The Grand Riemann Hypothesis

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Extended Abstract*

These lectures are a continuation of Bombieri’s series “The classical Theory of Zeta and L -Functions” (in this volume). Naturally there is some overlap between his and our presentations. My aim is to formulate the Riemann Hypothesis “GRH” in its most general setting and to demonstrate its importance and power as well as to indicate some of the progress that has been made around these conjectures. A particular theme being that a number of the striking applications of the GRH have been proven unconditionally by establishing suitably strong approximations thereof.

Lecture 1: $GL(1)$ and $GL(2)$, classical theory.

We start with Dirichlet L -functions. These are the zeta (or L) functions associated with $GL(1)$, the multiplicative group. After formulating the Riemann Hypothesis for these we give a number of applications to seemingly unrelated problems such as primality testing. The basic approximation to the Riemann Hypothesis for these is the Theorem of Bombieri and Vinogradov. After reviewing it we demonstrate its power by giving some applications in which it can be used as a substitute for the Riemann Hypothesis for such L -functions.

Next we discuss automorphic forms associated with $GL(2)$, that is classical automorphic forms on the upper half plane and their generalizations. This theory was initiated by Hecke and Maass and it continues to be a very active and central topic in number theory. A fundamental issue (that is trivial in the $GL(1)$ case) that arises in this $GL(2)$ context is the Ramanujan/Selberg Conjecture concerning sizes of Fourier coefficients or equivalently of Hecke and Laplace eigenvalues of such forms. This is critical to understanding L -functions and conversely understanding new L -functions associated to these forms via the principle of functoriality, play a critical role in understanding the Ramanujan Conjectures. These $GL(2)$ modular

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*The complete writeup of the lectures will appear in the next volume of MJM.

forms account for all zeta functions of degree 2 and each of these zeta functions is conjectured to satisfy a Riemann Hypothesis. Again there are striking applications.

Lecture 2: $GL(n)$, not so classical.

The general zeta function is an Euler product of degree n . Philosophically and practically this means that the study of zeta functions is tantamount to the study of automorphic forms on the general linear group $GL(n)$. Unlike the classical cases when n is 1 or 2, it is necessary to set up the theory from the point of view of adèle groups and their representations. We review this set up with the emphasis on the basic object of the theory that being an automorphic cusp form (or representation). Attached to such a form is a standard L -function (of degree n) whose analytic properties are similar to the Riemann Zeta function and in fact the proof that this is so follows Riemann's paper. The deeper properties of these L -functions and ones associated with them depend on the theory of Eisenstein Series and Whittaker functions. We review the relevant works and then formulate the most general form of GRH. Intertwined with this GRH are some basic problems which we review. The first is the generalized Ramanujan Conjecture for $GL(n)$. The second concerns the sizes of such L -functions at special points. A fundamental quantity associated with an L -function is its conductor (which is made out of its ramified primes and an archimedean part) and it measures the complexity of an L -function. One of the consequences of GRH is a uniform and sharp control of quantities associated with L -functions in terms of the conductor. The subconvex problem is an approximation to this sharp estimation and it often serves as a complete substitute in applications. We review what is known in this general setting and also discuss "weak subconvexity" which has been established recently.

Lecture 3: Applications of Subconvexity.

This lecture is devoted to applications. We show how $GL(2)$ subconvexity leads to a solution of Hilbert's 11th problem which is concerned with the representations of integers in a number field by a given integral quadratic form. The basic result being that for forms in 3 or more variables there is a local to global principle as long as one allows finitely many exceptions. The second application is to the "Quantum Unique Ergodicity Conjecture" concerning the equidistribution of the mass of modular form on $GL(2)$. This is one of the many problems that has arisen in the context of quantizing a chaotic dynamical system and in this setting it has a lovely application to the equidistribution of the zeros of such holomorphic forms of large weight.

The proofs of subconvexity are based on families of automorphic forms and L -functions. We use this occasion to formulate a definition of such a family and show how to compute a random matrix symmetry type associated with a given family. This symmetry type dictates the basic questions of distribution within the family.

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