 0$.

Therefore we find $\hat{x} \in \mathbb{R}^2 \setminus \{p_1, \dots, p_n\}$: $\tau v_1(\hat{x}) + (1 - \tau)v_2(\hat{x}) = \max_{\mathbb{R}^2} \tau v_1 + (1 - \tau)v_2$ and as above we have: $v_1(\hat{x}) \geq v_1(x_2) > 0$ and $v_2(\hat{x}) \geq v_2(x_1) > 0$.

But this is impossible, since

$$0 \leq -\Delta(\tau v_1 + (1 - \tau)v_2)(\bar{x}) = \alpha(\tau(1 - e^{v_1(\bar{x})}) + (1 - \tau)(1 - e^{v_2(\bar{x})})) < 0.$$

Part b) and c) follow as for the abelian case $\alpha = \beta$, only with obvious modifications, see the proof of Proposition 5 and 6 of [19] and Proposition 3.2.4 of [18]. $\square$

In analogy with the periodic case, it is easy to show that also (4.2) admits a variational formulation in the space $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ according to the new variables

$$z_1 = \tau u_1 + (1 - \tau)u_2 \text{ and } z_2 = u_1 - u_2. \quad (4.6)$$

Indeed, by a straightforward calculation we see that solutions of (4.2) correspond (via (4.6)) to critical points of the functional:

$$\begin{aligned} J(z_1, z_2) &= \frac{1}{2\alpha} \|\nabla z_1\|_2^2 + \frac{\tau(1 - \tau)}{2\beta} \|\nabla z_2\|_2^2 \\ &+ \tau \int_{\mathbb{R}^2} (e^{u_0} - 1) \left(e^{z_1 + (1 - \tau)z_2} - 1 \right) dx + \tau \int_{\mathbb{R}^2} \left(e^{z_1 + (1 - \tau)z_2} - 1 - (z_1 + (1 - \tau)z_2) \right) dx \\ &+ (1 - \tau) \int_{\mathbb{R}^2} (e^{z_1 - \tau z_2} - 1 - (z_1 - \tau z_2)) dx + \tau \int_{\mathbb{R}^2} g_0(x) (z_1 + (1 - \tau)z_2) dx. \end{aligned} \quad (4.7)$$

Notice that, $(1 - e^{u_0})$ and $g_0 \in L^2(\mathbb{R}^2)$, while by well known estimates (see e.g. the Appendix in [9]): $(e^z - 1) \in L^2(\mathbb{R}^2)$ and $(e^z - 1 - z) \in L^1(\mathbb{R}^2)$ for $z \in H^1(\mathbb{R}^2)$.

Thus the functional $J \in C^1(H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2))$ is lower semicontinuous and strictly convex in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$.

But now it is less obvious to check that J is also coercive and bounded from below.

To this purpose observe that, if we let for instance $\beta > \alpha$, then we can re-write J as follows:

$$J(z_1, z_2) = \frac{1}{2} \left(\frac{1}{\alpha} - \frac{1}{\beta} \right) \|\nabla z_1\|_2^2 + \tau J_1^\beta(z_1 + (1 - \tau)z_2) + (1 - \tau) J_2^\beta(z_1 - \tau z_2),$$

with

$$J_1^\beta(v) = \frac{1}{2\beta} \|\nabla v\|_2^2 + \int_{\mathbb{R}^2} e^{u_0} (e^v - 1 - v) + \int_{\mathbb{R}^2} (e^{u_0} - 1 + g_0)v$$

and

$$J_2^\beta(v) = \frac{1}{2\beta} \|\nabla v\|_2^2 + \int_{\mathbb{R}^2} (e^v - 1 - v), \quad v \in H^1(\mathbb{R}^2)$$

(note that J_1^β reduces to J_2^β when $u_0 \equiv 0 \equiv g_0$). Therefore we can apply Proposition 13 in [19] and conclude the existence of positive constants a, b : $J_1^\beta(v) \geq a\|v\|_{H^1} - b$ and $J_2^\beta(v) \geq a\|v\|_{H^1} - b$, $\forall v \in H^1(\mathbb{R}^2)$.

This ensures that, if $\alpha < \beta$, then J is coercive and bounded from below in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$.

On the other hand, when $\alpha > \beta$, then

$$J(z_1, z_2) = \frac{\tau(1-\tau)}{2} \left(\frac{1}{\alpha} - \frac{1}{\beta} \right) \|\nabla z_2\|_2^2 + \tau J_1^\alpha(z_1 + (1-\tau)z_2) + (1-\tau)J_2^\alpha(z_1 - \tau z_2)$$

where J_1^α and J_2^α are the same functionals above only with β replaced by α . So, also in this case J is coercive and bounded from below.

In any case J admits a *unique* critical point corresponding to its global minimum. We conclude:

Theorem 4.2. *Problem (4.2) admits a unique solution. When expressed in the variables (z_1, z_2) in (4.6), it corresponds to the global minimizer of J in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$.*

As for the periodic case, we wish to describe the behavior of the solution of (4.2) in the asymptotic regime:

$$\tau \rightarrow 0, \alpha \rightarrow +\infty \text{ but } \tau\alpha = m_0 > 0 \text{ and } \beta > 0 \text{ fixed,} \quad (4.8)$$

so that $\gamma = \frac{\beta}{\alpha} \rightarrow 0$. Observe once more that in the context of (2.4), the regime (4.8) corresponds to the interesting physical limit: $N \rightarrow +\infty$.

To this purpose, we shall need the following *a priori* estimates:

Proposition 4.3. *Let $(u_1, u_2) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ be a solutions of (4.2), and let the values $\lambda_1(= \lambda_1(\alpha, \beta, \tau))$ and $\lambda_2(= \lambda_2(\alpha, \beta, \tau))$ be defined in (4.5). We have:*

a) $u_2 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and there holds:

$$\|u_2\|_\infty \leq \log \left(1 + \frac{\tau|\alpha - \beta|}{\alpha - \tau(\alpha - \beta)_+} \right) \quad (4.9)$$

$$\|\nabla u_2\|_2^2 \leq \lambda_1 \tau |\alpha - \beta| \|u_2\|_\infty. \quad (4.10)$$

b)

$$\|\nabla u_1^+\|_2^2 + \frac{1}{2}(\tau\alpha + (1-\tau)\beta) \|u_1^+\|_2^2 \leq \frac{1}{2}(\tau\alpha + (1-\tau)\beta) \|u_0\|_2^2 \quad (4.11)$$

$$\|u_1^-\|_2^2 \leq 2\lambda_1(1 + 2\|\nabla u_1^-\|_2^2) \quad (4.12)$$

$$\|\nabla u_1^-\|_2^2 \leq A \left(1 + \sqrt{1 + \frac{1}{A}} \right) \quad (4.13)$$

with $A = 2\lambda_1 \left((1-\tau)|\alpha - \beta| (\lambda_2 e^{\|u_2\|_\infty})^{\frac{1}{2}} + \|g_0\|_2 \right)^2$.

Proof. We start to establish a) in case $\alpha > \beta$. Then $u_2 > 0$ in $\mathbb{R}^2$ (by part a) of Lemma 4.1) and there exists $x_2 \in \mathbb{R}^2$: $u_2(x_2) = \max_{\mathbb{R}^2} u_2$. Consequently,

$$0 \geq \Delta u_2(x_2) = ((1-\tau)\alpha + \tau\beta)(e^{u_2(x_2)} - 1) + \tau(\alpha - \beta)(e^{(u_0+u_1)(x_2)} - 1).$$

Thus

$$e^{\|u_2\|_{L^\infty}} = e^{u_2(x_2)} \leq 1 + \frac{\tau(\alpha - \beta)}{(1-\tau)\alpha + \tau\beta}$$

and (4.9) is established in this case.

In case $\alpha < \beta$, then $u_2 < 0$ and there exists $x_0 \in \mathbb{R}^2$: $u_2(x_0) = \min_{\mathbb{R}^2} u_2$. In addition,

$$0 \leq \Delta u_2(x_0) = ((1-\tau)\alpha + \tau\beta)(e^{u_2(x_0)} - 1) - \tau(\alpha - \beta)(e^{(u_0+u_1)(x_0)} - 1)$$

and we derive

$$e^{-\|u_2\|_{L^\infty}} = e^{u_2(x_0)} \geq 1 - \frac{\tau(\alpha - \beta)}{(1-\tau)\alpha + \tau\beta} = \frac{\alpha}{(1-\tau)\alpha + \tau\beta}$$

and (4.9) is established in this case as well.

To obtain (4.10) we multiply the second equation in (4.2) by u_2 and integrate over $\mathbb{R}^2$. By the usual integration by parts we derive:

$$\begin{aligned} \|\nabla u_2\|_2^2 &= ((1-\tau)\alpha + \tau\beta) \int_{\mathbb{R}^2} (1 - e^{u_2}) u_2 + \tau(\alpha - \beta) \int_{\mathbb{R}^2} (1 - e^{u_0+u_1}) u_2 \\ &\leq \tau|\alpha - \beta| \|u_2\|_\infty \int_{\mathbb{R}^2} (1 - e^{u_0+u_1}) \end{aligned}$$

and in view of (4.5) we deduce (4.10).

Concerning u_1 , we use the first equation in (4.2) to find

$$-\Delta u_1 + ((1-\tau)\beta + \tau\alpha)u_1 = ((1-\tau)\beta + \tau\alpha)(1 + u_1 - e^{u_0+u_1}) - (1-\tau)|\alpha - \beta||1 - e^{u_2}| - g_0$$

that we can multiply by u_1^+ and integrate over $\mathbb{R}^2$ to obtain:

$$\|\nabla u_1^+\|_2^2 + (\tau\alpha + (1-\tau)\beta) \|u_1^+\|_2^2 \leq (\tau\alpha + (1-\tau)\beta) \|u_0\|_2 \|u_1^+\|_2$$

from which (4.11) easily follows.

To establish (4.11) and (4.13), we use the first identity in (4.5) to obtain:

$$\lambda_1 = \int_{\mathbb{R}^2} (1 - e^{u_0+u_1}) \geq \int_{\{u_1 \leq 0\}} (1 - e^{u_0-u_1^-}) \geq \int_{\mathbb{R}^2} (1 - e^{-u_1^-}),$$

where in the last inequality we have used that $u_0 < 0$ in $\mathbb{R}^2$.

By virtue of the inequality: $\frac{t}{1+t} \leq (1 - e^{-t}) \forall t \geq 0$, we get:

$$\int_{\mathbb{R}^2} \left(\frac{u_1^-}{1 + u_1^-} \right) \leq \int_{\mathbb{R}^2} (1 - e^{-u_1^-}) \leq \lambda_1. \quad (4.14)$$

On the other hand, for every $u \in H^1(\mathbb{R}^2)$ the following estimate holds:

$$\|u\|_2^2 \leq 2(1 + \|\nabla u\|_2^2) \int_{\mathbb{R}^2} \left(\frac{|u|}{1 + |u|} \right)^2 \quad (4.15)$$

(for the proof see e.g. p. 139-140 of [19]), and (4.12) easily follows from (4.14) and (4.15).

Concerning (4.13), we use again the second equation in (4.2), which now we multiply by u_1^- , integrate over $\mathbb{R}^2$ and get:

$$\begin{aligned}\|\nabla u_1^-\|_2^2 &= (\tau\alpha + (1-\tau)\beta) \int_{\mathbb{R}^2} (e^{u_0+u_1} - 1)u_1^- + (1-\tau) \int_{\mathbb{R}^2} |e^{u_2} - 1|u_1^- + \int_{\mathbb{R}^2} g_0 u_1^- \\ &\leq ((1-\tau)|\alpha - \beta| \|e^{u_2} - 1\|_2 + \|g_0\|_2) \|u_1^-\|_2.\end{aligned}$$

Since we can estimate:

$$\|e^{u_2} - 1\|_2 \leq e^{\|u_2\|_\infty} \int_{\mathbb{R}^2} |e^{u_2} - 1| = e^{\|u_2\|_\infty} \lambda_2$$

we find:

$$\|\nabla u_1^-\|_2^2 \leq \left((1-\tau)|\alpha - \beta| \left(\lambda_2 e^{\|u_2\|_\infty} \right)^{\frac{1}{2}} + \|g_0\|_2 \right) \|u_1^-\|_2. \quad (4.16)$$

At this point we can insert in (4.16) the estimate (4.12) that we have already established for $\|u_1^-\|_2$, and by simple algebra arrive at (4.13). $\square$

With the estimate of Proposition 4.3 we can establish the following:

Theorem 4.4. *Under the asymptotic regime (4.8) we have:*

$$u_1 \rightarrow u_A, \quad u_2 \rightarrow 0 \text{ in } H^1(\mathbb{R}^2)$$

with u_A the unique solution of (4.3).

Proof. By the estimate (4.9) and (4.10) we easily see that, under (4.8), we have

$$\|u\|_\infty \rightarrow 0 \text{ and } \|\nabla u_2\|_2 \rightarrow 0. \quad (4.17)$$

Since, under (4.8) we can assume $\alpha > \beta$, we have: $u_2 > 0$ and

$$\|u_2\|_{L^1} \leq \int_{\mathbb{R}^2} (e^{u_2} - 1) \rightarrow 0. \quad (4.18)$$

So by using (4.17) together with (4.18), we conclude that $\|u_2\|_{L^2} \rightarrow 0$. In other words, $u_2 \rightarrow 0$ in $H^1(\mathbb{R}^2)$ as claimed.

Concerning u_1 , the estimates (4.11), (4.13) and (4.12) give that u_1 is uniformly bounded in $H^1(\mathbb{R}^2)$. Therefore along a sequence we can assume that $u_1 \rightharpoonup u_* \in H^1(\mathbb{R}^2)$, weakly in $H^1(\mathbb{R}^2)$, strongly in $L_{loc}^2(\mathbb{R}^2)$ and pointwise a.e. in $\mathbb{R}^2$.

On the other hand, recalling the equation for $u_1 - u_2$, namely:

$$-\Delta(u_1 - u_2) = \beta(1 - e^{u_0+u_1}) - \beta(1 - e^{u_2}) - g_0; \quad (4.19)$$

by taking the limit, we find that necessarily: $u_* = u_A$ the unique solution to (4.3). Furthermore from (4.19) and (4.3) we also get:

$$\begin{aligned}\|\nabla(u_1 - u_A)\|_2^2 &+ \beta \int_{\mathbb{R}^2} e^{u_0+u_A} (e^{u_1-u_A} - 1)(u_1 - u_A) \\ &\leq \int_{\mathbb{R}^2} \nabla u_2 \cdot \nabla(u_1 - u_A) + \beta \int_{\mathbb{R}^2} (e^{u_2} - 1)(u_1 - u_A) - \int_{\mathbb{R}^2} g_0(u_1 - u_A) \rightarrow 0.\end{aligned}$$

Hence, $\|\nabla(u_1 - u_A)\|_2^2 \rightarrow 0$ and $\int_{\mathbb{R}^2} e^{u_0+u_A} (e^{u_1-u_A} - 1)(u_1 - u_A) \rightarrow 0$.

In particular, $\int_{\mathbb{R}^2} e^{u_0+u_A} ((u_1 - u_A)^+)^2 \rightarrow 0$ and $\int_{\mathbb{R}^2} e^{u_0+u_A} \frac{((u_1 - u_A)^-)^2}{1 + (u_1 - u_A)^-} \rightarrow 0$.

By using the strong convergence: $u_1 \rightarrow u_A$ in $L^2_{loc}(\mathbb{R}^2)$ and the fact that $e^{u_0+u_A} \rightarrow 1$ as $|x| \rightarrow \infty$ (exponentially fast), we deduce that

$$\|(u_1 - u_A)^+\|_{L^2} \rightarrow 0 \text{ and } \int_{\mathbb{R}^2} \left(\frac{(u_1 - u_A)^-}{1 + (u_1 - u_A)^-} \right)^2 \rightarrow 0.$$

So once more we can use (4.15) to conclude that $\|u_1 - u_A\|_{L^2} \rightarrow 0$. Since the same strong convergence holds along any sequence, we conclude that, in the regime (4.8), $u_1 \rightarrow u_A$ in $H^1(\mathbb{R}^2)$ as claimed. $\square$

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Received: March 3, 2011.

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