Global existence results for eikonal equation with BV initial data

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Abstract. In this paper, we study a local and a non-local eikonal equations in one dimensional space describing the evolution of interfaces moving with non-signed velocity. For these equations, the global existence and uniqueness are available only of Lipschitz continuous viscosity solutions in some particular cases. In the present paper, we are interested in the study of the global in time existence of these equations, considering BV initial data. Based on a fundamental uniform BV estimate and the finite speed propagation property of these equations, we show, in a particular setting, global existence results of discontinuous viscosity solutions of this problem. An interesting application of these results is shown in the case of dislocation dynamics.

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1. Introduction and main results

1.1. Setting of the problem

In this article, we are interested in the local eikonal equation in one dimensional space given by

$$\begin{cases} \partial_t u(x, t) = c(x, t) |\partial_x u(x, t)| & \text{in } \mathbb{R} \times (0, T) \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}, \end{cases} \quad (1.1)$$

where $T > 0$, the solution u is a real-valued function, $\partial_t u$ and $\partial_x u$ stand, respectively, for its time and space derivatives. Here, the velocity c is assumed to satisfy, for all $T > 0$, the following regularity assumption:

$$c \in L^\infty(\mathbb{R} \times (0, T)) \cap L^\infty((0, T); BV(\mathbb{R})). \quad (1.2)$$

The goal of this work is to establish the global existence of discontinuous viscosity solutions of (1.1) assuming (1.2) and the following regularity on the initial data

$$u_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R}), \quad (1.3)$$

where $BV(\mathbb{R})$ is the space of functions of bounded variations given by

$$BV(\mathbb{R}) = \{f \in L^1_{loc}(\mathbb{R}) \mid TV(f) < +\infty\}$$

with $TV(f)$ the total variation of f defined as

$$TV(f) = \sup \left\{ \int_{\mathbb{R}} f(x) \phi'(x) dx; \quad \phi \in C^1_c(\mathbb{R}) \quad \text{and} \quad \|\phi\|_{L^\infty(\mathbb{R})} \leq 1 \right\}. \quad (1.4)$$

In the following, we take the space $BV(\mathbb{R})$ endowed with the semi-norm $|f|_{BV(\mathbb{R})} = TV(f)$. It is noted that BV functions are functions whose distributional derivative is a finite Radon measure.

We remark that Eq. (1.1) can be seen as the “level-set approach” equation associated to the motion of the front $\Gamma_t := \{x : u(x, t) = 0\}$ with a normal velocity $c(x, t)$ (see for instance Barles et al. [12]).

Our work is motivated by the applications to dynamics of defects in crystal, called dislocations, where the velocity of these defects is non-local and can change sign. In this framework, we can write

$$c(x, t) = c_0(\cdot, t) \star u(\cdot, t)(x),$$

where $\star$ denotes the convolution in space and $c_0(x, t)$ is a kernel which depends only on the physical properties of the crystal and on the choice of the dislocation, whose evolution we follow. Note that, in the special case of the application to dislocations, the kernel c_0 does not depend on time. However, to keep a general setting, we allow, in here, its dependence on the time variable. Furthermore, in this non-local and non-monotone “geometrical” equation, the inclusion principle, which plays a central role in the “level-set approach”, does not hold. Therefore, the uniqueness of solutions cannot be proved via standard viscosity solutions methods. We refer the reader to [5, 14, 15] for a complete overview of viscosity solutions.

Although this model of dislocation dynamics seems very simple, there are only few existence and uniqueness known results. We point out that, in the case of nonnegative velocities, the global existence and uniqueness were first obtained by Alvarez et al. [1] and then by Barles et al. [9] using different arguments. These uniqueness results were recently extended by Barles et al. in [8], using a new approach allowing to relax the assumptions of [1, 9]. Moreover, the proof proposed in [8] is simpler than that of [1, 9] and requires a mild regularity on the velocity.

In the general case, with unsigned velocity, this problem has first been investigated by Alvarez et al. in [2] where a short time existence and uniqueness result is proved under the assumption that the initial position of the dislocation is a Lipschitz graph. Recently, in a general setting, using a new stability result of Barles [6] in the framework of L^1 -viscosity solutions, global existence results of weak solutions were obtained by Barles et al. in [7]. Moreover, two

cases in which uniqueness can be obtained are presented in [7] which contains, as well, an interesting counter-example on the uniqueness of weak solutions. We also refer to Barles [4] for another counter-example on the uniqueness of discontinuous viscosity solution. Let us mention that the assumptions used in [7] were relaxed in [8].

From the numerical point of view, a new fast-marching algorithm for the eikonal equation in the case where the velocity can change sign, was proposed by Forcadell in [18], where the author proved the convergence and the comparison principle of the algorithm. Interesting applications of this algorithm, for dislocation dynamics computation and image segmentation, were also shown in [13] and [19] respectively.

In this paper, we present a global existence result for the eikonal equation (1.1) with BV initial data and without sign restrictions on the velocity. More precisely, we consider the case when the initial data and the velocity satisfy only the assumptions (1.2) and (1.3) without any better regularity. In its full generality, the fundamental issue of uniqueness for global solution remains open.

Let us return to the key steps followed to prove our existence results. First, we regularize by classical convolution the velocity $c(x, t)$ and the initial conditions in (1.1). This approximation brings us back to the study, for every $0 < \varepsilon < 1$, of the following equation

$$\begin{cases} \partial_t u_\varepsilon(x, t) = c_\varepsilon(x, t) |\partial_x u_\varepsilon(x, t)| & \text{in } \mathbb{R} \times (0, T) \\ u_\varepsilon(x, 0) = u_{0,\varepsilon}(x) & \text{in } \mathbb{R}, \end{cases} \quad (1.5)$$

where c_ε and $u_{0,\varepsilon}$ are the regularization of the functions c and u_0 respectively (see (4.7) for the definitions of c_ε and $u_{0,\varepsilon}$). Then, using the result of Ley [22], we prove that this regularized system has a unique Lipschitz continuous viscosity solution.

Afterwards, thanks to the L^∞ uniform estimate on u_ε inherited from the maximum principle, we show that the relaxed semi-limits of Barles and Perthame [10, 11]

$$\bar{u}(x, t) = \limsup^* u_\varepsilon(x, t) = \limsup_{(y,s) \xrightarrow[\varepsilon \rightarrow 0]{\longrightarrow} (x,t)} u_\varepsilon(y, s) \quad (1.6)$$

and

$$\underline{u}(x, t) = \liminf_* u_\varepsilon(x, t) = \liminf_{(y,s) \xrightarrow[\varepsilon \rightarrow 0]{\longrightarrow} (x,t)} u_\varepsilon(y, s) \quad (1.7)$$

are, respectively, discontinuous viscosity sub- and super-solutions of (1.1). Finally, reaching some ε -independent a priori estimates, we will be able to prove that $\bar{u}(\cdot, t) = \underline{u}(\cdot, t)$ almost everywhere in $\mathbb{R}$, for all $t > 0$. The latter proves that there exists a function u , defined as a strong limit of u_ε in $C([0, T]; L^1_{loc}(\mathbb{R}))$, that is almost everywhere discontinuous viscosity solution of (1.1). This turns out to be possible thanks to the uniform BV bound, obtained on u_ε , and the finite speed propagation property of the equation.

Related to our analysis, we obtain, as a consequence, the global existence of discontinuous viscosity solution of (1.1) for non-decreasing initial data. An interesting application in the case of dislocation dynamics is also shown.

1.2. Main results

In this subsection, we, first, present (see Theorem 1.1) a global existence result, in some weak sense, for discontinuous viscosity solution of (1.1). As consequence of this result, we show in Theorem 1.2 that this solution is a classical discontinuous viscosity solution of (1.1) in the case of non-decreasing functions. An application of these results to dislocation dynamics is then presented in Theorem 1.3. We refer to Definition 5.1 for a precise definition of discontinuous viscosity solutions. We can now announce our first main result.

Theorem 1.1. (Global existence result for local problem, in weak sense) *Suppose that the assumptions (1.2) and (1.3) are satisfied. Then, we have*

(i) *Existence and uniqueness of approximated problem:*

There exists a unique Lipschitz continuous viscosity solution u_ϵ of (1.5) satisfying, for all $T > 0$, the following uniform a priori estimates:

$$\|u_\epsilon\|_{L^\infty(\mathbb{R} \times (0, T))} \leq \|u_0\|_{L^\infty(\mathbb{R})} \quad (1.8)$$

$$\|\partial_x u_\epsilon\|_{L^\infty((0, T); L^1(\mathbb{R}))} \leq |u_0|_{BV(\mathbb{R})} \quad (1.9)$$

$$\|\partial_t u_\epsilon\|_{L^\infty((0, T); L^1(\mathbb{R}))} \leq \|c\|_{L^\infty(\mathbb{R} \times (0, T))} |u_0|_{BV(\mathbb{R})}. \quad (1.10)$$

(ii) *Sub- and super-solutions of the limit problem:*

Let u_ϵ be the solution of (1.5), constructed in (i), then the relaxed semi-limits $\bar{u}$ and $\underline{u}$, defined by (1.6)–(1.7), are respectively discontinuous viscosity sub- and super-solutions of (1.1).

(iii) *Convergence and existence of weak solution:*

Assume that u_ϵ satisfies (1.8), (1.9) and (1.10). Then, up to extract a subsequence, the function u_ϵ converges, as ϵ goes to zero, to a function

$$u \in L^\infty(\mathbb{R} \times (0, T)) \cap L^\infty((0, T); BV(\mathbb{R})) \cap C([0, T]; L^1_{loc}(\mathbb{R})) \quad (1.11)$$

strongly in $C([0, T]; L^1_{loc}(\mathbb{R}))$. Moreover, u satisfies, for all $T > 0$, the following estimates

$$\|u\|_{L^\infty(\mathbb{R} \times (0, T))} \leq \|u_0\|_{L^\infty(\mathbb{R})} \quad (1.12)$$

$$\|u\|_{L^\infty((0, T); BV(\mathbb{R}))} \leq |u_0|_{BV(\mathbb{R})} \quad (1.13)$$

$$\begin{aligned} \|u(\cdot, t) - u(\cdot, s)\|_{L^1(\mathbb{R})} &\leq (\|c\|_{L^\infty(\mathbb{R} \times (0, T))} |u_0|_{BV(\mathbb{R})}) |t - s|, \\ \text{for all } s, t &\in [0, T] \end{aligned} \quad (1.14)$$

and the following equality

$$\begin{aligned} u(\cdot, t) &= \bar{u}(\cdot, t) = \underline{u}(\cdot, t) \quad \text{except at most on a countable set in } \mathbb{R}, \\ \text{for all } t &\in [0, T]. \end{aligned} \quad (1.15)$$

The key point to establish this theorem is the uniform BV estimate on u_ϵ (1.9). We first consider the parabolic regularization of (1.1) and we show that the smooth solution admits the L^∞ bound (1.8) and the fundamental BV estimate (1.9). These estimates will allow us to pass to the limit when the

regularization vanishes, which will be provided from the stability properties of viscosity solutions (see Barles [5, Th 4.1]) and then show that the relaxed semi-limits $\bar{u}$ and $\underline{u}$ are, respectively, sub- and super-solutions of (1.1). These estimates also imply that the set of the discontinuity points, with respect to x , of the solution u is at most countable. Taking into account the finite speed propagation property of (1.1) and the time continuous estimate (1.10), it is then possible to show this property uniformly in time, which proves in particular (1.15).

We note that, the solution constructed in previous theorem, can be seen as a discontinuous viscosity solution but in some weak sense, since it verifies only an almost everywhere equality in space between $\bar{u}$ and $\underline{u}$, which is reflected by (1.15). In the following, such a solution will be called a weak discontinuity viscosity solution of (1.1).

Recall that in the framework of non-decreasing solutions, the eikonal equation (1.1) becomes a classical transport equation. Related to our analysis in Theorem 1.1, it is then possible to get the following theorem as a by-product.

Theorem 1.2. (Global existence result for non-decreasing solutions) *Assume that we have (1.2), (1.3) and that the initial data u_0 is non-decreasing. Then, Eq. (1.1) admits a discontinuous viscosity solution u , satisfying (1.11), (1.12), (1.13) and (1.14).*

It is worth pointing that our results can be easily extended to the generalized first order Hamilton–Jacobi equations with suitable assumptions on the hamiltonian, that ensure the BV estimate and the maximum principle. In this paper, we have considered the eikonal equation in order to simplify the presentation.

Now, we present an application of the previous results to the following non-local equation arising in the study of the dynamics of dislocation lines in crystals (see Alvarez et al. [2] and Rodney et al. [24] for modeling details). We consider the following equation:

$$\begin{cases} \partial_t u(x, t) = (c_0(\cdot, t) \star u(\cdot, t)(x)) |\partial_x u(x, t)| & \text{in } \mathbb{R} \times (0, T) \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}, \end{cases} \quad (1.16)$$

where c_0 is the kernel associated to the equation of linearized elasticity and “ $\star$ ” stands for the usual convolution in $\mathbb{R}$.

Applying Theorems 1.1 and 1.2, we will prove the following result.

Theorem 1.3. (Global existence results of dislocation equation) *Assume that u_0 satisfies (1.3) and $c_0 \in L^\infty((0, T); L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})) \cap C(\mathbb{R} \times (0, T))$. Then, we have*

(i) Global existence of weak viscosity solution:

There exists a weak discontinuous viscosity solution of (1.16) in the sense of Theorem 1.1, satisfying (1.11), (1.12), (1.13), (1.14) and (1.15).

(ii) Global existence of viscosity non-decreasing solution:

Assume that u_0 is non-decreasing. Then, Eq. (1.16) has a discontinuous viscosity solution u satisfying (1.11), (1.12), (1.13) and (1.14).

Remark 1.4. Note that, in previous theorem the meaning of discontinuous viscosity solution of (1.16) is defined through Definition 5.1, replacing in (5.2) and (5.3), the functions c^* and c_* by $c_0 \star \bar{u}$ and $c_0 \star \underline{u}$, respectively. Note also that, in this theorem, the kernel c_0 may dispose of the continuity condition if we use the framework of L^1 -viscosity solutions (see [6, 21]) instead of standard discontinuous viscosity solutions.

The proof of Theorem 1.3 is an application of Theorems 1.1 and 1.2. We refer the reader to Sect. 7 for the proof. Let us mention that, as recalled in the introduction, Eq. (1.16) is not only non-local but it is also non-monotone which means that the comparison principle does not hold and then we cannot apply directly the viscosity solutions theory. Nevertheless, in the special case where the kernel $c_0 \geq 0$, a comparison principle is expected (cf. Slepčev [26]), but this assumption is not natural in dislocations framework. Remark that, in Theorem 1.3, we only assume that $c_0 \in L^\infty((0, T); L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})) \cap C(\mathbb{R} \times (0, T))$ and then we prove global existence of solutions of (1.1) with non-signed kernel.

1.3. Organization of the paper

This paper is organized as follows: in Sect. 2, we regularize the approximated problem (1.5) by adding the viscosity term $\eta \partial_{xx}^2 u_\varepsilon$. Then, we show that this equation admits a global in time regular solution u_ε , which verifies some η -uniform Lipschitz estimates. In Sect. 3, we prove the (η, ε) -uniform BV estimate (1.9) and some other (η, ε) -uniform a priori estimates. In Sect. 4, we prove, thanks to the stability result for continuous viscosity solutions and by passing to the limit, as η goes to 0, the global existence result of Lipschitz continuous viscosity solution of the approximated problem (1.5) (announced in Theorem 1.1 (i)). In Sect. 5, by passing to the limit, as ε goes to 0, we prove Theorem 1.1 (ii), using the stability result for discontinuous viscosity solutions and the finite speed propagation property of the equation. In Sect. 6, we prove the main results Theorems 1.1 (iii) and 1.2. This will be achieved by using our (η, ε) -uniform a priori estimates, in particular the BV estimate, and also the finite speed propagation property of the equation. Finally, in Sect. 7, we give the proof of Theorem 1.3 as an application to the dislocation dynamics in crystals.

2. Global solution for parabolic regularized equation

In this section, we recall some preliminary global existence and uniqueness results for a parabolic equation obtained by regularization of problem (1.1) with smoothed data v_0 and smoothed velocity $\tilde{c}$. More precisely, we consider, for $0 < \eta \leq 1$, the following equation:

$$\begin{cases} \partial_t v_\eta(x, t) - \eta \partial_{xx}^2 v_\eta(x, t) = \tilde{c}(x, t) |\partial_x v_\eta(x, t)| & \text{in } \mathbb{R} \times (0, T) \\ v_\eta(x, 0) = v_0(x) & \text{in } \mathbb{R}, \end{cases} \quad (2.1)$$

where

$$v_0 \in W^{1,\infty}(\mathbb{R}) \cap C^\infty(\mathbb{R}) \quad \text{and} \quad \tilde{c} \in W^{1,\infty}(\mathbb{R} \times (0, T)) \cap C^\infty(\mathbb{R} \times (0, T)) \quad (2.2)$$

We also regularize the non-linear term $|\partial_x v_\eta|$ replacing it by a smooth function $\beta_\delta(\partial_x v_\eta)$, where β_δ is defined as follows:

$$\beta_\delta(x) = \sqrt{x^2 + \delta^2} \quad \text{for all } 0 < \delta \leq 1. \quad (2.3)$$

This brings us, finally, to consider the following parabolic equation:

$$\begin{cases} \partial_t v_\eta^\delta(x, t) - \eta \partial_{xx}^2 v_\eta^\delta(x, t) = \tilde{c}(x, t) \beta_\delta(\partial_x v_\eta^\delta(x, t)) & \text{in } \mathbb{R} \times (0, T) \\ v_\eta^\delta(x, 0) = v_0(x) & \text{in } \mathbb{R}. \end{cases} \quad (2.4)$$

For the regularized Eq. (2.4), we have the following result.

Theorem 2.1. (Existence and uniqueness of Lipschitz continuous solution of (2.1)) *Assume that (2.2) holds. Then, for all $T > 0$, we have*

(i) (Existence and uniqueness of (2.4)):

Equation (2.4) admits a unique solution $v_\eta^\delta \in C^\infty(\mathbb{R} \times (0, T)) \cap W^{2,\infty}(\mathbb{R} \times (0, T))$ satisfying the following estimates:

$$\|v_\eta^\delta\|_{L^\infty(\mathbb{R} \times (0, T))} \leq T\delta \|\tilde{c}\|_{L^\infty(\mathbb{R} \times (0, T))} + \|v_0\|_{L^\infty(\mathbb{R})} \quad (2.5)$$

$$\|\partial_x v_\eta^\delta\|_{L^\infty(\mathbb{R} \times (0, T))} \leq (\|\partial_x v_0\|_{L^\infty(\mathbb{R})} + \delta) e^{T\|\partial_x \tilde{c}\|_{L^\infty(\mathbb{R} \times (0, T))}}. \quad (2.6)$$

(ii) (Convergence):

Up to extract a subsequence, the function v_η^δ converges, as δ goes to 0, to a function v_η locally and strongly in $C([0, T]; W^{1,\infty}(\mathbb{R}))$. Moreover, v_η is the unique solution of (2.1) (in the distributional sense) belonging to the space $C([0, T]; W^{1,\infty}(\mathbb{R}))$ and satisfies, for all $0 < \eta \leq 1$, the following η -uniform estimates:

$$\|v_\eta\|_{L^\infty(\mathbb{R} \times (0, T))} \leq \|v_0\|_{L^\infty(\mathbb{R})} \quad (2.7)$$

$$\|\partial_x v_\eta\|_{L^\infty(\mathbb{R} \times (0, T))} \leq \|\partial_x v_0\|_{L^\infty(\mathbb{R})} e^{T\|\partial_x \tilde{c}\|_{L^\infty(\mathbb{R} \times (0, T))}}. \quad (2.8)$$

The lines of the proof of this theorem are very standard (see for instance El Hajj [17] for a similar technics). For this reason, we skip the details of the proof. We just point out that, in the first point (i), the existence result comes from a fixed point argument, while estimates (2.5)–(2.6) derive directly by applying the Maximum Principle Theorem for parabolic equations (see Lieberman [23, Th 2.10]). The regularity of the solution in (i) follows from a bootstrap argument. To prove (ii), we use the compactness lemma of Simon [25, Corollary 4.], inherited from estimates (2.5)–(2.6) and the standard parabolic estimates. The last estimates (2.7)–(2.8) come directly from (2.5)–(2.6) by passing to the limit as δ goes to 0. The uniqueness of the solution is an independent result valid for Lipschitz in space solutions.

3. A priori uniform estimates on the solution of (2.1)

In this section, we show two η -uniform estimates on the solution of equation (2.1) obtained in Theorem 2.1 (ii).

The first one concerns the BV estimate of the equation and is a key result.

Lemma 3.1. (*BV estimate*)

Assume that (2.2) holds and that v_0 and $\tilde{c}$ satisfy the following assumptions:

$$\partial_x v_0 \in L^1(\mathbb{R}) \quad \text{and} \quad \partial_x \tilde{c} \in L^\infty((0, T); L^1(\mathbb{R})). \quad (3.1)$$

Then, for all $0 < \eta \leq 1$, the solution v_η of (2.1), given by Theorem 2.1 (ii), satisfies (2.7), (2.8) and the following estimate:

$$\int_{\mathbb{R}} |\partial_x v_\eta(x, t)| dx \leq \int_{\mathbb{R}} |\partial_x v_0(x)| dx, \quad \text{for all } t \geq 0. \quad (3.2)$$

Proof of Lemma 3.1

First, we prove the above estimate on v_η^δ the solution of (2.4), constructed in Theorem 2.1 (i), and then, thanks to the uniqueness of solution, we deduce the announced result by passing to the limit as δ goes to 0. Indeed, differentiating the first equation in (2.4) with respect to x and then multiplying by $\beta'_\delta(\partial_x v_\eta^\delta)$, where β_δ is defined in (2.3), we obtain

$$\partial_t(\beta_\delta(\partial_x v_\eta^\delta)) = \eta \partial_{xx}^2(\partial_x v_\eta^\delta) \beta'_\delta(\partial_x v_\eta^\delta) + \partial_x(\tilde{c}(x, t) \beta_\delta(\partial_x v_\eta^\delta)) \beta'_\delta(\partial_x v_\eta^\delta). \quad (3.3)$$

Let $\phi \in C^\infty(\mathbb{R})$ be a cut-off function taking values in $[0, 1]$, supported on the interval $[-2, 2]$ and $\phi(x) \equiv 1$ on $[-1, 1]$. Multiplying (3.3) by $\phi_R(\cdot) = \phi(\frac{\cdot}{R})$, for $R > 0$, and integrating over the spatial variable, we get

$$\begin{aligned} \partial_t \left[\int_{-2R}^{2R} \beta_\delta(\partial_x v_\eta^\delta(x, t)) \phi_R(x) dx \right] &= \underbrace{\eta \int_{-2R}^{2R} \partial_{xx}^2(\partial_x v_\eta^\delta(x, t)) \beta'_\delta(\partial_x v_\eta^\delta(x, t)) \phi_R(x) dx}_{I_1} \\ &+ \underbrace{\int_{-2R}^{2R} \partial_x(\tilde{c}(x, t) \beta_\delta(\partial_x v_\eta^\delta(x, t))) \beta'_\delta(\partial_x v_\eta^\delta(x, t)) \phi_R(x) dx}_{I_2}. \end{aligned} \quad (3.4)$$

We note that these computations are justified because $v_\eta^\delta \in C^\infty(\mathbb{R} \times (0, T))$. In what follows, we will show that the terms I_1 and I_2 are bounded uniformly in η and in δ .

Step 1 (Estimate of I_1): From integration by parts, we have

$$\begin{aligned} I_1 &= -\eta \left[\int_{-2R}^{2R} (\partial_{xx}^2 v_\eta^\delta(x, t))^2 \beta''_\delta(\partial_x v_\eta^\delta(x, t)) \phi_R(x) dx \right. \\ &\quad \left. + \frac{1}{R} \int_{-2R}^{2R} \partial_{xx}^2 v_\eta^\delta(x, t) \beta'_\delta(\partial_x v_\eta^\delta(x, t)) \phi' \left(\frac{x}{R} \right) dx \right]. \end{aligned}$$

Thanks to the convexity of β_δ and the positivity of ϕ_R , we know that

$$(\partial_{xx}^2 v_\eta^\delta(x, t))^2 \beta''_\delta(\partial_x v_\eta^\delta(x, t)) \phi_R(x) \geq 0.$$

This implies that I_1 satisfies the following inequality

$$\begin{aligned} I_1 &\leq -\frac{\eta}{R} \int_{-2R}^{2R} \partial_{xx}^2 v_\eta^\delta(x, t) \beta'_\delta(\partial_x v_\eta^\delta(x, t)) \phi' \left(\frac{x}{R} \right) dx \\ &\leq -\frac{\eta}{R} \int_{-2R}^{2R} \partial_x (\beta_\delta(\partial_x v_\eta^\delta(x, t))) \phi' \left(\frac{x}{R} \right) dx. \end{aligned}$$

Integrating again by parts the right hand side of the above inequality, we obtain

$$I_1 \leq \frac{\eta}{R^2} \int_{-2R}^{2R} \beta_\delta(\partial_x v_\eta^\delta(x, t)) \phi'' \left(\frac{x}{R} \right) dx.$$

Using (2.6) with $M_\delta := (\|\partial_x v_0\|_{L^\infty(\mathbb{R})} + \delta) e^{T\|\partial_x \tilde{c}\|_{L^\infty(\mathbb{R} \times (0, T))}}$, we finally get that

$$I_1 \leq \frac{4\eta}{R} \beta_\delta(M_\delta) \|\phi''\|_{L^\infty(\mathbb{R})}. \quad (3.5)$$

Step 2 (Estimate of I_2): To estimate I_2 , we integrate by parts to get

$$\begin{aligned} I_2 &= - \underbrace{\int_{-2R}^{2R} \tilde{c}(x, t) \beta_\delta(\partial_x v_\eta^\delta(x, t)) \beta''_\delta(\partial_x v_\eta^\delta(x, t)) \partial_{xx}^2 v_\eta^\delta(x, s) \phi_R(x) dx}_{J_1} \\ &\quad - \underbrace{\frac{1}{R} \int_{-2R}^{2R} \tilde{c}(x, t) \beta_\delta(\partial_x v_\eta^\delta(x, t)) \beta'_\delta(\partial_x v_\eta^\delta(x, t)) \phi' \left(\frac{x}{R} \right) dx}_{J_2}. \end{aligned}$$

Furthermore, by the property of the function β_δ defined in (2.3), we can see that

$$\partial_{xx}^2 v_\eta^\delta \beta_\delta(\partial_x v_\eta^\delta) \beta''_\delta(\partial_x v_\eta^\delta) = \delta \partial_x \left(\arctan \left[\frac{\partial_x v_\eta^\delta}{\delta} \right] \right),$$

which shows that

$$J_1 = -\delta \int_{-2R}^{2R} \tilde{c}(x, t) \phi_R(x) \partial_x \left(\arctan \left[\frac{\partial_x v_\eta^\delta}{\delta}(x, t) \right] \right) dx.$$

Integrating by parts the above equality, we obtain

$$\begin{aligned} J_1 &= \delta \int_{-2R}^{2R} \partial_x \tilde{c}(x, t) \phi_R(x) \arctan \left(\frac{\partial_x v_\eta^\delta}{\delta}(x, t) \right) dx \\ &\quad + \frac{\delta}{R} \int_{-2R}^{2R} \tilde{c}(x, t) \phi' \left(\frac{x}{R} \right) \arctan \left(\frac{\partial_x v_\eta^\delta}{\delta}(x, t) \right) dx. \end{aligned}$$

Thereafter, using Hölder's inequality, we deduce that

$$J_1 \leq \delta \frac{\pi}{2} \|\tilde{c}\|_{W^{1,\infty}(\mathbb{R} \times (0, T))} (R \|\phi\|_{L^1(\mathbb{R})} + \|\phi'\|_{L^1(\mathbb{R})}). \quad (3.6)$$

However, since $\beta_\delta(\partial_x v_\eta^\delta)\beta'_\delta(\partial_x v_\eta^\delta) = \partial_x v_\eta^\delta$, we can rewrite J_2 as

$$J_2 = -\frac{1}{R} \int_{-2R}^{2R} \tilde{c}(x, t) \partial_x v_\eta^\delta(x, t) \phi' \left(\frac{x}{R} \right) dx.$$

Integrating by parts, we get

$$J_2 = \frac{1}{R} \int_{-2R}^{2R} \partial_x \tilde{c}(x, t) v_\eta^\delta(x, t) \phi' \left(\frac{x}{R} \right) dx + \frac{1}{R^2} \int_{-2R}^{2R} \tilde{c}(x, t) v_\eta^\delta(x, t) \phi'' \left(\frac{x}{R} \right) dx,$$

which gives, by Hölder's inequality, the following estimate

$$\begin{aligned} J_2 &\leq \frac{1}{R} \|\partial_x \tilde{c}\|_{L^\infty((0,T);L^1(\mathbb{R}))} \|v_\eta^\delta\|_{L^\infty(\mathbb{R} \times (0,T))} \|\phi'\|_{L^\infty(\mathbb{R})} \\ &\quad + \frac{1}{R} \|\phi''\|_{L^1(\mathbb{R})} \|v_\eta^\delta\|_{L^\infty(\mathbb{R} \times (0,T))} \|\tilde{c}\|_{L^\infty(\mathbb{R} \times (0,T))}. \end{aligned}$$

From estimate (2.5), we deduce that

$$J_2 \leq \frac{\overline{M}_\delta}{R} (\|\partial_x \tilde{c}\|_{L^\infty((0,T);L^1(\mathbb{R}))} \|\phi'\|_{L^\infty(\mathbb{R})} + \|\tilde{c}\|_{L^\infty(\mathbb{R} \times (0,T))} \|\phi''\|_{L^1(\mathbb{R})}), \quad (3.7)$$

where $\overline{M}_\delta = \|v_0\|_{L^\infty(\mathbb{R})} + \delta T \|\tilde{c}\|_{L^\infty(\mathbb{R} \times (0,T))}$. Combining (3.6) and (3.7), one gets

$$\begin{aligned} I_2 &\leq \delta \frac{\pi}{2} \|\tilde{c}\|_{W^{1,\infty}(\mathbb{R} \times (0,T))} (R \|\phi\|_{L^1(\mathbb{R})} + \|\phi'\|_{L^1(\mathbb{R})}) \\ &\quad + \frac{\overline{M}_\delta}{R} (\|\partial_x \tilde{c}\|_{L^\infty((0,T);L^1(\mathbb{R}))} \|\phi'\|_{L^\infty(\mathbb{R})} + \|\tilde{c}\|_{L^\infty(\mathbb{R} \times (0,T))} \|\phi''\|_{L^1(\mathbb{R})}). \end{aligned} \quad (3.8)$$

Step 3 (Passage to the limit): Integrating (3.4) in time on $(0, t)$, for $0 < t < T$, we get

$$\int_{-2R}^{2R} \beta_\delta(\partial_x v_\eta^\delta(x, t)) \phi_R(x) dx \leq \int_{-2R}^{2R} \beta_\delta(\partial_x v_0(x)) \phi_R(x) dx + T(I_1 + I_2).$$

Passing to the limit in the previous estimate, as δ goes to 0, and using the fact that $\partial_x v_\eta^\delta$ converges locally uniformly, as δ goes to 0, toward $\partial_x v_\eta$, we obtain

$$\begin{aligned} \int_{-2R}^{2R} |\partial_x v_\eta(x, t)| \phi_R(x) dx &\leq \int_{-2R}^{2R} |\partial_x v_0(x)| \phi_R(x) dx \\ &\quad + \frac{4\eta T}{R} \|\partial_x v_0\|_{L^\infty(\mathbb{R})} \|\phi''\|_{L^\infty(\mathbb{R})} e^{T \|\partial_x \tilde{c}\|_{L^\infty(\mathbb{R} \times (0,T))}} \\ &\quad + \frac{T}{R} \|v_0\|_{L^\infty(\mathbb{R})} (\|\partial_x \tilde{c}\|_{L^\infty((0,T);L^1(\mathbb{R}))} \|\phi'\|_{L^\infty(\mathbb{R})} \\ &\quad + \|\tilde{c}\|_{L^\infty(\mathbb{R} \times (0,T))} \|\phi''\|_{L^1(\mathbb{R})}) \end{aligned}$$

where we have used estimate (3.5) in the first line and (3.8) in second line. According to (2.8), (3.1) and the monotone convergence theorem, we get, by passing to the limit $R \rightarrow +\infty$ in the previous inequality

$$\int_{\mathbb{R}} |\partial_x v_\eta(x, t)| dx \leq \int_{\mathbb{R}} |\partial_x v_0(x)| dx,$$

which leads to the desired result. ■

The following estimate will provide the compactness in time of the solution v_η , given in Theorem 2.1 (ii), uniformly with respect to η .

Lemma 3.2. (Estimate on the time derivative of the solution) *Assume that (2.2) holds and let $W^{-1,\infty}(\mathbb{R})$ be the dual space of $W^{1,1}(\mathbb{R})$. Then, for all $0 < \eta \leq 1$ and for all $T > 0$, the solution v_η of (2.1), given by Theorem 2.1(ii), satisfies the following estimate:*

$$\|\partial_t v_\eta\|_{L^\infty((0,T);W^{-1,\infty}(\mathbb{R}))} \leq \widetilde{M}, \quad (3.9)$$

where $\widetilde{M}$ depends only on T , $\|\tilde{c}\|_{L^\infty(\mathbb{R} \times (0,T))}$, $\|\partial_x \tilde{c}\|_{L^\infty(\mathbb{R} \times (0,T))}$ and $\|v_0\|_{W^{1,\infty}(\mathbb{R})}$.

Proof of Lemma 3.2

The idea is somehow to bound $\partial_t v_\eta^\delta$ using the available bounds on the right hand side of the Eq. (2.4) and then passing to the limit $\delta \rightarrow 0$ to bound $\partial_t v_\eta$. The proof is given by duality. Multiplying the Eq. (2.4) by $\phi \in L^1((0,T);W^{1,1}(\mathbb{R}))$ and integrating on $\mathbb{R} \times (0,T)$, we get

$$\int_{\mathbb{R} \times (0,T)} \phi \partial_t v_\eta^\delta = \eta \overbrace{\int_{\mathbb{R} \times (0,T)} \phi \partial_{xx}^2 v_\eta^\delta}^{I_1} + \overbrace{\int_{\mathbb{R} \times (0,T)} \phi \tilde{c} \beta_\delta (\partial_x v_\eta^\delta)}^{I_2}.$$

Integrating by parts in I_1 , for all $0 < \eta \leq 1$, we obtain

$$\begin{aligned} |I_1| &\leq \left| \int_{\mathbb{R} \times (0,T)} \partial_x \phi \cdot \partial_x v_\eta^\delta \right| \leq \|\partial_x \phi\|_{L^1(\mathbb{R} \times (0,T))} \|\partial_x v_\eta^\delta\|_{L^\infty(\mathbb{R} \times (0,T))}, \\ &\leq \|\phi\|_{L^1((0,T);W^{1,1}(\mathbb{R}))} M_\delta, \end{aligned} \quad (3.10)$$

where we have used inequality (2.6) in the second line, noting

$$M_\delta := (\|\partial_x v_0\|_{L^\infty(\mathbb{R})} + \delta) e^{T\|\partial_x \tilde{c}\|_{L^\infty(\mathbb{R} \times (0,T))}}.$$

Similarly, for the term I_2 , we have

$$\begin{aligned} |I_2| &\leq \|\phi\|_{L^1(\mathbb{R} \times (0,T))} \|\tilde{c}\|_{L^\infty(\mathbb{R} \times (0,T))} \|\beta_\delta (\partial_x v_\eta^\delta)\|_{L^\infty(\mathbb{R} \times (0,T))}, \\ &\leq \|\phi\|_{L^1((0,T);W^{1,1}(\mathbb{R}))} \|\tilde{c}\|_{L^\infty(\mathbb{R} \times (0,T))} \beta_\delta(M_\delta). \end{aligned} \quad (3.11)$$

Collecting (3.10) and (3.11), we get that

$$\int_{\mathbb{R} \times (0,T)} \phi \partial_t v_\eta^\delta \leq C_\delta \|\phi\|_{L^1((0,T);W^{1,1}(\mathbb{R}))}.$$

where $C_\delta := M_\delta + \|\tilde{c}\|_{L^\infty(\mathbb{R} \times (0,T))} \beta_\delta(M_\delta)$. Then, by duality, we deduce that

$$\|\partial_t v_\eta^\delta\|_{L^\infty((0,T);W^{-1,\infty}(\mathbb{R}))} \leq C_\delta.$$

Finally, passing to the limit in the previous inequality, as δ goes to 0, and using the weakly- $\star$ convergence in $L^\infty((0,T);W^{-1,\infty}(\mathbb{R}))$, we get

$$\|\partial_t v_\eta\|_{L^\infty((0,T);W^{-1,\infty}(\mathbb{R}))} \leq \liminf_{\delta \rightarrow 0} \|\partial_t v_\eta^\delta\|_{L^\infty((0,T);W^{-1,\infty}(\mathbb{R}))} \leq \widetilde{M}$$

with

$$\widetilde{M} = \lim_{\delta \rightarrow 0} C_\delta = (1 + \|\tilde{c}\|_{L^\infty(\mathbb{R} \times (0, T))}) \|\partial_x v_0\|_{L^\infty(\mathbb{R})} e^{T \|\partial_x \tilde{c}\|_{L^\infty(\mathbb{R} \times (0, T))}},$$

which gives the stated result. ■

4. Global continuous solution for regularized eikonal equation

This section is devoted to the proof of Theorem 1.1 (i). For the sake of a clear presentation, we divide it into two subsections. In the first subsection, we prove the global existence of Lipschitz continuous viscosity solution of the eikonal equation (4.1), passing to the limit $\eta \rightarrow 0$ in equation (2.1). Using this result, we prove Theorem 1.1 (i) in the next subsection.

4.1. Passage to the limit when the parameter η tends to 0

In this subsection, we state a global in time existence and uniqueness result for the following equation:

$$\begin{cases} \partial_t v(x, t) = \tilde{c}(x, t) |\partial_x v(x, t)| & \text{in } \mathbb{R} \times (0, T) \\ v(x, 0) = v_0(x) & \text{in } \mathbb{R}. \end{cases} \quad (4.1)$$

We prove that this equation admits a unique Lipschitz continuous viscosity solution v , when the initial data and the velocity are Lipschitz continuous functions. Moreover, we show that v is the limit of v_η when $\eta \rightarrow 0$, where v_η is the solution of (2.1) given by Theorem 2.1 (ii). We also show that v satisfies Lipschitz estimates (2.7)–(2.8) if (2.2) holds and the BV estimate (3.2) if (3.1) holds.

Before showing the main theorem of this subsection, let us recall the definition of the continuous viscosity solution for (2.1) with $0 \leq \eta \leq 1$.

Definition 4.1. (*Continuous viscosity sub-solution, super-solution and solution*) A function $v \in C(\mathbb{R} \times [0, T))$ is a viscosity sub-solution of (2.1) if and only if $v(x, 0) \leq v_0(x)$ and for every $(x_0, t_0) \in \mathbb{R} \times (0, T)$ and for every test function $\phi \in C^2(\mathbb{R} \times (0, T))$, that is tangent from above to v at (x_0, t_0) , the following holds:

$$\partial_t \phi(x_0, t_0) - \tilde{c}(x_0, t_0) |\partial_x \phi(x_0, t_0)| - \eta \partial_{xx}^2 \phi(x_0, t_0) \leq 0.$$

A function $v \in C(\mathbb{R} \times [0, T))$ is a viscosity super-solution of (2.1) if and only if $v(x, 0) \geq v_0(x)$ and for every $(x_0, t_0) \in \mathbb{R} \times (0, T)$ and for every test function $\phi \in C^2(\mathbb{R} \times (0, T))$, that is tangent from below to v at (x_0, t_0) , the following holds:

$$\partial_t \phi(x_0, t_0) - \tilde{c}(x_0, t_0) |\partial_x \phi(x_0, t_0)| - \eta \partial_{xx}^2 \phi(x_0, t_0) \geq 0.$$

A continuous function v is a viscosity solution of (2.1) if and only if it is a sub- and a super-solution of (2.1).

We have the following existence and uniqueness result for the local equation (4.1).

Theorem 4.2. (Existence and uniqueness of Lipschitz continuous viscosity solution) *Assume that (2.2) holds, then, for all $T > 0$, we have:*

- (i) *The problem (4.1) admits a unique Lipschitz continuous viscosity solution on $\mathbb{R} \times (0, T)$, satisfying the following estimates:*

$$\|v\|_{L^\infty(\mathbb{R} \times (0, T))} \leq \|v_0\|_{L^\infty(\mathbb{R})} \quad (4.2)$$

$$\|\partial_x v\|_{L^\infty(\mathbb{R} \times (0, T))} \leq \|\partial_x v_0\|_{L^\infty(\mathbb{R})} e^{T\|\partial_x \tilde{c}\|_{L^\infty(\mathbb{R} \times (0, T))}}. \quad (4.3)$$

- (ii) *Assume, moreover, that (3.1) holds, then v satisfies the following BV estimate:*

$$\int_{\mathbb{R}} |\partial_x v(x, t)| dx \leq \int_{\mathbb{R}} |\partial_x v_0(x)| dx, \quad \text{for all } t \in [0, T) \quad (4.4)$$

and also the following:

$$\int_{\mathbb{R}} |\partial_t v(x, t)| dx \leq \|\tilde{c}\|_{L^\infty(\mathbb{R} \times (0, T))} \|\partial_x v_0\|_{L^1(\mathbb{R})}, \quad \text{for all } t \in [0, T). \quad (4.5)$$

To prove this theorem, we need the following stability lemma.

Lemma 4.3. (Stability of continuous viscosity solutions, Barles [5, Th 2.3]) *Let v_η be the solution of (2.1) constructed in Theorem 2.1 (ii) as the limit of v_η^δ when $\delta \rightarrow 0$. Then, we have:*

- (i) *The function v_η is also a continuous viscosity solution of (2.1).*
 (ii) *If v_η converges locally uniformly, as η goes to 0, to a function v , then v is a Lipschitz continuous viscosity solution of (4.1).*

To prove this lemma, it is sufficient to apply the stability result for continuous viscosity solution (see Barles [5, Th 2.3]) and the fact that all C^2 solutions of (2.4) are continuous viscosity solutions of (2.4).

Let us also recall the following compactness lemma.

Lemma 4.4. (Simon's Lemma [25, Corollary 4.]) *Let X , B and Y be three Banach spaces, where $X \hookrightarrow B$ with compact embedding and $B \hookrightarrow Y$ with continuous embedding. If $(\theta_n)_n$ is a sequence uniformly bounded in $L^\infty((0, T); X)$ and $(\partial_t \theta_n)_n$ is uniformly bounded in $L^r((0, T); Y)$ where $r > 1$, then, $(\theta_n)_n$ is relatively compact in $C((0, T); B)$.*

Proof of Theorem 4.2

Proof of (i): Let v_η be the solution of (2.1), constructed in Theorem 2.1 (ii). First, by Lemma 4.3 (i), we know that v_η is a continuous viscosity solution of (2.1).

Let K be a compact subset in $\mathbb{R}$. Using estimates (2.7), (2.8), (3.9) and applying Lemma 4.4 in the particular case where $X = W^{1, \infty}(K)$, $B = L^\infty(K)$ and $Y = W^{-1, \infty}(K) = (W^{1, 1}(K))'$, we can show that v_η is relatively compact in $L^\infty(K \times (0, T))$. Hence, as η goes to zero, we can extract a subsequence, still denoted by v_η , that converges locally uniformly to some limit v . Then, by Lemma 4.3 (ii), we can see that v is a Lipschitz continuous viscosity solution of (4.1). This proves the existence of a Lipschitz continuous viscosity solution of (4.1). However, the uniqueness of the solution is an independent result valid

for Lipschitz continuous viscosity solutions of (4.1) with Lipschitz velocity (see for instance Crandall et al. [16]).

Finally, we pass to the limit in the estimates (2.7) and (2.8), using the lower semi-continuity on $L^\infty(\mathbb{R} \times (0, T))$, we get (4.2) and (4.3).

Proof of (ii): From the estimate (3.2), we deduce that $(v_\eta(\cdot, t))_\eta$ is uniformly bounded in $BV(\mathbb{R})$ and then thanks to the lower semi-continuity of the total variation, we get

$$TV(v(\cdot, t)) \leq \liminf_{\eta \rightarrow 0} TV(v_\eta(\cdot, t)) \leq \int_{\mathbb{R}} |\partial_x v_0(x)| dx \quad \text{for all } t \geq 0.$$

Moreover, since v is Lipschitz continuous, we know that the L^1 -norm of $\partial_x v$ coincides with the total variation of v , which proves (4.4).

The last estimate (4.5) comes directly from estimate (4.4), using the Hölder's inequality and the fact that the Lipschitz continuous viscosity solution v satisfies the Eq. (4.1) almost everywhere. $\blacksquare$

Remark 4.5. (L^1 -Continuity in time) Under assumptions (2.2), (3.1), from estimate (4.5), we can remark that the solution v of (4.1), given by Theorem 4.2 (i), satisfies the following L^1 -Lipschitz estimate in time:

$$\|v(\cdot, t) - v(\cdot, s)\|_{L^1(\mathbb{R})} \leq \|\tilde{c}\|_{L^\infty(\mathbb{R} \times (0, T))} \|\partial_x v_0\|_{L^1(\mathbb{R})} |t - s| \quad \text{for all } t, s \geq 0.$$

4.2. Proof of Theorem 1.1 (i)

In this subsection, we prove the global existence and uniqueness of a Lipschitz continuous viscosity solution of Eq. (1.1) after the regularization of the velocity c and the initial data u_0 , which was announced in Theorem 1.1 (i). This approximation brings us to consider, for every $\epsilon > 0$, the following equation:

$$\begin{cases} \partial_t u_\epsilon(x, t) = c_\epsilon(x, t) |\partial_x u_\epsilon(x, t)| & \text{in } \mathbb{R} \times (0, T) \\ u_\epsilon(x, 0) = u_{0, \epsilon}(x) & \text{in } \mathbb{R} \end{cases} \quad (4.6)$$

where

$$u_{0, \epsilon}(x) = u_0 \star \rho_\epsilon^1(x) \quad \text{and} \quad c_\epsilon(x, t) = \hat{c} \star \rho_\epsilon^2(x, t) \quad \forall x \in \mathbb{R}, t \in \mathbb{R}, \quad (4.7)$$

with the function $\hat{c}$ is an extension in $\mathbb{R}^2$ of the function c by 0 and $\rho_\epsilon^1, \rho_\epsilon^2$ are the standard mollifiers defined as follows:

$$\begin{aligned} \rho_\epsilon^1(\cdot) &= \frac{1}{\epsilon} \rho^1\left(\frac{\cdot}{\epsilon}\right), \quad \text{such that } \rho^1 \in C_c^\infty(\mathbb{R}), \text{supp}\{\rho^1\} \subseteq B(0, 1), \rho^1 \geq 0, \\ &\text{and } \int_{\mathbb{R}} \rho^1 = 1 \\ \rho_\epsilon^2(\cdot, \cdot) &= \frac{1}{\epsilon^2} \rho^2\left(\frac{\cdot}{\epsilon}, \frac{\cdot}{\epsilon}\right), \quad \text{such that } \rho^2 \in C_c^\infty(\mathbb{R}^2), \text{supp}\{\rho^2\} \subseteq B(0, 1), \\ \rho^2 &\geq 0, \text{ and } \int_{\mathbb{R}^2} \rho^2 = 1. \end{aligned}$$

By assumptions (1.2), (1.3) and by classical properties of the mollifiers $\rho_\epsilon^1, \rho_\epsilon^2$, we can see that, for all $\epsilon > 0$, the functions $u_{0, \epsilon}$ and c_ϵ satisfy assumptions (2.2) and (3.1). Thus, we can apply Theorem 4.2, with $\tilde{c} = c_\epsilon$ and $v_0 = u_{0, \epsilon}$ and

prove that Eq. (4.6) admits a unique Lipschitz continuous viscosity solution u_ϵ satisfying the following estimates:

$$\|u_\epsilon\|_{L^\infty(\mathbb{R} \times (0, T))} \leq \|u_{0, \epsilon}\|_{L^\infty(\mathbb{R})} \quad (4.8)$$

$$\|\partial_x u_\epsilon(\cdot, t)\|_{L^1(\mathbb{R})} \leq \|\partial_x u_{0, \epsilon}\|_{L^1(\mathbb{R})}, \quad \text{for all } t \in [0, T) \quad (4.9)$$

$$\|\partial_t u_\epsilon(\cdot, t)\|_{L^1(\mathbb{R})} \leq \|c_\epsilon\|_{L^\infty(\mathbb{R} \times (0, T))} \|\partial_x u_{0, \epsilon}\|_{L^1(\mathbb{R})}, \quad \text{for all } t \in [0, T) \quad (4.10)$$

and also the following estimate (depending on ϵ)

$$\|\partial_x u_\epsilon\|_{L^\infty(\mathbb{R} \times (0, T))} \leq \|\partial_x u_{0, \epsilon}\|_{L^\infty(\mathbb{R})} e^{T \|\partial_x c_\epsilon\|_{L^\infty(\mathbb{R} \times (0, T))}}. \quad (4.11)$$

Moreover, by classical properties of the mollifiers, we know that (see Ambrosio et al. [3, Th 2.2 (b)], for the third estimate):

$$\begin{aligned} \|u_{0, \epsilon}\|_{L^\infty(\mathbb{R})} &\leq \|u_0\|_{L^\infty(\mathbb{R})} \\ \|c_\epsilon\|_{L^\infty(\mathbb{R} \times (0, T))} &\leq \|c\|_{L^\infty(\mathbb{R} \times (0, T))} \\ \|\partial_x u_{0, \epsilon}(x)\|_{L^1(\mathbb{R})} &\leq TV(u_0) = |u_0|_{BV(\mathbb{R})} \end{aligned}$$

which joint to (4.8), (4.9) and (4.10) imply (1.8), (1.9) and (1.10). $\blacksquare$

5. Existence of sub and super discontinuous viscosity solutions

This section is devoted to the proof of Theorem 1.1 (ii). In order to present it, we start by passing to the limit $\epsilon \rightarrow 0$ in Eq. (4.6), using estimate (1.8) and the stability result of discontinuous viscosity solution. This is based on the technical method of relaxed semi-limits introduced by Barles and Perthame ([10, 11]). This section will be divided in two subsections. First, in Sect. 5.1, we recall some useful results for viscosity solutions and then in Sect. 5.2, we give the proof of Theorem 1.1(ii).

5.1. Some useful results for viscosity solutions

In the following, we denote by f^* and f_* the respective upper and lower semi-continuous envelopes of a locally bounded function f defined on an open domain in $\mathbb{R}^n$ and given by

$$f^*(X) = \limsup_{Y \rightarrow X} f(Y) \quad \text{and} \quad f_*(X) = \liminf_{Y \rightarrow X} f(Y). \quad (5.1)$$

First, let us recall the definition of discontinuous viscosity solutions of the eikonal equation (1.1).

Definition 5.1. (*Discontinuous viscosity sub-solution, super-solution and solution*) Assume that c is locally bounded on $\mathbb{R} \times (0, T)$ and u_0 is locally bounded on $\mathbb{R}$.

(1) (Discontinuous viscosity sub-solution) An upper semi-continuous function v on $\mathbb{R} \times [0, T)$ is a discontinuous viscosity sub-solution of (1.1) if it satisfies:

- (i) $v(x, 0) \leq (u_0)^*(x)$, for all $x \in \mathbb{R}$.
- (ii) For every $(x_0, t_0) \in \mathbb{R} \times (0, T)$ and for every test function $\phi \in C^1(\mathbb{R} \times (0, T))$, tangent from above to v at the point (x_0, t_0) , we have:

$$\partial_t \phi(x_0, t_0) - c^*(x_0, t_0) |\partial_x \phi(x_0, t_0)| \leq 0. \quad (5.2)$$

(2) (Discontinuous viscosity super-solution)

A lower semi-continuous function v on $\mathbb{R} \times [0, T]$ is a discontinuous viscosity super-solution of (1.1) if it satisfies:

- (i) $v(x, 0) \geq (u_0)_*(x)$, for all $x \in \mathbb{R}$.
- (ii) For every $(x_0, t_0) \in \mathbb{R} \times (0, T)$ and for every test function $\phi \in C^1(\mathbb{R} \times (0, T))$, tangent from below to v at the point (x_0, t_0) , we have:

$$\partial_t \phi(x_0, t_0) - c_*(x_0, t_0) |\partial_x \phi(x_0, t_0)| \geq 0. \quad (5.3)$$

(3) (Discontinuous viscosity solution)

Finally, we say that a locally bounded function v defined on $\mathbb{R} \times [0, T]$ is a discontinuous viscosity solution of (1.1) if its upper semi-continuous (respectively lower semi-continuous) envelope is a viscosity sub-solution (respectively super-solution).

We also need to recall the following stability result of discontinuous viscosity solutions.

Theorem 5.2. (Stability discontinuous result, [5, Th 4.1]) *Assume that, for $\epsilon > 0$, u_ϵ is an upper semi-continuous viscosity sub-solution (resp. a lower semi-continuous viscosity super-solution) of*

$$\partial_t u_\epsilon = c_\epsilon(x, t) |\partial_x u_\epsilon| \quad \text{in } \mathbb{R} \times (0, T),$$

where $(c_\epsilon)_\epsilon$ is a sequence of uniformly locally bounded functions on $\mathbb{R} \times (0, T)$. Assume that the functions u_ϵ are uniformly locally bounded on $\mathbb{R} \times (0, T)$. Then, $\bar{u}$ defined by (1.6) (resp. $\underline{u}$ defined by (1.7)) is a viscosity sub-solution (resp. super-solution) of the equation

$$\partial_t u = \bar{c}(x, t) |\partial_x u| \quad \text{in } \mathbb{R} \times (0, T)$$

where $\bar{c}(x, t) = \limsup^* c_\epsilon(x, t)$ (resp. of the equation

$$\partial_t u = \underline{c}(x, t) |\partial_x u| \quad \text{in } \mathbb{R} \times (0, T)$$

where $\underline{c}(x, t) = \liminf_* c_\epsilon(x, t)$).

Remark 5.3. The previous stability result allows us to pass to the limit in the non-linear term, in viscosity sense, by disposing only L^∞ uniform bounds on the solution and on the velocity. Note that, in the case where c is continuous with respect to x and L^1 in time, it is possible to get a better stability result using the L^1 -viscosity solutions theory introduced by Ishii [21], see Barles [6].

We end this subsection by showing the following finite speed propagation property, valid on the continuous viscosity solutions of Eq. (4.1).

Lemma 5.4. (Finite speed propagation property) *Under the assumption (2.2), if v is the unique continuous viscosity solution of (4.1), given by Theorem 4.2 (i), then, v satisfies, for all $h \geq 0$, the following estimate:*

$$\inf_{|y-x| \leq t \|\tilde{c}\|_{L^\infty(\mathbb{R} \times (0, T))}} v(y, h) \leq v(x, t+h) \leq \sup_{|y-x| \leq t \|\tilde{c}\|_{L^\infty(\mathbb{R} \times (0, T))}} v(y, h), \quad (5.4)$$

for all $(x, t) \in \mathbb{R} \times [0, T-h]$.

Proof of Lemma 5.4

Let us start by proving the right hand side of (5.4), in viscosity sense, namely

$$v(x, t+h) \leq \sup_{|y-x| \leq t \|\tilde{c}\|_{L^\infty(\mathbb{R} \times (0, T))}} v(y, h). \quad (5.5)$$

Let $v^h(x, t) = v(x, t+h)$. Then, we can see that

$$v_t^h(x, t) = \partial_t v(x, t+h) = \tilde{c}(x, t+h) |\partial_x v(x, t+h)| = \tilde{c}^h(x, t) |v_x^h(x, t)|.$$

Thus, v^h is a viscosity solution of

$$\partial_t w = \tilde{c}^h |\partial_x w|, \quad w(x, 0) = v(x, h).$$

Since $\tilde{c}^h(x, t) \leq \|\tilde{c}\|_{L^\infty(\mathbb{R} \times (0, T))}$, we can deduce that v^h is viscosity sub-solution of the following equation:

$$\partial_t w = C_1 |\partial_x w|, \quad w(x, 0) = v(x, h), \quad \text{with } C_1 = \|\tilde{c}\|_{L^\infty(\mathbb{R} \times (0, T))}. \quad (5.6)$$

Furthermore, if we note

$$\alpha(x, t) = \sup_{|y-x| \leq C_1 t} v(y, h),$$

then, by Lax–Oleinik formula (See [5, Lemma 2.1]), we know that α is the unique viscosity continuous solution of (5.6). Using the comparison principle (see [12, Th 1.1]), we deduce that

$$v(x, t+h) \leq \alpha(x, t), \quad \text{on } \mathbb{R} \times (0, T),$$

which implies (5.5). The same proof is done for the inequality

$$\inf_{|y-x| \leq C_1 t} v(y, h) \leq v(x, t+h),$$

by considering the following equation $\partial_t w = -C_1 |\partial_x w|$. ■

5.2. Proof of Theorem 1.1 (ii)

We only prove the result for the sub-solution case, the super-solution case is proved analogously. Let u_ϵ be the solution of (4.6), constructed in Theorem 1.1 (i). We have to prove that the relaxed semi-limit $\bar{u}$ defined in (1.6) is a discontinuous viscosity sub-solution of (1.1), in the sense of Definition 5.1. We do this in two steps.

Step 1 (Meaning of the initial data): It is sufficient to prove the following inequality

$$\bar{u}(x, 0) \leq (u_0)^*(x) \quad \text{for all } x \in \mathbb{R}. \quad (5.7)$$

From the definition of $\bar{u}$, we know that there exists a sequence $(\epsilon_n, x_{\epsilon_n}, t_{\epsilon_n}) \rightarrow (0, x, 0)$, when $n \rightarrow +\infty$ such that

$$\bar{u}(x, 0) = \lim_{n \rightarrow +\infty} u_{\epsilon_n}(x_{\epsilon_n}, t_{\epsilon_n}).$$

Using Lemma 5.4 with $h = 0$ and $t = t_{\epsilon_n}$, we get

$$\begin{aligned} u_{\epsilon_n}(x_{\epsilon_n}, t_{\epsilon_n}) &\leq \sup_{|y-x_{\epsilon_n}| \leq t_{\epsilon_n} \|c_{\epsilon_n}\|_{L^\infty(\mathbb{R} \times (0, T))}} u_{\epsilon_n}(y, 0) \\ &\leq \sup_{|y-x_{\epsilon_n}| \leq t_{\epsilon_n} \|c\|_{L^\infty(\mathbb{R} \times (0, T))}} \left(\int_{\mathbb{R}} u_0(z) \rho_{\epsilon_n}^1(y-z) dz \right) \\ &\leq \sup_{|y-x_{\epsilon_n}| \leq t_{\epsilon_n} \|c\|_{L^\infty(\mathbb{R} \times (0, T))}} \left(\sup_{|z-y| \leq \epsilon_n} u_0(z) \right), \end{aligned}$$

where we have used in the second and the third lines the definition of the functions u_{0, ϵ_n} , c_{ϵ_n} in (4.7) and also the classical properties of the mollifiers. Furthermore, the convergence, as $n \rightarrow +\infty$, of $(\epsilon_n, x_{\epsilon_n}, t_{\epsilon_n})_n$ to $(0, x, 0)$, implies that for all $\alpha > 0$, there exists $n_\alpha > 0$, such that, for all $n \geq n_\alpha$, we have:

$$\epsilon_n \leq \alpha, \quad |x_{\epsilon_n} - x| \leq \alpha \quad \text{and} \quad t_{\epsilon_n} \leq \alpha.$$

Thus, for every $\alpha > 0$ and $n \geq n_\alpha$, we have

$$u_{\epsilon_n}(x_{\epsilon_n}, t_{\epsilon_n}) \leq \sup_{|z-x| \leq \alpha(2+\|c\|_{L^\infty(\mathbb{R} \times (0, T))})} u_0(z). \quad (5.8)$$

First, we pass to the limit, $n \rightarrow +\infty$, in the previous inequality to obtain

$$\bar{u}(x, 0) \leq \sup_{|z-x| \leq \alpha(2+\|c\|_{L^\infty(\mathbb{R} \times (0, T))})} u_0(z).$$

Then, we pass to the limit, $\alpha \rightarrow 0$, to complete the proof of (5.7).

Step 2 (Meaning of the equation): We will show that $\bar{u}$ is a discontinuous viscosity sub-solution of the following equation

$$\partial_t u = c^*(x, t) |\partial_x u| \quad \text{in } \mathbb{R} \times (0, T). \quad (5.9)$$

Indeed, using the definition of c_ϵ in (4.7) and the classical properties of the mollifiers, we know that

$$\|c_\epsilon\|_{L^\infty(\mathbb{R} \times (0, T))} \leq \|c\|_{L^\infty(\mathbb{R} \times (0, T))}.$$

This proves, according to estimate (1.8) and the stability result of discontinuous viscosity solution, announced in Theorem 5.2, that $\bar{u}$ is a discontinuous viscosity sub-solution of the equation

$$\partial_t u = \bar{c}(x, t) |\partial_x u| \quad \text{in } \mathbb{R} \times (0, T),$$

where $\bar{c}(x, t) = \limsup^* c_\epsilon(x, t)$. Moreover, we also know that there exists a sequence $(\epsilon_n, x_{\epsilon_n}, t_{\epsilon_n}) \rightarrow (0, x, t)$, as n goes to $+\infty$, such that

$$\bar{c}(x, t) = \lim_{n \rightarrow +\infty} c_{\epsilon_n}(x_{\epsilon_n}, t_{\epsilon_n}).$$

From (4.7), we can see that

$$\begin{aligned} c_{\epsilon_n}(x_{\epsilon_n}, t_{\epsilon_n}) &= \int_{\mathbb{R} \times (0, T)} c(y, \tau) \rho_{\epsilon_n}^2(x_{\epsilon_n} - y, t_{\epsilon_n} - \tau) dy d\tau \\ &\leq \sup_{\substack{|y - x_{\epsilon_n}| \leq \epsilon_n \\ |\tau - t_{\epsilon_n}| \leq \epsilon_n}} c(y, \tau), \end{aligned}$$

using the fact that $\rho_{\epsilon_n}^2 \geq 0$ and $\int_{\mathbb{R}^2} \rho_{\epsilon_n}^2 = 1$. Thanks to the convergence, as $n \rightarrow +\infty$, of $(\epsilon_n, x_{\epsilon_n}, t_{\epsilon_n})_n$ to $(0, x, t)$, we can deduce, as in (5.8), that for every $\alpha > 0$ there exists $n_\alpha > 0$, such that, for all $n \geq n_\alpha$, we have

$$c_{\epsilon_n}(x_{\epsilon_n}, t_{\epsilon_n}) \leq \sup_{\substack{|y - x| \leq 2\alpha \\ |\tau - t| \leq 2\alpha}} c(y, \tau).$$

Now, we pass to the limit, $n \rightarrow +\infty$, firstly and then $\alpha \rightarrow 0$ we get $\bar{c}(x, t) \leq c^*(x, t)$. This proves that $\bar{u}$ is a discontinuous viscosity sub-solution of (5.9). ■

6. Link between the sub and the super discontinuous viscosity solutions

In this section, we will prove Theorem 1.1 (iii) and Theorem 1.2. First, we show, in Sect. 6.1, some preliminary results that we need in Sect. 6.2 to complete the proof of Theorem 1.2. Finally, in Sect. 6.3, we complete the proof of Theorem 1.1 (iii).

6.1. Preliminary results

First of all, let us recall some properties valid on bounded $BV(\mathbb{R})$ -functions.

Lemma 6.1. (Properties of BV -functions, [3])

Let f be a bounded $BV(\mathbb{R})$ -function. Then, the following hold:

(i) f is continuous except at most on a countable set.

(ii) The right and left limits

$$f(x^+) = \lim_{\substack{y \rightarrow x, \\ y > x}} f(y), \quad f(x^-) = \lim_{\substack{y \rightarrow x, \\ y < x}} f(y)$$

exist at every point $x \in \mathbb{R}$. Moreover, there exists a unique right-continuous function f_r (resp. left-continuous function f_l) coinciding with f except on a countable set.

(iii) There exists a pair of non-decreasing functions $f^1, f^2 \in L^\infty(\mathbb{R})$ such that $f = f^1 - f^2$.

The following lemma shows a local estimate valid on sequences of non-decreasing functions converging locally and strongly in $L^1(\mathbb{R})$.

Lemma 6.2. (Sequences of non-decreasing functions)

(i) Sequences of non-decreasing functions strongly convergent in $L^1_{loc}(\mathbb{R})$:

Let $(\phi_\epsilon)_\epsilon$ be a sequence of non-decreasing defined functions on $\mathbb{R}$ such that $\phi_\epsilon \rightarrow \phi$ strongly in $L^1_{loc}(\mathbb{R})$, as $\epsilon \rightarrow 0$, with ϕ is a non-decreasing defined function on $\mathbb{R}$. Then, for all $a > 0$ and $0 < \delta \leq \frac{a}{2}$, there exists $\epsilon_a^\delta > 0$, such that, for every $0 < \epsilon \leq \epsilon_a^\delta$, the following estimate holds:

$$-\delta + \phi(x - \delta) \leq \phi_\epsilon(x) \leq \delta + \phi(x + \delta), \quad \forall x \in [-a, a]. \quad (6.1)$$

(ii) Sequences of non-decreasing functions strongly convergent in $C([0, T]; L^1_{loc}(\mathbb{R}))$: Let $(\phi_\epsilon)_\epsilon$ be a sequence of defined functions on $\mathbb{R} \times [0, T]$ such that, for all $t \in [0, T]$, the function $\phi_\epsilon(\cdot, t)$ is non-decreasing on $\mathbb{R}$. Assume, moreover, that $\phi_\epsilon \rightarrow \phi$ strongly in $C([0, T]; L^1_{loc}(\mathbb{R}))$, as $\epsilon \rightarrow 0$, with, for all $t \in [0, T]$, the function $\phi(\cdot, t)$ is defined and non-decreasing on $\mathbb{R}$. Then, for all $a > 0$ and $0 < \delta \leq \frac{a}{2}$, there exists $\epsilon_{a,T}^\delta > 0$, such that, for every $0 < \epsilon \leq \epsilon_{a,T}^\delta$, the following estimate holds:

$$-\delta + \phi(x - \delta, t) \leq \phi_\epsilon(x, t) \leq \delta + \phi(x + \delta, t), \quad \forall x \in [-a, a], \quad \forall t \in [0, T]. \quad (6.2)$$

Proof of Lemma 6.2

Proof of (i): We will just prove the right hand side of estimate (6.1). Let $a > 0$, since $\phi_\epsilon \rightarrow \phi$ strongly in $L^1_{loc}(\mathbb{R})$, then for all $0 < \delta \leq \frac{a}{2}$, there exists $\epsilon_a^\delta > 0$, such that, for all $0 < \epsilon \leq \epsilon_a^\delta$, we have

$$\|\phi_\epsilon - \phi\|_{L^1(-2a, 2a)} \leq \delta^2. \quad (6.3)$$

Furthermore, for all $x \in [-a, a]$, $\delta \in (0, \frac{a}{2}]$ and $y \in (x, x + \delta)$, as ϕ_ϵ and ϕ are non-decreasing functions, we can see that

$$\phi_\epsilon(x) - \phi(x + \delta) \leq \phi_\epsilon(y) - \phi(y).$$

Integrating with respect to y over the interval $(x, x + \delta)$, we get

$$\phi_\epsilon(x) - \phi(x + \delta) \leq \frac{1}{\delta} \|\phi_\epsilon - \phi\|_{L^1(-2a, 2a)}.$$

From (6.3), the above estimate implies that, for all $0 < \delta \leq \frac{a}{2}$ and $0 < \epsilon \leq \epsilon_a^\delta$

$$\phi_\epsilon(x) \leq \delta + \phi(x + \delta), \quad \text{for all } x \in [-a, a].$$

The left hand side of (6.1) is proved in the same way.

Proof of (ii): To prove (6.2), we repeat the same procedures as in the proof of (i), using the strong convergence in $C([0, T]; L^1_{loc}(\mathbb{R}))$ of ϕ_ϵ . ■

Finally, in the following lemma we show a local estimate valid on sequences of bounded and BV functions converging locally and strongly in $L^1(\mathbb{R})$.

Lemma 6.3. (Sequence of BV($\mathbb{R}$) functions) Let $(\phi_\epsilon)_\epsilon$ be a sequence of functions, defined on $\mathbb{R}$, uniformly bounded in $L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ and strongly convergent to $\phi \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ in $L^1_{loc}(\mathbb{R})$, with ϕ a right-continuous function. Then, there exists a subsequence $(\phi_{\epsilon'})_{\epsilon'}$ such that, for all $a > 0$ and for all

$0 < \delta \leq \frac{a}{2}$, there exists $\epsilon_a^\delta > 0$, such that, for all $0 < \epsilon' \leq \epsilon_a^\delta$, the following estimate holds:

$$-2\delta + \phi^1(x - \delta) - \phi^2(x + \delta) \leq \phi_{\epsilon'}(x) \leq 2\delta + \phi^1(x + \delta) - \phi^2(x - \delta),$$

$$\forall x \in [-a, a] \quad (6.4)$$

where ϕ^1 and ϕ^2 are two bounded, right-continuous and non-decreasing functions on $\mathbb{R}$ satisfying $\phi = \phi^1 - \phi^2$.

Proof of Lemma 6.3

The proof of this lemma follows from Lemma 6.2 (i) and the properties (ii) and (iii) of Lemma 6.1. As done before, we will just show the right hand side of (6.4). Indeed, let $(\phi_\epsilon)_\epsilon$ be a sequence of functions uniformly bounded in $BV(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then, there exist two bounded and non-decreasing functions ϕ_ϵ^1 and ϕ_ϵ^2 satisfying

$$\phi_\epsilon = \phi_\epsilon^1 - \phi_\epsilon^2 \quad (6.5)$$

and defined as

$$\phi_\epsilon^1(x) = TV_{(-\infty, x]}(\phi_\epsilon) + \phi_\epsilon(x), \quad \phi_\epsilon^2(x) = TV_{(-\infty, x]}(\phi_\epsilon), \quad (6.6)$$

where $TV_{(-\infty, x]}(\phi_\epsilon)$ is the total variation of ϕ_ϵ on $(-\infty, x]$. Furthermore, since the sequence $(\phi_\epsilon^1)_\epsilon$ is uniformly bounded in $L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$, then, by Helly's compactness Theorem (see [3, Th 3.23]) and using the weak- $\star$ convergence in $L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$, we can extract a subsequence $(\phi_{\epsilon'}^1)_{\epsilon'}$ such that

$$\phi_{\epsilon'}^1 \longrightarrow \phi^1, \quad \text{strongly in } L_{loc}^1(\mathbb{R}),$$

with $\phi^1 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$. According to the property (ii) of Lemma 6.1, we know that ϕ^1 coincides with a right-continuous function almost everywhere in $\mathbb{R}$. Since the two functions are equal in $L_{loc}^1(\mathbb{R})$, we can choose ϕ^1 as a right-continuous function. Therefore, the function ϕ^1 will be a defined and non-decreasing function on $\mathbb{R}$.

Moreover, using the equality (6.5) and the strong convergence of $(\phi_\epsilon)_\epsilon$ in $L_{loc}^1(\mathbb{R})$, we also get

$$\phi_{\epsilon'}^2 \longrightarrow \phi^2, \quad \text{strongly in } L_{loc}^1(\mathbb{R}),$$

where ϕ^2 is a right-continuous and non-decreasing function on $\mathbb{R}$ satisfying $\phi^2 = \phi^1 - \phi$. Thus, by Lemma 6.2 (i), we deduce that, for every $a > 0$ and for all $0 < \delta \leq \frac{a}{2}$, there exists $\epsilon_{a,1}^\delta$ and $\epsilon_{a,2}^\delta$ such that the following hold:

$$\text{for all } 0 < \epsilon' \leq \epsilon_{a,1}^\delta, \quad \phi_{\epsilon'}^1(x) \leq \delta + \phi^1(x + \delta), \quad \forall x \in [-a, a]$$

and

$$\text{for all } 0 < \epsilon' \leq \epsilon_{a,2}^\delta, \quad -\delta + \phi^2(x - \delta) \leq \phi_{\epsilon'}^2(x), \quad \forall x \in [-a, a].$$

Collecting the two previous estimates with equality (6.5), we obtain that, for all $0 < \delta \leq \frac{a}{2}$ and $0 < \epsilon' \leq \epsilon_a^\delta = \inf(\epsilon_{a,1}^\delta, \epsilon_{a,2}^\delta)$

$$\phi_{\epsilon'}(x) \leq 2\delta + \phi^1(x + \delta) - \phi^2(x - \delta), \quad \forall x \in [-a, a].$$

The left hand side of (6.4) is obtained similarly. ■

Remark 6.4. In the previous lemma, we decided to choose, for a uniformly bounded sequence in $BV(\mathbb{R})$, a right-continuous limit among the strong $L^1_{loc}(\mathbb{R})$ limits. We can also replace it by a left-continuous limit, which is not an issue since, it is only necessary to construct a limit in $L^1_{loc}(\mathbb{R})$, that is defined and non-decreasing everywhere in $\mathbb{R}$.

6.2. Proof of Theorem 1.2

Let u_ϵ be the solution of (4.6), constructed in Theorem 1.1 (i). From estimates (1.8), (1.9) and (1.10), we can say that, for all compact $K_0 \subset \mathbb{R}$, $(u_\epsilon)_\epsilon$ is uniformly bounded in $L^\infty((0, T); BV(K_0)) \cap L^\infty(K_0 \times (0, T))$ and $(\partial_t u_\epsilon)_\epsilon$ is uniformly bounded in $L^\infty((0, T); L^1(K_0))$. Using Simon's lemma (see Lemma 4.4), in the particular case $X = BV(K_0)$, $B = Y = L^1(K_0)$ and the following compact embedding $BV(K_0) \hookrightarrow L^1(K_0)$, we can extract a subsequence, denoted by $(u_{\epsilon_n, K_0})_{\epsilon_n, K_0}$, that converges strongly in $L^\infty((0, T); L^1(K_0))$ to some limit u , as $n \rightarrow +\infty$. By a standard diagonalization procedure, we can extract a subsequence $(u_{\epsilon_n})_{\epsilon_n}$ (independent on K) that converges to the limit u strongly in $C([0, T]; L^1(K))$ for all compact $K \subset \mathbb{R}$. Now, thanks to estimate (1.8), we can extract a subsequence, still denoted by $(u_{\epsilon_n})_{\epsilon_n}$, satisfying the following convergences:

$$\left\{ \begin{array}{ll} u_{\epsilon_n} \longrightarrow u, & \text{strongly in } C([0, T]; L^1(K)), \text{ for all compact } K \subset \mathbb{R}, \\ u_{\epsilon_n} \longrightarrow u, & \text{weakly-} \star \text{ in } L^\infty(\mathbb{R} \times (0, T)), \\ u_{\epsilon_n} \longrightarrow u, & \text{weakly-} \star \text{ in } L^\infty((0, T); BV(\mathbb{R})). \end{array} \right. \quad (6.7)$$

Taking the $\liminf$ in estimates (1.8), (1.9) and using the lower semi-continuity of $\|\cdot\|_{L^\infty(\mathbb{R})}$ and $|\cdot|_{BV(\mathbb{R})}$, we can prove that u satisfies (1.11), (1.12) and (1.13). Moreover, using estimate (1.10), Remark 4.5 and the strong convergence in $C([0, T]; L^1_{loc}(\mathbb{R}))$ of u_{ϵ_n} , we can check that u satisfies also the L^1 -Lipschitz estimate in time (1.14). Since, for all $t \in [0, T]$, the function $u(\cdot, t) \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$, then, by property (ii) of Lemma 6.1, we know that this function coincides with a right-continuous function almost everywhere in $\mathbb{R}$ and consequently in $L^1_{loc}(\mathbb{R})$. This allows us to consider, in the following, a right-continuous limit with respect to the space variable.

Now, we will prove that u is a discontinuous viscosity solution of (1.1). Since we have proved in Theorem 1.1 (ii) that $\bar{u}$ and $\underline{u}$ are respectively discontinuous viscosity sub- and super-solutions, then it is sufficient to show that

$$\bar{u}(x, t) = u^*(x, t) \quad \text{and} \quad \underline{u}(x, t) = u_*(x, t) \quad \text{for all } (x, t) \in \mathbb{R} \times [0, T] \quad (6.8)$$

We only show the proof of the first equality, the second is proved similarly.

Step 1. We will prove the following inequality

$$\bar{u}(x, t) \leq u^*(x, t). \quad (6.9)$$

Let $a > 0$, $x \in [-\frac{a}{2}, \frac{a}{2}]$ and $t \in [0, T]$. In fact, by the definition of $\bar{u}$, we know that there exists a sequence $(\epsilon_m, x_{\epsilon_m}, t_{\epsilon_m}) \rightarrow (0, x, t)$, when $m \rightarrow +\infty$, such that

$$\bar{u}(x, t) = \lim_{m \rightarrow +\infty} u_{\epsilon_m}(x_{\epsilon_m}, t_{\epsilon_m}).$$

For all $\alpha > 0$, we can state that, there exists $m_\alpha > 0$, such that, for all $m \geq m_\alpha$, we have

$$|x_{\epsilon_m} - x| \leq \alpha \quad \text{and} \quad |t_{\epsilon_m} - t| \leq \alpha.$$

Using Lemma 5.4, with

$$h_\alpha = \begin{cases} t - \alpha & \text{if } t > 0, \\ 0 & \text{if } t = 0, \end{cases}$$

we deduce that, for all $m \geq m_\alpha$ and $\alpha > 0$ such that $h_\alpha \geq 0$,

$$\begin{aligned} u_{\epsilon_m}(x_{\epsilon_m}, t_{\epsilon_m}) &\leq \sup_{|y - x_{\epsilon_m}| \leq (t_{\epsilon_m} - h_\alpha) \|c_{\epsilon_m}\|_{L^\infty(\mathbb{R} \times (0, T))}} u_{\epsilon_m}(y, h_\alpha) \\ &\leq \sup_{|y - x_{\epsilon_m}| \leq 2\alpha \|c\|_{L^\infty(\mathbb{R} \times (0, T))}} u_{\epsilon_m}(y, h_\alpha) \\ &\leq \sup_{|y - x| \leq \alpha(2\|c\|_{L^\infty(\mathbb{R} \times (0, T))} + 1)} u_{\epsilon_m}(y, h_\alpha), \end{aligned}$$

where we have used in the second line the definition of the function c_{ϵ_m} in (4.7) and the classical properties of the mollifiers. Moreover, from the maximum principle of the eikonal equation (4.6) and since the initial data is non-decreasing, we know that u_{ϵ_m} is non-decreasing (with respect to x) and therefore, for all $m \geq m_\alpha$,

$$u_{\epsilon_m}(x_{\epsilon_m}, t_{\epsilon_m}) \leq u_{\epsilon_m}(x + \alpha(2\|c\|_{L^\infty(\mathbb{R} \times (0, T))} + 1), h_\alpha).$$

Now, using the fact that u_{ϵ_m} is non-decreasing (with respect to x) and satisfies estimates (1.8), (1.9), (1.10), we can extract a subsequence, still denoted by $(u_{\epsilon_m})_{\epsilon_m}$, that converges in the sense of (6.7) to a function u , where, for all $t \in [0, T)$, the function $u(\cdot, t)$ is defined and non-decreasing on $\mathbb{R}$. By the previous inequality and Lemma 6.2 (ii), we obtain that, for all $0 < \alpha \leq \alpha_a$, where

$$\alpha_a(\|c\|_{L^\infty(\mathbb{R} \times (0, T))}, t) = \begin{cases} \min\left(\frac{a}{2(2\|c\|_{L^\infty(\mathbb{R} \times (0, T))} + 1)}, \frac{t}{2}\right) & \text{if } t > 0, \\ \frac{a}{2(2\|c\|_{L^\infty(\mathbb{R} \times (0, T))} + 1)}, & \text{if } t = 0, \end{cases}$$

there exists $m_{a,T}^\alpha > 0$, such that, for every $m \geq m_{a,T}^\alpha$, we have

$$u_{\epsilon_m}(x_{\epsilon_m}, t_{\epsilon_m}) \leq u(x + 2\alpha(\|c\|_{L^\infty(\mathbb{R} \times (0, T))} + 1), h_\alpha) + \alpha.$$

Passing to the limit $m \rightarrow +\infty$ and then $\alpha \rightarrow 0$, we obtain (6.9).

Step 2. It remains to show that

$$u^*(x, t) \leq \bar{u}(x, t). \quad (6.10)$$

Consider $a > 0$, $x \in [-\frac{a}{2}, \frac{a}{2}]$ and $t \in [0, T)$. In fact, from the definition of u^* we know that there exists a sequence $(x_{\epsilon_m}, t_{\epsilon_m}) \rightarrow (x, t)$, when $m \rightarrow +\infty$, such that

$$u^*(x, t) = \lim_{m \rightarrow +\infty} u(x_{\epsilon_m}, t_{\epsilon_m}).$$

Similarly, as in Step 1, we can state that, for all $\alpha > 0$, there exists $m_\alpha > 0$, such that, for all $m \geq m_\alpha$, we have

$$|x_{\epsilon_m} - x| \leq \alpha \quad \text{and} \quad |t_{\epsilon_m} - t| \leq \alpha.$$

However, using Lemma 6.2 (ii), we know that, for all $0 < \alpha \leq \frac{a}{2}$, there exists $k_{a,T}^\alpha > 0$ and a subsequence $0 < \alpha_k \leq \alpha$ such that, for every $k \geq k_{a,T}^\alpha$,

$$u(x_{\epsilon_m}, t_{\epsilon_m}) \leq u_{\alpha_k}(x_{\epsilon_m} + \alpha, t_{\epsilon_m}) + \alpha \leq \sup_{\substack{\alpha_k \leq \alpha, |s-t| \leq 2\alpha \\ |y-x| \leq 2\alpha}} u_{\alpha_k}(y, s) + \alpha.$$

Passing to the limit $m \rightarrow +\infty$ and then $\alpha \rightarrow 0$, we obtain (6.10). $\blacksquare$

6.3. Proof of Theorem 1.1 (iii)

Let u_ϵ be the solution of (4.6), constructed in Theorem 1.1 (i). As explained at the beginning of the proof of Theorem 1.2, we can extract a subsequence u_{ϵ_n} satisfying (6.7) with a limit u that verifies (1.11), (1.12), (1.13) and (1.14). Moreover, for all $t \in [0, T]$, $u(\cdot, t)$ is a right-continuous function on $\mathbb{R}$. It remains to show equality (1.15). For better presentation, we'll perform this in three steps.

Step 1 (Regularity in time estimate): Let $T > 0$, $a > 0$ and set $\gamma_c = 2(\|c\|_{L^\infty(\mathbb{R} \times (0, T))} + 1)$. First, we will show that there are two bounded and non-decreasing functions u^1, u^2 satisfying $u = u^1 - u^2$ and the following inequalities

$$\begin{aligned} -2h + u^1(x - h\gamma_c, t) - u^2(x + h\gamma_c, t) &\leq \underline{u}(x, t + h) \\ &\leq \bar{u}(x, t + h) \leq 2h + u^1(x + h\gamma_c, t) - u^2(x - h\gamma_c, t) \end{aligned} \quad (6.11)$$

for all $x \in [-\frac{a}{2}, \frac{a}{2}]$, $t \in [0, T]$ and for all $h > 0$ verifying

$$h \leq \frac{a}{2(2\|c\|_{L^\infty(\mathbb{R} \times (0, T))} + 1)} \quad \text{and} \quad t + h < T. \quad (6.12)$$

We begin with the proof of the right inequality in (6.11), namely,

$$\bar{u}(x, t + h) \leq 2h + u^1(x + h\gamma_c, t) - u^2(x - h\gamma_c, t). \quad (6.13)$$

Indeed, consider $h > 0$ satisfying (6.12), by the definition of $\bar{u}$, we know that there exists a sequence $(\epsilon_m, x_{\epsilon_m}, t_{\epsilon_m}^h) \rightarrow (0, x, t + h)$, when $m \rightarrow +\infty$, such that,

$$\bar{u}(x, t + h) = \lim_{m \rightarrow +\infty} u_{\epsilon_m}(x_{\epsilon_m}, t_{\epsilon_m}^h).$$

Now, since $(x_{\epsilon_m}, t_{\epsilon_m}^h) \rightarrow (x, t + h)$ when $m \rightarrow +\infty$, we can see that there exists $m_h > 0$, such that, for all $m \geq m_h$, we have:

$$|x_{\epsilon_m} - x| < h \quad \text{and} \quad |t_{\epsilon_m}^h - t - h| < h.$$

Using Lemma 5.4, we deduce that, for all $m \geq m_h$,

$$\begin{aligned} u_{\epsilon_m}(x_{\epsilon_m}, t_{\epsilon_m}^h) &\leq \sup_{|y-x_{\epsilon_m}| \leq (t_{\epsilon_m}^h - t)\|c_{\epsilon_m}\|_{L^\infty(\mathbb{R} \times (0, T))}} u_{\epsilon_m}(y, t) \\ &\leq \sup_{|y-x_{\epsilon_m}| \leq 2h\|c\|_{L^\infty(\mathbb{R} \times (0, T))}} u_{\epsilon_m}(y, t) \\ &\leq \sup_{|y-x| \leq h(2\|c\|_{L^\infty(\mathbb{R} \times (0, T))} + 1)} u_{\epsilon_m}(y, t), \end{aligned} \quad (6.14)$$

where we have used in the second line the definition of the function c_{ϵ_m} in (4.7) and the classical properties of the mollifiers. Since, for all $t \in [0, T]$, the sequence $u_{\epsilon_m}(\cdot, t)$ is uniformly bounded in $L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ and converges

strongly in $L^1_{loc}(\mathbb{R})$, we can deduce, from Lemma 6.3, that there exists a subsequence $u_{\epsilon_n}(\cdot, t)$ and a positive constant $n_{a,t}^h$, such that, for all $n \geq n_{a,t}^h$, we have

$$u_{\epsilon_n}(y, t) \leq 2h + u^1(y + h, t) - u^2(y - h, t), \quad \forall y \in [-a, a], \quad (6.15)$$

where u^1 and u^2 are two bounded, right-continuous and non-decreasing functions (with respect to x) satisfying $u = u^1 - u^2$. Collecting (6.14) and (6.15), we obtain that, for all $h > 0$ satisfying (6.12) and for all $n \geq n_{a,t}^h$,

$$u_{\epsilon_n}(x_{\epsilon_n}, t_{\epsilon_n}^h) \leq 2h + u^1(x + h\gamma_c, t) - u^2(x - h\gamma_c, t).$$

We pass to the limit $n \rightarrow +\infty$ to get (6.13). Similarly, using the finite speed propagation property, specifically the left inequality in (5.4), and the fact that $u_{\epsilon_n}(\cdot, t)$ is uniformly bounded in $L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$, we can prove the left inequality in (6.11), namely,

$$-2h + u^1(x - h\gamma_c, t) - u^2(x + h\gamma_c, t) \leq \underline{u}(x, t + h). \quad (6.16)$$

Step 2 (Right and left continuity): Let $T > 0$ and $t \in [0, T)$. Since $u^1(\cdot, t)$, $u^2(\cdot, t)$ are bounded and non-decreasing functions on $\mathbb{R}$, then, from property (ii) of Lemma 6.1, we know that, the right and left limits of these functions exist at every point $x \in \mathbb{R}$. This implies that, for all $\alpha > 0$ and $x \in [-\frac{\alpha}{2}, \frac{\alpha}{2}]$, there exists $h_{a,t}^\alpha > 0$, such that, for all $0 < z \leq h_{a,t}^\alpha$, we have

$$\begin{cases} u^1(x + z, t) \leq \frac{\alpha}{4} + u_r^1(x, t) \\ u^2(x + z, t) \leq \frac{\alpha}{4} + u_r^2(x, t) \end{cases} \quad \text{and} \quad \begin{cases} -u^1(x - z, t) \leq \frac{\alpha}{4} - u_l^1(x, t) \\ -u^2(x - z, t) \leq \frac{\alpha}{4} - u_l^2(x, t) \end{cases} \quad (6.17)$$

where $u_r^1(\cdot, t)$, $u_r^2(\cdot, t)$, are right-continuous functions on $\mathbb{R}$ and $u_l^1(\cdot, t)$, $u_l^2(\cdot, t)$ are left-continuous functions on $\mathbb{R}$. Note that, in here, the choice of the constant $h_{a,t}^\alpha$ does not depend on x , that is a consequence of Heine–Cantor Theorem.

Now, let $T > 0$, $t \in [0, T)$ and $\alpha > 0$, we can observe that, if we denote

$$\bar{h}_{a,t,T}^\alpha = \min \left(\frac{h_{a,t}^\alpha}{\gamma_c}, \frac{a}{2(2\|c\|_{L^\infty(\mathbb{R} \times (0,T))} + 1)}, \frac{\alpha}{4}, \frac{T-t}{2} \right),$$

then, for all $0 < h \leq \bar{h}_{a,t,T}^\alpha$ the assumption (6.12) holds. Therefore, we obtain

$$\bar{u}(x, t + h) \leq 2h + u^1(x + h\gamma_c, t) - u^2(x - h\gamma_c, t) \leq \alpha + u_r^1(x, t) - u_l^2(x, t), \quad (6.18)$$

where we have used (6.13) in the first inequality and (6.17) in the second one. Similarly, using (6.16) and (6.17), we can prove that, for all $x \in [-\frac{\alpha}{2}, \frac{\alpha}{2}]$, $t \in [0, T)$, $\alpha > 0$ and $0 < h \leq \bar{h}_{a,t,T}^\alpha$, we have

$$-\alpha + u_l^1(x, t) - u_r^2(x, t) \leq \underline{u}(x, t + h). \quad (6.19)$$

Step 3 (Link between $\bar{u}$ and $\underline{u}$): Let $T > 0$, $x \in [-\frac{\alpha}{2}, \frac{\alpha}{2}]$, $t \in [0, T)$ and $\alpha > 0$. In Step 2, we proved that, there exists a positive constant $\bar{h}_{a,t,T}^\alpha$, such that, for all $0 < h \leq \bar{h}_{a,t,T}^\alpha$, we have

$$\begin{aligned}\bar{u}(x, t+h) &\leq \alpha + u_r^1(x, t) - u_l^2(x, t) \\ -\underline{u}(x, t+h) &\leq \alpha - u_l^1(x, t) + u_r^2(x, t).\end{aligned}\quad (6.20)$$

Since $\bigcup_{t \in [0, T]} [t, t + \bar{h}_{a, t, T}^\alpha]$ is a cover of $[0, \frac{T}{2}]$, then there is a finite number N_a^α of ordered intervals, satisfying

$$\begin{aligned}\bigcup_{0 \leq i \leq N_a^\alpha} [\tau_{a, i}^\alpha, \tau_{a, i}^\alpha + \bar{h}_{a, \tau_{a, i}^\alpha, T}^\alpha] &\supset \left[0, \frac{T}{2}\right] \quad \text{with} \quad \tau_{a, 0}^\alpha = 0 \quad \text{and} \\ \tau_{a, i+1}^\alpha &= \tau_{a, i}^\alpha + \bar{h}_{a, \tau_{a, i}^\alpha, T}^\alpha \quad \text{for} \quad i = 1, \dots, N_a^\alpha - 1.\end{aligned}$$

This expression joint to (6.20) and the fact that $\mathbb{R} = \bigcup_{a \in \mathbb{Q}} [-\frac{a}{2}, \frac{a}{2}]$ shows that, for all $x \in \mathbb{R}$, $\tau \in [0, \frac{T}{2}]$ and for all positive constant $\alpha \in \mathbb{Q}$, there exist two indices $a_0 \in \mathbb{Q}$ and $0 \leq j \leq N_{a_0}^\alpha$, such that,

$$\begin{aligned}\bar{u}(x, \tau) &\leq \alpha + u_r^1(x, \tau_{a_0, j}^\alpha) - u_l^2(x, \tau_{a_0, j}^\alpha) \\ -\underline{u}(x, \tau) &\leq \alpha - u_l^1(x, \tau_{a_0, j}^\alpha) + u_r^2(x, \tau_{a_0, j}^\alpha).\end{aligned}\quad (6.21)$$

Moreover, from property (ii) of Lemma 6.1, we know that, for all positive constants $\alpha, a \in \mathbb{Q}$ and $0 \leq i \leq N_a^\alpha$, the functions $u_r^1(\cdot, \tau_{a, i}^\alpha)$, $u_l^1(\cdot, \tau_{a, i}^\alpha)$ (resp. $u_r^2(\cdot, \tau_{a, i}^\alpha)$, $u_l^2(\cdot, \tau_{a, i}^\alpha)$) coincide with $u^1(\cdot, \tau_{a, i}^\alpha)$ (resp. $u^2(\cdot, \tau_{a, i}^\alpha)$) except on a countable set on $\mathbb{R}$, denoted $D_{a, i}^\alpha$. Now, we define the following countable set

$$D = \bigcup_{a, \alpha \in \mathbb{Q}} \bigcup_{0 \leq i \leq N_a^\alpha} D_{a, i}^\alpha.$$

Thanks to (6.21), we can see that, for all $x \notin D$, $\tau \in [0, \frac{T}{2}]$ and for all positive constant $\alpha \in \mathbb{Q}$, there exist two indices $a_0 \in \mathbb{Q}$ and $0 \leq j \leq N_{a_0}^\alpha$, such that

$$\begin{aligned}\bar{u}(x, \tau) &\leq \alpha + u^1(x, \tau_{a_0, j}^\alpha) - u^2(x, \tau_{a_0, j}^\alpha) \leq \alpha + u(x, \tau_{a_0, j}^\alpha) \\ -\underline{u}(x, \tau) &\leq \alpha - u^1(x, \tau_{a_0, j}^\alpha) + u^2(x, \tau_{a_0, j}^\alpha) \leq \alpha - u(x, \tau_{a_0, j}^\alpha).\end{aligned}\quad (6.22)$$

Adding the previous inequalities, we deduce that, for all rational number $\alpha > 0$, $x \notin D$ and $\tau \in [0, \frac{T}{2}]$,

$$0 \leq \bar{u}(x, \tau) - \underline{u}(x, \tau) \leq 2\alpha.$$

Passing to the limit $\alpha \rightarrow 0$ and replacing T by $2T$, we get

$$\bar{u}(\cdot, \tau) = \underline{u}(\cdot, \tau) \quad \text{except at most on a countable set in } \mathbb{R}, \quad \text{for all } \tau \in [0, T]. \quad (6.23)$$

This equality allows us to link the sub-solution $\bar{u}$ and the super-solution $\underline{u}$. It remains to show that

$$u(\cdot, \tau) = \bar{u}(\cdot, \tau) \quad \text{except at most on a countable set in } \mathbb{R}, \quad \text{for all } \tau \in [0, T].$$

To do this, it is sufficient to use, the right continuity of the functions $u(\cdot, \tau)$, $u^1(\cdot, \tau)$ and Lemma 6.2 (i). Indeed, let $\alpha > 0$, the right continuity of the

functions $u(\cdot, \tau)$, $u^1(\cdot, \tau)$, implies that, for all $x \in [-\frac{a}{2}, \frac{a}{2}]$, there exists $\alpha_{a,\tau}^1 > 0$, such that, for all $0 < \delta \leq \alpha_{a,\tau}^1$, we have

$$\begin{aligned} u(x, \tau) &\leq \alpha + u(x + \delta, \tau) \\ &= \alpha + u^1(x + \delta, \tau) - u^2(x + \delta, \tau) \\ &\leq 2\alpha + u^1(x, \tau) - u^2(x + \delta, \tau) \end{aligned} \quad (6.24)$$

where u^1, u^2 are the right-continuous non-decreasing functions, given in (6.15). However, using Lemma 6.2 (i), we know that, for all $0 < \delta \leq \frac{a}{2}$, there exists $k_{a,\tau}^\alpha > 0$ and a subsequence $0 < \epsilon_k \leq \delta$ such that for every $k \geq k_{a,\tau}^\alpha$,

$$\begin{cases} u^1(x, \tau) \leq \frac{\delta}{2} + u_{\epsilon_k}^1(x + \frac{\delta}{2}, \tau) \\ -u^2(x + \delta, \tau) \leq \frac{\delta}{2} - u_{\epsilon_k}^2(x + \frac{\delta}{2}, \tau) \end{cases} \quad (6.25)$$

where the sequences $u_{\epsilon_k}^1$ and $u_{\epsilon_k}^2$ satisfy the following equality $u_{\epsilon_k} = u_{\epsilon_k}^1 - u_{\epsilon_k}^2$ and are constructed as in (6.6). Finally, bringing together the two inequalities (6.24) and (6.25), we see that, for all $0 < \delta \leq \min(\frac{a}{2}, \alpha_{a,\tau}^1)$ and $k \geq k_{a,\tau}^\alpha$, we have

$$u(x, \tau) \leq 2\alpha + \delta + u_{\epsilon_k}\left(x + \frac{\delta}{2}, \tau\right) \leq 2\alpha + \delta + \sup_{\substack{\epsilon_k \leq \delta, |s-\tau| \leq \delta \\ |y-x| \leq 2\delta}} u_{\epsilon_k}(y, s).$$

To complete the proof, we pass to the limit $\delta \rightarrow 0$ and then $\alpha \rightarrow 0$, to get $u(x, \tau) \leq \bar{u}(x, \tau)$. Similarly, we can show that $\underline{u}(x, \tau) \leq u(x, \tau)$, which joint to (6.23) proves the desired result. ■

Note that, equality (1.15) implies that the functions $\bar{u}$, $\underline{u}$ also satisfy estimates (1.12), (1.13) and (1.14).

7. Application to dislocation dynamics

In this section, we study a simple model for dislocation dynamics. A perfect crystal, for small deformations, is well described by the equations of linear elasticity. The real crystals contain in particular some line defects called dislocations (we refer to Hirth et al. [20] for a physical presentation of dislocations). The dislocation dynamics is one of the main explanation of the plastic deformation of metals. When we apply an exterior stress, these dislocations lines can move in a slip plane of the crystal. The dislocation dynamics is given by a normal velocity proportional to the Peach–Koehler force, calculated from the equations of linear elasticity. Here, we are interested in the study of a one-dimensional sub-model of the particular model introduced by Rodney et al. [24], initially proposed in the two-dimensional case. More precisely, we consider a simple geometry where the dislocations are parallel lines moving in a same plane (xy) . This plane is embedded in a three dimensional elastic crystal. The particular geometry of this problem leads to study a one-dimensional model given by the following non-local eikonal equation

$$\begin{cases} \partial_t u(x, t) = (c_0(\cdot, t) \star u(\cdot, t)(x)) |\partial_x u(x, t)| & \text{in } \mathbb{R} \times (0, T) \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}, \end{cases} \quad (7.1)$$

where the function c_0 is a kernel associated to the equations of linearized elasticity and $\star$ denotes the convolution in space. Let us give some references for this model: we refer the reader to Barles et al. in [7] where weak existence and uniqueness results are provided in a particular setting of L^1 -viscosity solutions. We also note that, with some positivity condition on the velocity, existence and uniqueness results were first obtained by Alvarez et al. in [1] and then generalized by Barles et al. in [8]. In this section, applying Theorems 1.1 and 1.2, we prove a global existence result of this equation (announced in Theorem 1.3), without any positivity condition on the velocity and considering a $BV(\mathbb{R})$ initial data. To do this, we divide this section into two subsections. In the first one, after regularizing the initial data, we prove the existence of Lipschitz continuous viscosity solution of the regularized problem. Then, using equality (1.15) and the stability result of discontinuous viscosity solutions, we will be able to pass to the limit and prove the result.

7.1. Existence, uniqueness and a priori estimates of the regularized problem

In this subsection we consider, for $\epsilon > 0$, the following regularized problem of (7.1)

$$\begin{cases} \partial_t u_\epsilon(x, t) = (c_0(\cdot, t) \star u_\epsilon(\cdot, t)(x)) |\partial_x u_\epsilon(x, t)| & \text{in } \mathbb{R} \times (0, T) \\ u_\epsilon(x, 0) = u_{0,\epsilon}(x) & \text{in } \mathbb{R} \end{cases} \quad (7.2)$$

where $u_{0,\epsilon}$ is the regularization of u_0 defined in (4.7). First, we define, for all $\epsilon > 0$ and $T > 0$, the set

$$X_T^\epsilon = \left\{ v_\epsilon \in L^\infty(\mathbb{R} \times (0, T)), \quad \text{s.t.} \quad \begin{cases} \|v_\epsilon\|_{L^\infty(\mathbb{R} \times (0, T))} \leq \|u_0\|_{L^\infty(\mathbb{R})} \\ \|\partial_x v_\epsilon\|_{L^\infty(\mathbb{R} \times (0, T))} \leq L_\epsilon(T) \\ \|\partial_x v_\epsilon\|_{L^\infty((0, T); L^1(\mathbb{R}))} \leq |u_0|_{BV(\mathbb{R})} \\ \|\partial_t v_\epsilon\|_{L^\infty((0, T); L^1(\mathbb{R}))} \leq K \end{cases} \right\}$$

with

$$L_\epsilon(T) = \|\partial_x u_{0,\epsilon}\|_{L^\infty(\mathbb{R})} e^{\|c_0\|_{L^\infty(\mathbb{R} \times (0, T))} |u_0|_{BV(\mathbb{R})} T} \text{ and } K = \|c_0\|_{L^\infty((0, T); L^1(\mathbb{R}))} \|u_0\|_{L^\infty(\mathbb{R})} |u_0|_{BV(\mathbb{R})}.$$

Now, for $v_\epsilon \in X_T^\epsilon$, we define $u_\epsilon = G(v_\epsilon)$ as the unique Lipschitz continuous viscosity solution of the following local problem

$$\begin{cases} \partial_t u_\epsilon(x, t) = (c_0(\cdot, t) \star v_\epsilon(\cdot, t)(x)) |\partial_x u_\epsilon(x, t)| & \text{in } \mathbb{R} \times (0, T) \\ u_\epsilon(x, 0) = u_{0,\epsilon}(x) & \text{in } \mathbb{R}. \end{cases} \quad (7.3)$$

We will show that $G : X_T^\epsilon \longrightarrow X_T^\epsilon$ is a strict contraction for T small enough. First, we will prove that G is well defined. We can see, using Young's inequality, that, if $v_\epsilon \in X_T^\epsilon$, then, the velocity $c_0 \star v_\epsilon$ satisfies (1.2), which implies by Theorem 1.1 (i) that Eq. (7.3) admits a unique Lipschitz continuous viscosity solution u_ϵ that satisfies estimates (1.8), (1.9), (1.10) and also estimate (4.11) replacing c_ϵ by $c_0 \star v_\epsilon$. This shows, by Young's inequality, that $u_\epsilon \in X_T^\epsilon$.

It, thus, remains to show that G is a contraction. Let $u_\epsilon^i = G(v_\epsilon^i)$, $i = 1, 2$ and set $K = \|v_\epsilon^1 - v_\epsilon^2\|_{L^\infty(\mathbb{R} \times (0, T))}$. We remark that u_ϵ^2 is a viscosity sub-solution of

$$\partial_t u_\epsilon(x, t) = (c_0(\cdot, t) \star v_\epsilon^1(\cdot, t)(x)) |\partial_x u_\epsilon(x, t)| + \|c_0\|_{L^\infty((0, T); L^1(\mathbb{R}))} L_\epsilon(T) K.$$

Moreover, $u_\epsilon^1 + \|c_0\|_{L^\infty((0, T); L^1(\mathbb{R}))} L_\epsilon(T) K t$ is a viscosity solution of the same equation. By comparison principle, we deduce, then

$$\|u_\epsilon^1 - u_\epsilon^2\|_{L^\infty(\mathbb{R} \times (0, T))} \leq T \|c_0\|_{L^\infty((0, T); L^1(\mathbb{R}))} L_\epsilon(T) \|v_\epsilon^1 - v_\epsilon^2\|_{L^\infty(\mathbb{R} \times (0, T))}.$$

This shows that, for T small enough, G is a contraction on X_T^ϵ which is a closed set. So, by fixed point theorem, there exists a unique Lipschitz continuous viscosity solution of (7.2) in X_T^ϵ , for all $T > 0$, such that $T \|c_0\|_{L^\infty((0, T); L^1(\mathbb{R}))} L_\epsilon(T) \leq \frac{1}{2}$. By iterating this process, we can construct a solution for all $T > 0$.

7.2. Passage to the limit and proof of Theorem 1.3

The aim of this section is to prove that the relaxed semi-limits $\bar{u}$ and $\underline{u}$ (defined in (1.6)–(1.7)) are, respectively, discontinuous viscosity sub-solution and super-solutions of Eq. (7.1), in the sense of Definition 5.1, replacing in (5.2) and (5.3), the functions c^* and c_* by $c_0 \star \bar{u}$ and $c_0 \star \underline{u}$, respectively.

Note that, Theorem 1.3 (i) follows directly from the stability result of discontinuous viscosity solutions (see Theorem 5.2) and equality (1.15). In fact, if u_ϵ is the solution of (7.2), given in the previous subsection, then, using the continuity of c_0 , we can see that the relaxed semi-limit $\bar{u}$ (resp. $\underline{u}$) is discontinuous viscosity sub-solution (resp. super-solution) of the following equation

$$\begin{aligned} \partial_t u(x, t) &= (c_0^+(\cdot, t) \star \bar{u}(\cdot, t)(x) - c_0^-(\cdot, t) \star \underline{u}(\cdot, t)(x)) |\partial_x u(x, t)| \\ (\text{resp. } \partial_t u(x, t) &= (c_0^+(\cdot, t) \star \underline{u}(\cdot, t)(x) - c_0^-(\cdot, t) \star \bar{u}(\cdot, t)(x)) |\partial_x u(x, t)|), \end{aligned}$$

where c_0^+ and c_0^- are, respectively, the positive and the negative parts of c_0 , defined as

$$c_0^+ = \frac{|c_0| + c_0}{2} \quad \text{and} \quad c_0^- = \frac{|c_0| - c_0}{2}.$$

However, from equality (1.15), we know that

$$\begin{aligned} c_0^+(\cdot, t) \star \bar{u}(\cdot, t)(x) - c_0^-(\cdot, t) \star \underline{u}(\cdot, t)(x) &= (c_0^+(\cdot, t) - c_0^-(\cdot, t)) \star \bar{u}(\cdot, t) \\ &= c_0(\cdot, t) \star \bar{u}(\cdot, t)(x) \end{aligned}$$

and

$$\begin{aligned} c_0^+(\cdot, t) \star \underline{u}(\cdot, t)(x) - c_0^-(\cdot, t) \star \bar{u}(\cdot, t)(x) &= (c_0^+(\cdot, t) - c_0^-(\cdot, t)) \star \underline{u}(\cdot, t) \\ &= c_0(\cdot, t) \star \underline{u}(\cdot, t)(x). \end{aligned}$$

Therefore, we obtain that $\bar{u}$ and $\underline{u}$ are, respectively, discontinuous viscosity sub and super-solutions of (7.1). To prove Theorem 1.3 (ii), we repeat the same thing using equality (6.8) instead of (1.15), which is only valid in the case of non-decreasing solutions. $\blacksquare$

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